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AN APPROACH TO RELIABILITY EVALUATION OF GENERAL NETWORKS WITH REPAIRABLE COMPONENTS

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1. Introduction and Summary[1)]

On account of increasing needs to evaluate reliabilities of multiplexed systems under various maintenance conditions, the so-called repair man problems have been investigated by many contributors. They have constructed a lot of probability models and have given their solutions corresponding to the specified restrictions of the models. Most of them have supposed special structures of their systems and assumed that at least one of the life-time and repair-time distributions of each component of the systems is exponential. This assumption makes it possible to treat the problems by the familiar methods of Markov or Markov renewal processes except the case of two-unit stand-by redundant system. This last was exceptionally solved by

1) The summary of the paper was presented at the 1972 autumn meeting of the Operations Research Society of Japan held in Sapporo, Sept. 25-27.

Gnedenko *et al.* [3] and by Srinivasan [5] without assuming the exponentiality.

The purpose of the present paper is to show a method of evaluating the reliability of general systems, assuming neither exponential life-time nor exponential repair-time distributions of the components. Naturally, we assume that the system under consideration is of monotonic structure and that all the paths and cuts of the structure are clearly identified, where the terms monotonic structure, paths and cuts are defined just as in Barlow and Proschan [1, p. 203]. That is, the monotonic structure is such that the failure of the structure can never be recovered by the additional failures of its operative components and the function of the structure can never be disturbed by the recoveries of its inoperative components under the primary convention that the functioning [failure] of all components assures at least the functioning [failure] of the structure, and "a path is a minimal set of components which by functioning ensure the functioning of the structure" and "a cut is a minimal set of components which by failing guarantee the failure of the structure" [1]. These definitions of the path and cut correspond to the ones of the minimal and the maximal state vectors of the monotonic structure introduced previously by Mine [4], and they are rather generalized than the usual ones in a graph or network with binary function and with binary components. Thus, the notion of the system of the monotonic structure with the paths and cuts defined here is rather extended than the corresponding strict one of a two-terminal network constructed from binary components with its paths and cuts. However, let us call the system characterized above simply as the (generalized two-terminal) network in the subsequent descriptions of the paper, since even if it is not the network of the strict sense it is stochastically equivalent to the hypothesized two-terminal unique-branch network whose branch connects the terminals directly and functions if and only if the monotonic structure does, *i.e.*, the hypothesized network operates if and only if there exists at least a path alive in the monotonic structure. It should be noted here that the terms, the network and paths and cuts, must be interpreted in the present paper as the extended meaning just mentioned above.

Let us consider three ranks, rank-1, rank-2 and rank-3, of the components specified as follows:

Rank-1 component is repairable but has neither warm nor cold stand-by components inside or outside of the network; rank-2 component is repairable and has a_i additional cold stand-by components of its own outside the network ($a_i>0$); rank-3 component is unrepairable but has b_i additional cold stand-by components of its own outside the network ($b_i \geqslant 0$), where cold stand-by components are put, one by one, into the network in place of the failed component of the same kind as far as they are provided and the term cold stand-by components here means that they neither fail nor age in their stand-by situations, and similarly warm stand-by components means that they fail less frequently in the stand-by situations than in the operative conditions.

For simplicity of investigation let us first assume that all the components of the network are of rank-1 until we generalize the network so as to include rank-2 and rank-3 components in the final section of the paper.

In Section 2 we shall give the major assumptions and definitions that are used often throughout the paper. Section 3 contains the algorithm and examples of enumerating the mutually exclusive critical triplets that are defined in the previous section and play an important role in the subsequent sections. In Section 4 some lemmas shall be proved to achieve the goal of the present paper. The main results are given in Section 5. Theorem 5.1 gives the method of evaluating the network reliability, $R(w)$, by a kind of sequential approximations and the theorem also clarifies the error bound at each stage of the approximation procedures. Theorem 5.2 is the convergence theorem which asserts that the sequential procedure approaches to the true value of the reliability $R(w)$. Theorems 5.3 and 5.4 guarantee the rapidness of the convergence and give a simple form of the error bound at each stage of the evaluations in case of highly reliable networks. Section 6 is the numerical example of the reliability evaluations, where we apply the method developed in the previous sections to the simplest network with known reliability, and exemplify that our first and second approximations are very close to the true value as far as the network is highly reliable. Finally, in Section 7 we shall give some remarks on the extensions of the model and method shown in the previous sections. We shall discuss briefly the treaties of rank-2 and rank-3 components. It will be noted that the so-called intermittently used systems are easily treated by a slight modification of the network and it will be remarked also that the network can be analyzed under alternative initial conditions imposed on the components of the network.

2. Assumptions and Definitions

Our *Assumptions* are as follows:

1° Given an arbitrary system with monotonic structure constructed from n components denoted here by $i=1, 2, \cdots, n$, all the paths and cuts, $A_1, A_2, \cdots, A_r$ and $B_1, B_2, \cdots, B_s$ respectively, of the system or the network in our terminology are clearly identified, where the terms, the network and paths and cuts, should be interpreted as noted in the previous section.

2° In the network each component i has a life-time (operating time length) distribution $F_i(t)$ and a repair-time distribution $G_i(t)$, both of which are continuous functions of $-\infty<t<\infty$ and satisfy

$$F_i(0)=G_i(0)=0 \qquad (i=1, 2, \cdots, n).$$

The life-times and repair-times of the components $i=1, 2, \cdots, n$ are mutually independent.

3° If any component of the network fails (becomes inoperative), it is instantly sent to a repair station. Each station repairs failed components one by one successively, following the rule of first-in-first-out. The components are recovered completely by the repair and the repaired components become operative immediately in the network.

4° The number, m, of the repair stations satisfies

$$m \geqslant m_0 = \max_{j=1,\cdots,r} \{n - |A_j|\}, \tag{2.1}$$

where $|A|$ denotes the number of elements in the set A, and the repair-times of the components are irrespective of the repair stations undertaken.

5° Each component $i=1, 2, \cdots, n$ is newly installed at time 0 in the network.

Remark 2.1. In Assumptions 1°—5° given above we have implicitely assumed that all the components of the network are of rank-1 in the sense of the previous section. As mentioned in the section, we shall mainly analyze the reliability of the network under the conditions 1°—5°, and then in the final section we shall give some remarks on the extension of the model to treat rank-2 components with $a_i=1$ and rank-3 components with $b_i \geqslant 0$.

Remark 2.2. The existence of the density, $f_i(t)$, of $F_i(t)$ is neither assumed nor essential in our model. However, for convenience of notations let us denote $dF_i(t)=F_i(t)-F_i(t-0)$ as $f_i(t)dt$ as if it were assumed in the following sections for each $i=1, 2, \cdots, n$. Therefore, if the density does not exist, then, $f_i(t)dt$ and $f_i(t-s)dt$ should be replaced by the orthographies $dF_i(t)$ and $d_tF_i(t-s)$, respectively, for each i.

Now, define the notations that are used frequently in the subsequent sections as follows:

$$\bar{F}_i(t)=1-F_i(t), \qquad \bar{G}_i(t)=1-G_i(t),$$

$$H_i^{(1)}(t)=F_i*G_i(t), \qquad H_i(t)=\sum_{k=0}^{\infty} H_i^{(k)}(t),$$

where $H_i^{(k)}(t)$ is the kth convolution of $H_i^{(1)}(t)$ for every positive integer k and $H_i^{(0)}(t)$ is the unit distribution whose density is concentrated at the origin $t=0$, for each $i=1, 2, \cdots, n$. And denote the integrals $\int_{0-0}^{t} dH_i(x)$ and $\int_{s-0}^{t} d_x H_i(x-s)$ simply as $\int_{0}^{t} dH_i(x)$ and $\int_{s}^{t} d_x H_i(x-s)$, respectively, omitting "-0" in the lower bounds of the integrations, in the subsequent sections for each i.

Remark 2.3. The network fails if and only if there exists at least a cut whose components are all inoperative, *i.e.*, there exists no path whose components are all operative.

Definition 2.1. We define the system reliability, $R(w)$, of the network as the probability that the time-to-first-failure, W, of the system is greater than w, *i.e.*,

$$P(W>w)=R(w),$$

where w is an arbitrarily given real number.

In order to evaluate the system reliability $R(w)$ under Assumptions 1°—5°, we shall need the critical components and critical triplets of the component sets defined as follows:

Definition 2.2. If and only if the component i_α and the sets C_α and D_α satisfy the following three conditions

(2.2) $i_\alpha \in N=\{1, 2, \cdots, n\}$, $i_\alpha \notin C_\alpha$, $i_\alpha \notin D_\alpha$, $C_\alpha D_\alpha=\phi$, $\{i_\alpha\}+C_\alpha+D_\alpha=N$;

(2.3) $\{i_\alpha\}+C_\alpha$ contains at least one path A_i but C_α contains no path of the network;

(2.4) D_α contains no cut but $\{i_\alpha\}+D_\alpha$ contains at least one cut B_j of the network;

then, we call i_α the *critical component* and $[I_\alpha, C_\alpha, D_\alpha]$ the *critical triplet* of the component sets, putting

(2.5) $I_\alpha=\{i_\alpha\}$.

Definition 2.3. Two critical triplets $[I_\alpha, C_\alpha, D_\alpha]$ and $[I_\beta, C_\beta, D_\beta]$ are *mutually exclusive* [*duplicated*] if and only if at least one [none] of the inequalities

$$I_\alpha \neq I_\beta\,, \qquad C_\alpha \neq C_\beta\,, \qquad D_\alpha \neq D_\beta$$

holds. The critical triplets of a class $[I_\alpha, C_\alpha, D_\alpha]$ $(\alpha=1, 2, \cdots, \nu;\ \nu \geqslant 2)$ are mutually exclusive if and only if any two triplets taken from the class are mutually exclusive.

Given the network satisfying Assumption 1°, we can enumerate all the mutually exclusive critical triplets (m.e.c.t.) of the component sets. Now that m.e.c.t. of the network are completely identified, the system failure of the network is certainly characterized by the critical triplet through which the system becomes inoperative in the sense of the following definition.

Definition 2.4. The network fails via $[I_\alpha, C_\alpha, D_\alpha]$ in the interval $(t, t+dt)$, if and only if the n components of the network satisfy the following three conditions:

(2.6) $\qquad i_\alpha \in I(t;\, dt)\,,$

(2.7) $\qquad i \in C(t) \quad$ for each component $i \in C_\alpha$,

(2.8) $\qquad i \in D(t) \quad$ for each component $i \in D_\alpha$,

where $i \in I(t;\, dt)$ denotes that component i is operative at time t and becomes inoperative in the interval $(t, t+dt)$, while $i \in C(t)$ denotes that component i is operative at time t as well as throughout the interval $(t, t+dt)$, and $i \in D(t)$ denotes that component i is inoperative at time t as well as throughout the interval $(t, t+dt)$.

3. Enumeration of M.E.C.T.—Algorithm and Examples

In this section we shall give an algorithm of enumerating all of m.e.c.t. and show some examples of its application.

Algorithm of Enumerating All of M.E.C.T.

Step 1. For each component $i \in N$ enumerate all the pairs of the sets A_j, B_k such that

(3.1) $\qquad \{i\} = A_j B_k\,,$

which implies that i is the unique component of the intersection $A_j B_k$ of the sets A_j and B_k, where $A_j \in \{A_1, A_2, \cdots, A_r\}$ and $B_k \in \{B_1, B_2, \cdots, B_s\}$. Let all of the distinct pairs of such sets be denoted by

(3.2) $A_{ij}B_{ij} \quad (j=1, 2, \cdots, \nu_i)$,

where $A_{ij} \in \{A_1, A_2, \cdots, A_r\}$ and $B_{ij} \in \{B_1, B_2, \cdots, B_s\}$ for each $j \in \{1, 2, \cdots, \nu_i\}$.

Step 2. Put

(3.3) $C^0_{ij} = A_{ij} - \{i\}, \quad D^0_{ij} = B_{ij} - \{i\}$,

(3.4) $N_{ij} = N - \{i\} - C^0_{ij} - D^0_{ij}, \quad |N_{ij}| = \nu_{ij}$

for each $j \in \{1, 2, \cdots, \nu_i\}$.

(i) If $\nu_{ij}=0$, then, put

(3.5) $C^1_{ij} = C^0_{ij}, \quad D^1_{ij} = D^0_{ij}$,

so as to obtain a critical triplet

(3.6) $[\{i\}, C^1_{ij}, D^1_{ij}]$.

(ii) If $\nu_{ij}>0$, then, add each of the ν_{ij} components in N_{ij} either to C^0_{ij} or to D^0_{ij} so as to get $2^{\nu_{ij}}$ exclusive pairs of sets,

$$\{C^k_{ij}, D^k_{ij}\} \quad (k=1, 2, \cdots, 2^{\nu_{ij}})$$

such that satisfy

$$\{i\} + C^k_{ij} + D^k_{ij} = N,$$

$$C^0_{ij} \subset C^k_{ij}, \quad D^0_{ij} \subset D^k_{ij}, \quad C^k_{ij} D^k_{ij} = \phi$$

for each $k=1, 2, \cdots, 2^{\nu_{ij}}$. From these, one can immediately obtain m.e.c.t.

(3.7) $[\{i\}, C^k_{ij}, D^k_{ij}] \quad (k=1, 2, \cdots, 2^{\nu_{ij}})$,

which are generated from C^0_{ij} and D^0_{ij} of (3.3).

Step 3. (i) Perform Steps 2 (i) and (ii) for each $j \in \{1, 2, \cdots, \nu_i\}$ so as to obtain the critical triplets

(3.8) $[\{i\}, C^k_{ij}, D^k_{ij}] \quad (k=1, 2, \cdots, 2^{\nu_{ij}};\ j=1, 2, \cdots, \nu_i)$.

(ii) Among the triplets (3.8) exclude the duplicated ones so as to obtain m.e.c.t.

(3.9) $[\{i\}, C_{ij}, D_{ij}] \quad (j=1, 2, \cdots, \mu_i)$,

where μ_i is the number of m.e.c.t. with the common critical component i, and hence $1 \leqslant \mu_i \leqslant \sum_{j=1}^{\nu_i} 2^{\nu_{ij}}$.

Step 4. (i) Perform Steps 1 through 3 for each $i \in N$ so as to obtain all of m.e.c.t.

(3.10) $[\{i\}, C_{ij}, D_{ij}]$ $(j=1, 2, \cdots, \mu_i;\ i=1, 2, \cdots, n)$.

(ii) Rewrite all of m.e.c.t. (3.10) as

(3.11) $[I_\alpha, C_\alpha, D_\alpha]$ $(\alpha=1, \cdots, \nu)$,

where $\{i_\alpha\}=I_\alpha$ and $\nu=\sum_{i=1}^{n}\mu_i$ is the total number of m.e.c.t. of the network.

Remark 3.1. Each of the triplets (3.11) trivially satisfies the conditions of Definition 2.2, and (3.11) is a class of m.e.c.t. owing to the algorithm given above. Conversely, any critical triplet defined in Definition 2.2 for a network is obviously included in the class (3.11).

For simplicity of notations, let us denote the critical triplet $[\{i_\alpha\}, \{i_1, \cdots, i_k\}, \{i_{k+1}, \cdots, i_l\}]$ as $[i_\alpha;\ i_1, \cdots, i_k;\ i_{k+1}, \cdots, i_l]$ in the following examples.

Example 3.1. Two-component-paralleled redundant system with $N=\{1, 2\}$. This is one of the simplest networks. See Figure 3.1. In this case we obviously have

paths: $A_1=\{1\}$, $A_2=\{2\}$;
cut : $B_1=\{1, 2\}$; and $m_0=1$.

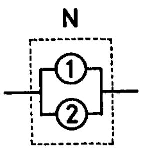

Fig. 3.1. Two-component-paralleled redundant system $N=\{1, 2\}$.

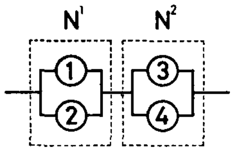

Fig. 3.2. Series system of two two-component-paralleled redundant subsystems $N^1=\{1, 2\}$ and $N^2=\{3, 4\}$.

Step 1. Critical components: $\{i_1\}=\{1\}=A_1B_1=A_{11}B_{11}$ $(\nu_1=1)$,
$\{i_2\}=\{2\}=A_2B_1=A_{21}B_{21}$ $(\nu_2=1)$,

where $A_{11}=A_1$, $B_{11}=B_1$; $A_{21}=A_2$, $B_{21}=B_1$.

Step 2. $C_{11}^0=A_{11}-\{1\}=\phi$, $D_{11}^0=B_{11}-\{1\}=\{2\}$, $N_{11}=\phi$, $\nu_{11}=0$;
$C_{21}^0=A_{21}-\{2\}=\phi$, $D_{21}^0=B_{21}-\{2\}=\{1\}$, $N_{21}=\phi$, $\nu_{21}=0$.

(i) For $i=j=1$ we get a critical triplet $[1;\ \phi;\ 2]$.

Step 2 (ii) and *Step 3* are vacant since $\nu_1=1$ and $\nu_{11}=0$.
Step 4 (i). For $i=2$ we get another triplet [2; ϕ; 1] and finally the two ones

(3.12) $\quad$ [1; ϕ; 2] and [2; ϕ; 1]

of the network. (ii) They are obviously all of m.e.c.t. of the network.

Example 3.2. 2-out-of-3 system with $N=\{1, 2, 3\}$. In this case we have

$$\text{paths: } A_1=\{1, 2\},\quad A_2=\{2, 3\},\quad A_3=\{3, 1\};$$
$$\text{cuts : } B_1=A_1,\quad B_2=A_2,\quad B_3=A_3;\quad \text{and}\quad m_0=1.$$

Step 1. Critical components: $\{i_1\}=\{1\}=A_1B_3=A_3B_1 \quad (\nu_1=2)$,
$\{i_2\}=\{2\}=A_2B_1=A_1B_2 \quad (\nu_2=2)$,
$\{i_3\}=\{3\}=A_3B_2=A_2B_3 \quad (\nu_3=2)$.

Step 2. For $i=j=1$ we get $C^0_{11}=\{2\}$, $D^0_{11}=\{3\}$, $N_{11}=\phi$, $\nu_{11}=0$;
(i) $C^1_{11}=C^0_{11}$, $D^1_{11}=D^0_{11}$, and the critical triplet

(3.13) $\quad$ [1; 2; 3].

Step 2 (ii) is vacant since $\nu_{11}=0$.
Step 3 (i) For $i=1$, $j=2$ we have $C^0_{12}=\{3\}$, $D^0_{12}=\{2\}$, $N_{12}=\phi$, $\nu_{12}=0$, $C^1_{12}=C^0_{12}$, $D^1_{12}=D^0_{12}$ and the critical triplet

(3.14) $\quad$ [1; 3; 2].

(ii) The triplets (3.13) and (3.14) are mutually exclusive.
Step 4 (i) For $i=2$ we obtain the triplets

(3.15) $\quad$ [2; 1; 3], [2; 3; 1],

and for $i=3$ the triplets

(3.16) $\quad$ [3; 1; 2], [3; 2; 1],

which are mutually exclusive.
(ii) Finally we get the six m.e.c.t. (3.13)—(3.16).

Example 3.3. Series system of two two-component-paralleled redundant subsystems $N^1=\{1, 2\}$ and $N^2=\{3, 4\}$. See Figure 3.2. In this case we have

$$\text{paths: } A_1=\{1, 3\},\quad A_2=\{1, 4\},\quad A_3=\{2, 3\},\quad A_4=\{2, 4\};$$
$$\text{cuts : } B_1=\{1, 2\},\quad B_2=\{3, 4\};\quad \text{and}\quad m_0=2.$$

Step 1. Critical components: $\{1\}=A_1B_1=A_2B_1 \quad (\nu_1=2)$,

$$\{2\}=A_3B_1=A_4B_1 \quad (\nu_2=2),$$
$$\{3\}=A_1B_2=A_3B_2 \quad (\nu_3=2),$$
$$\{4\}=A_2B_2=A_4B_2 \quad (\nu_4=2).$$

Step 2. For $i=j=1$ we get $C_{11}^0=A_{11}-\{1\}=\{3\}$, $D_{11}^0=B_{11}-\{1\}=\{2\}$, $N_{11}=\{4\}$, $\nu_{11}=1$; $C_{12}^0=A_{12}-\{1\}=\{4\}$, $D_{12}^0=B_{12}-\{1\}=\{2\}$, $N_{12}=\{3\}$, $\nu_{12}=1$, where $A_{11}=A_1$, $B_{11}=B_1$, $A_{12}=A_2$ and $B_{12}=B_1$.

(ii) $C_{11}^1=C_{11}^0+\{4\}=\{3, 4\}$, $D_{11}^1=D_{11}^0=\{2\}$; $C_{11}^2=C_{11}^0=\{3\}$, $D_{11}^2=D_{11}^0+\{4\}=\{2, 4\}$. From these we obtain the triplets

(3.17) [1; 3, 4; 2], [1; 3; 2, 4].

Step 3 (i) For $i=1$, $j=2$ we have $C_{12}^1=C_{12}^0+\{3\}=\{3, 4\}$, $D_{12}^1=D_{12}^0=\{2\}$; $C_{12}^2=C_{12}^0=\{4\}$, $D_{12}^2=D_{12}^0+\{3\}=\{2, 3\}$, and, hence, the triplets

(3.18) [1; 3, 4; 2], [1; 4; 2, 3].

(ii) Among the four triplets (3.17) and (3.18), [1; 3, 4; 2] are duplicated and so we exclude one of them so as to obtain the following three m.e.c.t.

(3.19) [1; 3, 4; 2], [1; 3; 2, 4], [1; 4; 2, 3]

with the common critical component 1.

Step 4 (i) By the same procedure we have the nine more m.e.c.t.

[2; 3, 4; 1], [2; 3; 4, 1], [2; 4; 3, 1];
(3.20) [3; 1, 2; 4], [3; 1; 4, 2], [3; 2; 1, 4];
[4; 1, 2; 3], [4; 1; 2, 3], [4; 2; 1, 3].

(ii) We denote the twelve m.e.c.t. (3.19) and (3.20) as

$$[I_\alpha, C_\alpha, D_\alpha] \quad (\alpha=1, 2, \cdots, 12),$$

respectively, which are the whole to be obtained for the network of Fig. 3.2.

Remark 3.2. As seen in Examples 3.1—3.3 the labor of enumerating m.e.c.t. increases exponentially as the number of the components does. However, the procedure is so straight forwords that we can easily accomplish the work with the aid of a computer if n is not large. It should also be remarked here that the system reliability evaluation will be exceedingly simplified if the network is decomposed in series to a certain number of subnetworks where the series decomposition of the network is characterized by the following two conditions: (i) the subnetworks are defined as the networks in the sense of Sections 1 and 2 on their respective disjoint subsets of the set N and (ii) the original network operates when and only when all of the subnetworks operate. We shall see later in Example 6.2 how efficient the decomposition is not only in enumerating m.e.c.t. but also in evaluating the reliability of a series system.

4. Preliminary Lemmas

In this section we shall show the lemmas to be needed in the following sections. In the primal model defined by Assumptions 1°—5° there exist $m \geqslant m_0$ repair stations to maintain the n components of the network. In this model, if $n > m \geqslant m_0$, then we add $(n-m)$ hypothetical repair stations to it to equalize the number of repair stations with that of the components of the network. We shall call this the *modified model,* where the other assumptions are not altered from the primal one but the number of the repair stations. Now we have the following lemma.

Lemma 4.1. The reliability of the network of the primal model equals that of the modified one.

Proof. The network of the primal model is operative as long as at least one of the paths of the network is active, where the number of inoperative components is less than m_0 owing to Assumption 4°. As we have $m \geqslant m_0$ repair stations in the primal model, it behaves just as the modified one as far as the number of failed components is less than m_0. These facts assure the statement of the lemma.

Remark 4.1. Lemma 4.1 implies that the system reliability is independent of the number m of the repair stations if $m \geqslant m_0$, and that it is sufficient for us to analyze the stochastic feature of the modified model, which is easier to handle than the primal one since the former consists of n independent alternating renewal processes. For the alternating renewal process see Cox [2, Chap. 7].

Lemma 4.2. Under Assumptions 1°—5° any failure of the network can, almost surely, be characterized by one and only one of m.e.c.t. enumerated by the algorithm in the previous section.

Proof. By the independence of the lifetimes and repair-times and by the continuity of the distribution functions in Assumption 2° we can neglect the probability that two or more failures and/or recoveries of the components may occur in the interval $(t, t+dt)$, compared with the probability that one component failure occurs in the interval $(t, t+dt)$. Therefore, any failure of the network can almost surely be identified by a critical triplet, which is included in the class (3.11) because of the construction algorithm of m.e.c.t. given in the previous section.

Now let us define the random variables:

$$Z(t_i,\ t_{i+1})=\begin{cases}1 & \text{if the network is operative at time } t_i \text{ and fails in the interval } (t_i,\ t_{i+1}),\\ 0 & \text{otherwise,}\end{cases} \tag{4.1}$$

for $i=1, 2, \cdots, l$, where

$$0=t_1<t_2<\cdots<t_{l+1}=w\,. \tag{4.2}$$

First, we shall evaluate

$$\pi_1(w)=\lim_{\Delta_l\to 0} E\{\sum_{i=1}^{l} Z(t_i,\ t_{i+1})\}\,, \tag{4.3}$$

where $\Delta_l=\max_{i=1,\cdots,l}\{t_{i+1}-t_i\}$. From Lemma 4.2 and the definition (4.1) we obviously have

$$\begin{aligned} E\{Z(t,\ t+dt)\}&=P\{Z(t,\ t+dt)=1\}\\ &=P\{\text{the network is operative at time } t \text{ and becomes inoperative in the interval } (t,\ t+dt)\}\\ &=\sum_{\alpha=1}^{\nu} P\{\text{the network fails via } [I_\alpha,\ C_\alpha,\ D_\alpha] \text{ in the interval } (t,\ t+dt)\}, \end{aligned} \tag{4.4}$$

neglecting the infinitesimal terms of higher orders. Here let us define the set function

$$\delta_i(M)=\begin{cases}1 & \text{if component } i\in M,\\ 0 & \text{otherwise,}\end{cases} \tag{4.5}$$

and we obtain

(4.6) $$\delta_i(I_\alpha)+\delta_i(C_\alpha)+\delta_i(D_\alpha)=1$$

for each $i=1, 2, \cdots, n$. From Definition 2.4 and Remark 4.1 the probability, $Q_\alpha(t)dt$, that the network fails via $[I_\alpha, C_\alpha, D_\alpha]$ in the interval $(t, t+dt)$ can be evaluated by

(4.7) $$\begin{aligned} Q_\alpha(t)dt &= \prod_{i=1}^{n} P\{i \in I(t;\ dt)\}^{\delta_i(I_\alpha)} P\{i \in C(t)\}^{\delta_i(C_\alpha)} P\{i \in D(t)\}^{\delta_i(D_\alpha)} \\ &= P\{i_\alpha \in I(t;\ dt)\} \prod_{i\in C_\alpha} P\{i \in C(t)\} \prod_{j\in D_\alpha} P\{j \in D(t)\}, \end{aligned}$$

where $P\{\cdots\}^{\delta_i(\cdot)}$ represents the $\delta_i(\cdot)$th power of $P\{\cdots\}$ and

(4.8) $$P\{i \in I(t;\ dt)\}=\int_0^t f_i(t-u)dH_i(u)\cdot dt,$$

(4.9) $$P\{i \in C(t)\}=\int_0^t \bar{F}_i(t-u)dH_i(u),$$

(4.10) $$P\{i \in D(t)\}=\int_0^t dH_i(u)\int_u^t f_i(v-u)\bar{G}_i(t-v)dv,$$

neglecting the infinitesimal terms of higher orders, and $\prod_{i\in C_\alpha}$ and $\prod_{j\in D_\alpha}$ on the right side of (4.7) indicate the products of the probabilities over all the components $i \in C_\alpha$ and $j \in D_\alpha$, respectively. These facts imply the following lemma.

Lemma 4.3. In the modified model $\pi_1(w)$ defined by (4.3) can be evaluated as

(4.11) $$\begin{aligned} \pi_1(w) &= \sum_{\alpha=1}^{\nu}\int_0^w Q_\alpha(t)dt \\ &= \sum_{\alpha=1}^{\nu}\int_0^w \prod_{i\in C_\alpha} P\{i \in C(t)\} \prod_{j\in D_\alpha} P\{j \in D(t)\} P\{i_\alpha \in I(t;\ dt)\} \\ &= \sum_{\alpha=1}^{\nu}\int_0^w [\prod_{i\in C_\alpha}\int_0^t \bar{F}_i(t-u_i)dH_i(u_i) \\ &\quad \times \prod_{j\in D_\alpha}\int_0^t dH_j(u_j)\int_{u_j}^t f_j(v_j-u_j)\bar{G}_j(t-v_j)dv_j \\ &\quad \times \int_0^t f_{i_\alpha}(t-u)dH_{i_\alpha}(u)]dt. \end{aligned}$$

Second, we shall evaluate

(4.12) $$\pi_2(w)=\lim_{\Delta_l\to 0} E\{\sum_{i=1}^{l}\sum_{j=i+1}^{l} Z(t_i,\ t_{i+1})Z(t_j,\ t_{j+1})\}\ ,$$

where the notations are defined in (4.1) and (4.2), putting conventionally as $\sum_{j=k}^{l} a_j=0$ for $k>l$. For the purpose we remark that

(4.13) $$\begin{aligned} &E\{Z(s,\ s+ds)Z(t,\ t+dt)\} \\ &=P\{Z(s,\ s+ds)=Z(t,\ t+dt)=1\} \\ &=\sum_{\alpha=1}^{\nu}\sum_{\beta=1}^{\nu} P\{\text{the network fails via } [I_\alpha,\ C_\alpha,\ D_\alpha] \text{ in the interval } (s,\ s+ds) \text{ and fails also via } [I_\beta,\ C_\beta,\ D_\beta] \text{ in the interval } (t,\ t+dt)\}, \end{aligned}$$

just as in (4.4), for every $0\leqslant s<t$, neglecting the infinitesimal terms of higher orders. On evaluating the probability we should note that for arbitrarily given $0\leqslant s<t$ each component i of the network can take only one of the following 3^2 mutually exclusive states:

(4.14a) $i\in I(s;\ ds)I(t;\ dt)$,

(4.14b) $i\in I(s;\ ds)C(t)$,

(4.14c) $i\in I(s;\ ds)D(t)$,

(4.15a) $i\in C(s)I(t;\ dt)$,

(4.15b) $i\in C(s)C(t)$,

(4.15c) $i\in C(s)D(t)$,

(4.16a) $i\in D(s)I(t;\ dt)$,

(4.16b) $i\in D(s)C(t)$,

(4.16c) $i\in D(s)D(t)$,

where $i\in I(s;\ ds)I(t;\ dt)$, for instance, denotes the product event that component $i\in I(s;\ ds)$ and also $i\in I(t;\ dt)$, and the other notations (4.14b)—(4.16c) are obviously defined just as shown above. In the modified model the probabilities of these events are given by

(4.17a) $$\begin{aligned} &P\{i\in I(s;\ ds)I(t;\ dt)\} \\ &=\int_0^s d_uH_i(u)f_i(s-u)\int_s^t d_xG_i(x-s) \\ &\quad\times\int_s^t d_yH_i(y-x)f_i(t-y)\cdot ds\,dt\ , \end{aligned}$$

(4.17b) $$P\{i \in I(s;\ ds)C(t)\} = \int_0^s d_u H_i(u) f_i(s-u) \int_s^t d_x G_i(x-s) \int_s^t d_y H_i(y-x) \bar{F}_i(t-y) \cdot ds\,,$$

(4.17c) $$P\{i \in I(s;\ ds)D(t)\} = \int_0^s d_u H_i(u) f_i(s-u) \int_s^t d_x H_i(x-s) \bar{G}_i(t-x) \cdot ds\,,$$

(4.18a) $$P\{i \in C(s)I(t;\ dt)\} = \int_0^s d_u H_i(u) \{f_i(t-u) + \int_s^t d_x F_i(x-u) \int_s^t d_y G_i(y-x) \times \int_s^t d_z H_i(z-y) f_i(t-z)\} \cdot dt\,,$$

(4.18b) $$P\{i \in C(s)C(t)\} = \int_0^s d_u H_i(u) \{\bar{F}_i(t-u) + \int_s^t d_x F_i(x-u) \int_s^t d_y G_i(y-x) \times \int_s^t d_z H_i(z-y) \bar{F}_i(t-z)\}\,,$$

(4.18c) $$P\{i \in C(s)D(t)\} = \int_0^s d_u H_i(u) \int_s^t d_x F_i(x-u) \int_s^t d_y H_i(y-x) \bar{G}_i(t-y),$$

(4.19a) $$P\{i \in D(s)I(t;\ dt)\} = \int_0^s d_u H_i(u) \int_0^s d_v F_i(v-u) \int_s^t d_x G_i(x-v) \times \int_s^t d_y H_i(y-x) f_i(t-y) \cdot dt\,,$$

(4.19b) $$P\{i \in D(s)C(t)\} = \int_0^s d_u H_i(u) \int_0^s d_v F_i(v-u) \int_s^t d_x G_i(x-v) \times \int_s^t d_y H_i(y-x) \bar{F}_i(t-y)\,,$$

(4.19c) $$P\{i \in D(s)D(t)\} = \int_0^s d_u H_i(u) \int_0^s d_v F_i(v-u) \{\bar{G}_i(t-v) + \int_s^t d_x G_i(x-v) \times \int_s^t d_y H_i(y-x) \int_s^t d_z F_i(z-y) \bar{G}_i(t-z)\}\,,$$

respectively, neglecting the infinitesimal terms of higher orders. The probability, $Q_{\alpha\beta}(s, t)dsdt$, that the modified network fails via $[I_\alpha, C_\alpha, D_\alpha]$ in the interval $(s, s+ds)$ and fails also via $[I_\beta, C_\beta, D_\beta]$ in the interval $(t, t+dt)$ can be formulated as

$$
\begin{aligned}
(4.20)\qquad & Q_{\alpha\beta}(s, t)dsdt \\
&= \prod_{i=1}^{n} [P\{i \in I(s;\, ds)I(t;\, dt)\}^{\delta_i(I_\alpha I_\beta)} \\
&\quad \times P\{i \in I(s;\, ds)C(t)\}^{\delta_i(I_\alpha C_\beta)} \\
&\quad \times P\{i \in I(s;\, ds)D(t)\}^{\delta_i(I_\alpha D_\beta)} \\
&\quad \times P\{i \in C(s)I(t;\, dt)\}^{\delta_i(C_\alpha I_\beta)} \\
&\quad \times P\{i \in C(s)C(t)\}^{\delta_i(C_\alpha C_\beta)} \\
&\quad \times P\{i \in C(s)D(t)\}^{\delta_i(C_\alpha D_\beta)} \\
&\quad \times P\{i \in D(s)I(t;\, dt)\}^{\delta_i(D_\alpha I_\beta)} \\
&\quad \times P\{i \in D(s)C(t)\}^{\delta_i(D_\alpha C_\beta)} \\
&\quad \times P\{i \in D(s)D(t)\}^{\delta_i(D_\alpha D_\beta)}]
\end{aligned}
$$

for $0 \leqslant s < t$, neglecting the infinitesimal terms of higher orders. Notice that owing to (2.2) and (4.5) we have

$$
\begin{aligned}
(4.21)\qquad & \delta_{i_\alpha}(I_\alpha I_\beta) + \delta_{i_\alpha}(I_\alpha C_\beta) + \delta_{i_\alpha}(I_\alpha D_\beta) = \delta_{i_\alpha}(I_\beta) + \delta_{i_\alpha}(C_\beta) + \delta_{i_\alpha}(D_\beta) = 1\,, \\
& \delta_{i_\beta}(I_\alpha I_\beta) + \delta_{i_\beta}(C_\alpha I_\beta) + \delta_{i_\beta}(D_\alpha I_\beta) = \delta_{i_\beta}(I_\alpha) + \delta_{i_\beta}(C_\alpha) + \delta_{i_\beta}(D_\alpha) = 1\,,
\end{aligned}
$$

since $I_\alpha = \{i_\alpha\}$, $I_\beta = \{i_\beta\}$. Then, define the functions $\varphi_{\alpha\beta}$ and $\psi_{\beta\alpha}$ as follows:

$$
\begin{aligned}
(4.22)\qquad \varphi_{\alpha\beta}(s, t) &= \delta_{i_\alpha}(I_\beta) + \delta_{i_\alpha}(C_\beta)\int_s^t d_x G_{i_\alpha}(x-s)\int_s^t d_y H_{i_\alpha}(y-x)\bar{F}_{i_\alpha}(t-y) \\
&\quad + \delta_{i_\alpha}(D_\beta)\int_s^t d_x H_{i_\alpha}(x-s)\bar{G}_{i_\alpha}(t-x)\,,
\end{aligned}
$$

$$
\begin{aligned}
(4.23)\qquad \psi_{\beta\alpha}(s, t) &= \delta_{i_\beta}(I_\alpha)\int_s^t d_x G_{i_\beta}(x-s)\int_s^t d_z H_{i_\beta}(z-x) f_{i_\beta}(t-z) \\
&\quad + \delta_{i_\beta}(C_\alpha)\left\{\int_0^s d_z H_{i_\beta}(z) f_{i_\beta}(t-z) + \int_0^s d_u H_{i_\beta}(u)\right. \\
&\quad \times \int_s^t d_x F_{i_\beta}(x-u) \\
&\quad \left.\times \int_s^t d_y G_{i_\beta}(y-x)\int_s^t d_z H_{i_\beta}(z-y) f_{i_\beta}(t-z)\right\}
\end{aligned}
$$

$$+\delta_{i_\beta}(D_\alpha)\int_0^s d_uH_{i_\beta}(u)\int_0^s d_xF_{i_\beta}(x-u)\int_s^t d_yG_{i_\beta}(y-x)$$
$$\times\int_s^t d_zH_{i_\beta}(z-y)f_{i_\beta}(t-z)$$

for every $0\leqslant s<t$. From (4.13), (4.20), (4.21), (4.22) and (4.23) we get the following lemma.

Lemma 4.4. In the modified model $\pi_2(w)$ defined by (4.12) can be evaluated as

$$\pi_2(w)=\sum_{\alpha=1}^{\nu}\sum_{\beta=1}^{\nu}\int_0^w ds\int_s^w Q_{\alpha\beta}(s,t)dt \tag{4.24}$$
$$=\sum_{\alpha=1}^{\nu}\sum_{\beta=1}^{\nu}\int_0^w ds\int_s^w dt\Big[\int_0^s d_uH_{i_\alpha}(u)f_{i_\alpha}(s-u)\varphi_{\alpha\beta}(s,t)\psi_{\beta\alpha}(s,t)$$
$$\times\prod_{i\in C_\alpha C_\beta}P\{i\in C(s)C(t)\}\prod_{j\in C_\alpha D_\beta}P\{j\in C(s)D(t)\}$$
$$\times\prod_{k\in D_\alpha C_\beta}P\{k\in D(s)C(t)\}\prod_{h\in D_\alpha D_\beta}P\{h\in D(s)D(t)\}\Big].$$

Third, we shall sketch the method of evaluating

$$\pi_k(w)=\lim_{\Delta_l\to 0}E\{\sum_{i_1=1}^{l}\sum_{i_2=i_1+1}^{l}\cdots\sum_{i_k=i_{k-1}+1}^{l}Z(t_{i_1},t_{i_1+1}) \tag{4.25}$$
$$\times Z(t_{i_2},t_{i_2+1})\cdots Z(t_{i_k},t_{i_k+1})\}$$

for every integer $k\geqslant 2$. Let $i\in C(s;u)$ denote that component i has been operative up to time s since the last recovery of the component occurred in the interval $(u,u+du)$, and similarly let $i\in D(s;u)$ denote that component i has been inoperative up to time s since the last failure of the component occurred in the interval $(u,u+du)$, for every $0\leqslant u<s$. Then, one obtains the following 3^2 conditional probabilities

$$P\{i\in I(t;dt)|i\in I(s;ds)\} \tag{4.26a}$$
$$=\int_s^t d_xG_i(x-s)\int_s^t d_yH_i(y-x)f_i(t-y)\cdot dt,$$

$$P\{i\in C(t)|i\in I(s;ds)\} \tag{4.26b}$$
$$=\int_s^t d_xG_i(x-s)\int_s^t d_yH_i(y-x)\bar{F}_i(t-y),$$

$$P\{i\in D(t)|i\in I(s;ds)\}=\int_s^t d_xH_i(x-s)\bar{G}_i(t-x), \tag{4.26c}$$

(4.27a)
$$P\{i \in I(t;\, dt) | i \in C(s;\, u)\} = \frac{1}{\bar{F}_i(s-u)} \left\{ f_i(t-u) + \int_s^t d_x F_i(x-u) \int_s^t d_y G_i(y-x) \int_s^t d_z H_i(z-y) f_i(t-z) \right\} \cdot dt\,,$$

(4.27b)
$$P\{i \in C(t) | i \in C(s;\, u)\} = \frac{1}{\bar{F}_i(s-u)} \left\{ \bar{F}_i(t-u) + \int_s^t d_x F_i(x-u) \int_s^t d_y G_i(y-x) \int_s^t d_z H_i(z-y) \bar{F}_i(t-z) \right\}\,,$$

(4.27c)
$$P\{i \in D(t) | i \in C(s;\, u)\} = \frac{1}{\bar{F}_i(s-u)} \left\{ \int_s^t d_x F_i(x-u) \int_s^t d_z H_i(z-x) \bar{G}_i(t-z) \right\}\,,$$

(4.28a)
$$P\{i \in I(t;\, dt) | i \in D(s;\, u)\} = \frac{1}{\bar{G}_i(s-u)} \left\{ \int_s^t d_x G_i(x-u) \int_s^t d_z H_i(z-x) f_i(t-z) \right\} \cdot dt\,,$$

(4.28b)
$$P\{i \in C(t) | i \in D(s;\, u)\} = \frac{1}{\bar{G}_i(s-u)} \left\{ \int_s^t d_x G_i(x-u) \int_s^t d_z H_i(z-x) \bar{F}_i(t-z) \right\}\,,$$

(4.28c)
$$P\{i \in D(t) | i \in D(s;\, u)\} = \frac{1}{\bar{G}_i(s-u)} \left\{ \bar{G}_i(t-u) + \int_s^t d_x G_i(x-u) \int_s^t d_y F_i(y-x) \int_s^t d_z H_i(z-y) \bar{G}_i(t-z) \right\}\,,$$

for every $0 \leqslant u < s < t$. Here, remark that $P\{i \in C(s)C(t)\}$ in (4.18b), for instance, can be obtained from $P\{i \in C(s)\}$ and $P\{i \in C(t) | i \in C(s;\, u)\}$ as follows

(4.29)
$$P\{i \in C(s)C(t)\} = \int_0^s d_u H_i(u) \bar{F}_i(s-u) \cdot \frac{1}{\bar{F}_i(s-u)} \left\{ \bar{F}_i(t-u) + \int_s^t d_x F_i(x-u) \int_s^t d_y G_i(y-x) \int_s^t d_z H_i(z-y) \bar{F}_i(t-z) \right\}\,,$$

which appears as if it were the formal multiplication of $P\{i \in C(s)\}$ by

$P\{i \in C(t)|i \in C(s;\ u)\}$, although it should be written rigorously as

$$P\{i \in C(s)C(t)\}=E_u[P\{i \in C(s;\ u)\}P\{i \in C(t)|i \in C(s;\ u)\}]\,. \tag{4.30}$$

Apparently, all the probabilities (4.17a)—(4.19c) can be reproduced through the formal multiplications of (4.8)—(4.10) by (4.26a)—(4.28c), respectively, where one has only to use the common notation u between the multipliers and their multiplicands, just as exemplified in (4.29). Let us call u the conjunction variable, and call the multiplication just defined the formal multiplication with the conjunction variable u. Using the same argument as above, all of the 3^3 probabilities $P\{i \in I(s;\ ds)I(t;\ dt)I(w;\ dw)\}$, $P\{i \in I(s;\ ds)I(t;\ dt)C(w)\}$, $\cdots$, $P\{i \in D(s)D(t)D(w)\}$ evaluating $\pi_3(w)$ can be obtained from (4.17a)—(4.19c) and (4.26a)—(4.28c). For example, the formal multiplication of $P\{i \in C(s)C(t)\}$ by $P\{i \in D(w)|i \in C(t;\ z)\}$ with conjunction variable z yields

$$\begin{aligned}
&P\{i \in C(s)C(t)D(w)\}\\
&=\Big\{\int_0^s d_zH_i(z)\bar{F}_i(t-z)\\
&\quad+\int_0^s d_uH_i(u)\int_s^t d_xF_i(x-u)\int_s^t d_yG_i(y-x)\int_s^t d_zH_i(z-y)\\
&\quad\times\bar{F}_i(t-z)\Big\}\cdot\frac{1}{\bar{F}_i(t-z)}\int_t^w d_\xi F_i(\xi-z)\int_t^w d_\eta H_i(\eta-\xi)\bar{G}_i(w-\eta)\\
&=\Big\{\int_0^s d_zH_i(z)\\
&\quad+\int_0^s d_uH_i(u)\int_s^t d_xF_i(x-u)\int_s^t d_yG_i(y-x)\int_s^t d_zH_i(z-y)\Big\}\\
&\quad\times\int_t^w d_\xi F_i(\xi-z)\int_t^w d_\eta H_i(\eta-\xi)\bar{G}_i(w-\eta)
\end{aligned}$$

for every $0\leqslant s<t<w$. These facts hold for all the probabilities evaluating $\pi_k(w)$ $(k\geqslant2)$, because each component of the network forms an alternating renewal process in our modified model. Thus, we have the following lemma.

Lemma 4.5. Under the assumptions of the previous lemma, given the

3^{k-1} probabilities $P\{i \in I(s_1;\ ds_1)I(s_2;\ ds_2)\cdots I(s_{k-1};\ ds_{k-1})\}$, $P\{i \in I(s_1;\ ds_1)I(s_2;\ ds_2)\cdots I(s_{k-2};\ ds_{k-2})C(s_{k-1})\}$, $\cdots$, $P\{i \in D(s_1)D(s_2)\cdots D(s_{k-1})\}$ for evaluating $\pi_{k-1}(w)$, the 3^k probabilities $P\{i \in I(s_1;\ ds_1)I(s_2;\ ds_2)\cdots I(s_k;\ ds_k)\}$, $P\{i \in I(s_1;\ ds_1)I(s_2;\ ds_2)\cdots I(s_{k-1};\ ds_{k-1})C(s_k)\}$, $\cdots$, $P\{i \in D(s_1)D(s_2)\cdots D(s_k)\}$ for evaluating $\pi_k(w)$ can be obtained through the formal multiplications of the given 3^{k-1} probabilities by the conditional probabilities (4.26a)—(4.28c) whose arguments s and t are replaced by s_{k-1} and s_k, respectively, with the conjunction variable u. Moreover, the joint probability, $Q_{\alpha_1\alpha_2\cdots\alpha_k}(s_1, s_2, \cdots, s_k)ds_1ds_2\cdots ds_k$, that the network fails via critical triplets $[I_{\alpha_i}, C_{\alpha_i}, D_{\alpha_i}]$ $(i=1, 2, \cdots, k)$ in the intervals (s_i, s_i+ds_i) $(i=1, 2, \cdots, k)$, respectively, is given by

(4.31)
$$\begin{aligned} &Q_{\alpha_1\alpha_2\cdots\alpha_k}(s_1, s_2, \cdots, s_k)ds_1ds_2\cdots ds_k \\ &= \prod_{i=1}^{n} [P\{i \in I(s_1;\ ds_1)I(s_2;\ ds_2)\cdots I(s_k;\ ds_k)\}^{\delta_i(I_{\alpha_1}I_{\alpha_2}\cdots I_{\alpha_k})} \\ &\quad \times P\{i \in I(s_1;\ ds_1)\cdots I(s_{k-1};\ ds_{k-1})C(s_k)\}^{\delta_i(I_{\alpha_1}\cdots I_{\alpha_{k-1}}C_{\alpha_k})} \\ &\quad \cdots\cdots \\ &\quad \times P\{i \in D(s_1)D(s_2)\cdots D(s_k)\}^{\delta_i(D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k})}]\,, \end{aligned}$$

neglecting the infinitesimal terms of higher orders, and $\pi_k(w)$ is evaluated as

(4.32)
$$\begin{aligned} \pi_k(w) = &\sum_{\alpha_1=1}^{\nu}\sum_{\alpha_2=1}^{\nu}\cdots\sum_{\alpha_k=1}^{\nu}\int_0^w ds_1\int_{s_1}^w ds_2\cdots\int_{s_{k-1}}^w ds_k \\ &\times Q_{\alpha_1\alpha_2\cdots\alpha_k}(s_1, s_2, \cdots, s_k) \end{aligned}$$

for every $0 \leqslant s_1 < s_2 < \cdots < s_k < w$ and for every integer $k \geqslant 2$.

It should be remarked here that owing to Lemmas 4.3—4.5 the algorithm of evaluating $\pi_k(w)$ is simple enough at least for small k, but the computation becomes very tedious for large values of k. However, it may fortunately suffice for us to evaluate $\pi_k(w)$ only for small values of k as far as we concern highly reliable networks, as seen in the subsequent sections.

5. System Reliability Evaluations

In this section we shall show a method of evaluating the reliability $P\{W>w\}=R(w)$ of the network under Assumptions 1°—5°, using the results obtained in the previous sections. Let us put

$$R_k(w)=\sum_{i=0}^{k}(-1)^i\pi_i(w) \tag{5.1}$$

for $k=1, 2, \cdots$, where $\pi_i(w)$ $(i=1, 2, \cdots)$ are given in Lemmas 4.3—4.5 and we define $\pi_0(w)$ as $\pi_0(w)\equiv 1$.

Theorem 5.1. Under Assumptions 1°—5° the reliability $R(w)$ of the network can be evaluated by

$$R_{2k-1}(w)\leqslant R(w)\leqslant R_{2k}(w) \tag{5.2}$$

for every positive integer k.

Proof. By means of the random variables $Z(t_i, t_{i+1})$ defined in (4.1), the system reliability $R(w)$ can be written as

$$R(w)=E[\prod_{i=1}^{l}\{1-Z(t_i, t_{i+1})\}]\,. \tag{5.3}$$

On the other hand, if $0\leqslant z_i\leqslant 1$ $(i=1, 2, \cdots, l)$, then the following inequalities

$$\begin{aligned} &1-\sum{}'z_{i_1}+\sum{}'z_{i_1}z_{i_2}-\cdots+(-1)^{2k-1}\sum{}'z_{i_1}z_{i_2}\cdots z_{i_{2k-1}} \\ &\leqslant \prod_{i=1}^{l}(1-z_i) \\ &\leqslant 1-\sum{}'z_{i_1}+\sum{}'z_{i_1}z_{i_2}-\cdots+(-1)^{2k}\sum{}'z_{i_1}z_{i_2}\cdots z_{i_{2k}} \end{aligned} \tag{5.4}$$

hold for every integer $l\geqslant 2$ and $k=1, 2, \cdots, [l/2]$, where $\sum{}'z_{i_1}z_{i_2}\cdots z_{i_k}$ indicates $\sum_{i_1=1}^{l}\sum_{i_2=i_1+1}^{l}\cdots\sum_{i_k=i_{k-1}+1}^{l}z_{i_1}z_{i_2}\cdots z_{i_k}$ with $i_0=0$ and $[\cdot]$ is Gauss notation. These inequalities are easily proved by the method of induction. As the definition (4.1) ascertains that $0\leqslant Z(t_i, t_{i+1})\leqslant 1$ $(i=1, 2, \cdots, l)$, we can obviously apply (5.4) to the evaluation of the right side of (5.3) which yields the statement of the theorem owing to Lemmas 4.1—4.5.

Remark 5.1. Theorem 5.1 implies that

$$0\leqslant R(w)-R_{2k-1}(w)\leqslant \pi_{2k}(w)\,, \tag{5.5}$$

$$0\leqslant R_{2k}(w)-R(w)\leqslant \pi_{2k+1}(w) \tag{5.6}$$

hold for every positive integer k. These formulae show that if $\pi_{2k}(w)$ or $\pi_{2k+1}(w)$ is less than a certain preassigned small value, $\varepsilon(>0)$, then the reliability $R(w)$ of the network can be approximated sufficiently well by $R_{2k-1}(w)$ or $R_{2k}(w)$, respectively. In this respect the following theorem gives us more insight on the precision and convergence of our evaluation procedure for the reliability. On stating the theorem, let us define $M(w)$ as the number of the system failures of the modified model observed in the interval $(0, w]$ and put

$$p_i(w)=P\{M(w)=i\} \qquad (i=0, 1, \cdots). \tag{5.7}$$

Theorem 5.2. Under Assumptions 1°—5° there hold the following three statements:[2)]

(i) The random variable $M(w)$ (defined in the modified model) has finite moments of all orders for an arbitrarily fixed $w\geqslant 0$.

(ii) The quantities $\pi_k(w)$ (obtained in the modified model) can be rewritten as

$$\pi_k(w)=\sum_{i=k}^{\infty}\binom{i}{k}p_i(w) \qquad (k=1, 2, \cdots). \tag{5.8}$$

(iii) The sequence $\{R_k(w)\}$ converges to $R(w)$ as $k\to\infty$, or, more precisely, the relation

$$R(w)-R_k(w)=(-1)^{k+1}\sum_{i=k+1}^{\infty}\binom{i-1}{k}p_i(w)\to 0 \qquad (k\to\infty) \tag{5.9}$$

holds, where the convergence to zero is uniform for every $w\in[0, w_0]$ with an arbitrarily fixed positive constant w_0.

Proof. As we have assumed the continuity of $F_i(x)$ and $F_i(0)=0$ $(i\in N)$ in Assumption 2°, we can choose a constant $\tau>0$ such that

$$\sum_{i=1}^{n}F_i(\tau)\equiv\varepsilon_1(\tau)<\frac{1}{2}. \tag{5.10}$$

Let us put

$$m_1=[nw_0/\tau]$$

and then m_1 is a non-negative integer, where w_0 is an arbitrarily

[2)] The statements (ii) and (iii) are closely related to the formulae [(5.3) and (3.2)] in Chap. IV of W. Feller, An Introduction to Probability Theory and Its Applications, Vol. 1 (2nd ed.), Wiley, 1957. In the latter, as the number of events is finite, no convergence problem arises, while in the former statements (ii) and (iii) the convergence properties are essential.

chosen positive constant such that $0<w\leqslant w_0$. Now, we can prove

(5.11) $$P\{M(w)\geqslant i\}\leqslant\{\varepsilon_1(\tau)\}^{i-m_1} \qquad (i=m_1,\ m_1+1,\ \cdots)$$

for every $w\in(0,\ w_0]$. We remark here that, whenever the network fails, at least one component of the network becomes inoperative at the instant, owing to Lemma 4.2. Then, $M(w)\geqslant i\geqslant m_1$ implies that at least i components fail in the interval $(0,\ w]$ and the total sum of their life-times is not greater than $nw\leqslant nw_0$. Therefore, $M(w)\geqslant i\geqslant m_1$ also implies that at least $(i-m_1)$ components have smaller life-times than τ, because, otherwise, (m_1+1) or more components should have the greater life-times than τ and consequently the sum of them should exceed nw_0, which is apparently a contradiction. This fact yields (5.11), from which the kth moment of $M(w)$ is evaluated as

(5.12) $$\sum_{i=0}^{\infty} i^k p_i(w)\leqslant \sum_{i=0}^{m_1} i^k+\sum_{i=m_1+1}^{\infty} i^k\{\varepsilon_1(\tau)\}^{i-m_1}.$$

The right side of the above is certainly finite for every positive integer k and every $w\in(0,\ w_0]$, since $\varepsilon_1(\tau)<1$ from (5.10) and $m_1<\infty$ from the definition. Thus, we have proved the statement (i) of the theorem.

In order to prove the statement (ii) of the theorem, define the random variables

(5.13) $$Y_k^{(l)}=\sum{}' z_{i_1}z_{i_2}\cdots z_{i_k} \qquad (k=1,\ 2,\ \cdots,\ l)$$

where $z_i=Z(t_i,\ t_{i+1})$ $(i=1,\ 2,\ \cdots,\ l)$ are defined by (4.1) and $\sum'$ are defined just as in (5.4). Then, the identity

(5.14) $$\begin{aligned} Y_1^{(l)}Y_k^{(l)}&=(k+1)Y_{k+1}^{(l)}+\sum{}' z_{i_1}^2 z_{i_2}\cdots z_{i_k}+\sum{}' z_{i_1}z_{i_2}^2 z_{i_3}\cdots z_{i_k}\\ &\quad+\cdots+\sum{}' z_{i_1}z_{i_2}\cdots z_{i_{k-1}}z_{i_k}^2\\ &=(k+1)Y_{k+1}^{(l)}+k\,Y_k^{(l)} \end{aligned}$$

holds for each $k=1, 2, \cdots, l-1$, since $z_i^2=z_i$ $(i=1, 2, \cdots, l)$ from (4.1). From (5.14) we obviously have

(5.15) $$Y_k^{(l)}=\frac{1}{k!}Y_1^{(l)}(Y_1^{(l)}-1)\cdots(Y_1^{(l)}-k+1) \qquad (k=1,\ 2,\ \cdots,\ l).$$

Notice here that the probability that two or more system failures occur at the same instant of $(0,\ w]$ can be neglected owing to the continuity assumption on $F_i(x)$ $(i \in N)$ and, hence, that $P\{\lim_{\Delta_l \to 0} Y_1^{(l)} = M(w)\} = 1$ holds. By means of this fact and Lebesgue's convergence theorem we get

$$\lim_{\Delta_l \to 0} P\{Y_1^{(l)} = i\} = P\{\lim_{\Delta_l \to 0} Y_1^{(l)} = i\} = P\{M(w) = i\} = p_i(w) \quad (i = 0, 1, \cdots). \tag{5.16}$$

From (5.13), (5.15) and (5.16) we finally obtain

$$\pi_k(w) = \lim_{\Delta_l \to 0} E\{Y_k^{(l)}\} = \frac{1}{k!} \sum_{i=k}^{\infty} i(i-1)\cdots(i-k+1) p_i(w), \tag{5.17}$$

where the existence of the right side is guaranteed by the statement (i), and this proves the statement (ii) of the theorem.

To prove the statement (iii) of the theorem, remark that Lemma 4.1 yields

$$R(w) = p_0(w) = 1 - \sum_{i=1}^{\infty} p_i(w).$$

Then, the equality in (5.9) can be proved inductively. First, for $k=1$ the equality is asserted directly as

$$R(w) - R_1(w) = \{1 - \sum_{i=1}^{\infty} p_i(w)\} - \{1 - \sum_{i=1}^{\infty} i p_i(w)\}$$
$$= \sum_{i=2}^{\infty} (i-1) p_i(w).$$

Second, let us assume that the equality in (5.9) holds for an integer $k > 1$, and then we have

$$R(w) - R_{k+1}(w) = \{R(w) - R_k(w)\} - (-1)^{k+1} \pi_{k+1}(w)$$
$$= \frac{(-1)^{k+2}}{(k+1)!} \sum_{i=k+2}^{\infty} (i-1)(i-2)\cdots(i-k-1) p_i(w),$$

which asserts the equality in (5.9) for k replaced by $k+1$. This completes the inductive proof of the equality in (5.9). The convergence to zero and its uniformity with respect to $w \in [0, w_0]$ are obviously verified by the statement (i) and its proof given above, as follows: if $\varepsilon_1(\tau) = 0$, then the statement (iii) is trivially true; and if $\eta \equiv \varepsilon_1(\tau) > 0$, then we have for every integer $k \geq m_1$ and every $w \in (0, w_0]$

$$\sum_{i=k+1}^{\infty}\binom{i-1}{k}p_i(w)\leqslant\sum_{i=k+1}^{\infty}\binom{i-1}{k}\eta^{i-m_1}$$

$$=a_k\frac{d^k}{d\eta^k}\sum_{i=k+1}^{\infty}\eta^{i-1}\qquad(a_k\equiv\eta^{k+1-m_1}/k!)$$

$$=a_k\frac{d^k}{d\eta^k}\frac{\eta^k}{1-\eta}$$

$$=a_k\sum_{i=0}^{k}\binom{k}{i}\left(\frac{d^i}{d\eta^i}\eta^k\right)\left(\frac{d^{k-i}}{d\eta^{k-i}}\frac{1}{1-\eta}\right)$$

$$=\eta^{-m_1}\left(\frac{\eta}{1-\eta}\right)^{k+1},$$

which converges to zero as k increases since $\eta<1/2$ and $0<\eta/(1-\eta)<1$. Thus, we have accomplished the proof of the theorem.

Remark 5.2. In (5.8) and (5.9), using the inequality

$$\frac{1}{k!}\sum_{i=k+1}^{\infty}(i-1)(i-2)\cdots(i-k)p_i(w)$$

$$\leqslant\frac{1}{(k+1)!}\sum_{i=k+1}^{\infty}i(i-1)\cdots(i-k)p_i(w)$$

$$=\pi_{k+1}(w),$$

we can derive the reliability evaluations (5.2), (5.5) and (5.6). This gives another proof of Theorem 5.1 and Remark 5.1. Furthermore, it is interesting to note that, defining the moment generating function

$$P(s,w)=\sum_{k=0}^{\infty}p_k(w)s^k,$$

we get formally from Taylor expansion of $P(s, w)$

$$R(w)=P(0,w)$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}P^{(k)}(1,w)$$

$$=1+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{i=k}^{\infty}i(i-1)\cdots(i-k+1)p_i(w)$$

$$=1+\sum_{k=1}^{\infty}(-1)^k\pi_k(w)$$

$$=\lim_{k\to\infty}R_k(w),$$

where $P^{(k)}(s, w)$ denotes the kth partial derivative of $P(s, w)$ with respect to s and the existence of the limit on the right side of the above is ascertained by the statement (iii) of Theorem 5.2, or by the fact that the convergence radius of $\sum p_k(w)s^k$ is greater than 2.

For practical applications of these results it is needed to evaluate more precisely the order of the magnitude of $\pi_k(w)$ for every $k=1, 2, \cdots$, or, to evaluate more simply the rapidness of the convergence of $R_k(w)$ to $R(w)$. We shall try to achieve this goal in the sequel of the section under some additional assumptions.

Theorem 5.3. In addition to Assumptions 1°—5°, if the inequality

$$\sum_{i=1}^{n} F_i(w) \equiv \varepsilon_1(w) < \frac{1}{2} \tag{5.18}$$

is satisfied, then $\pi_k(w)$ is evaluated as

$$\pi_k(w) \leqslant \frac{\varepsilon_0(w)}{\{1-\varepsilon_1(w)\}^2} \left\{\frac{\varepsilon_1(w)}{1-\varepsilon_1(w)}\right\}^{k-1} \tag{5.19}$$

for every $k=1, 2, \cdots$, where

$$\varepsilon_0(w) = \min\{\varepsilon_1(w),\ \varepsilon_2(w)\},\quad \varepsilon_2(w) = \sum_{j=1}^{s} \prod_{i \in B_j} F_i(w)\,. \tag{5.20}$$

Proof. Let i_k be the critical component of the kth failure of the modified network observed in the interval $(0, w]$. Then, we obviously have

$$\begin{aligned} p_1(w) &\leqslant \sum_{i_1=1}^{n} P\{\text{component } i_1 \text{ of the network fails in the interval } (0, w]\} \\ &= \varepsilon_1(w)\,. \end{aligned} \tag{5.21}$$

On the otherhand, each failure of the network accompanies at least a cut, B_j, of the network such that all elements of B_j are inoperative at the instant. Therefore, we also have

$$\begin{aligned} p_1(w) &\leqslant \sum_{j=1}^{s} P\{\text{all components of } B_j \text{ fail in the interval } (0, w]\} \\ &\leqslant \sum_{j=1}^{s} \prod_{i \in B_j} F_i(w) = \varepsilon_2(w)\,. \end{aligned} \tag{5.22}$$

From (5.20), (5.21) and (5.22) we get

$$p_1(w) \leqslant \varepsilon_0(w)\,.$$

Remark that we can neglect the probability that two or more failures of the components occur at the same instant and

$$P\{\text{component } i_1 \text{ fails and next component } i_2 \text{ also fails both in the interval } (0, w]\} \leqslant F_{i_1}(w)F_{i_2}(w)$$

not only for $i_1 \neq i_2$ but also for $i_1 = i_2$, because in both cases the two life-lengths are independent in our (modified) model. Then, we get

$$(5.23) \qquad p_2(w) \leqslant \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} P\{\text{component } i_1 \text{ fails and next component } i_2 \text{ also fails both in the interval } (0, w]\} \leqslant \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} F_{i_1}(w)F_{i_2}(w) = \varepsilon_1^2(w) .$$

Similarly, we get

$$(5.24) \qquad p_2(w) \leqslant \sum_{j=1}^{s} \sum_{i_2=1}^{n} P\{\text{all components of } B_j \text{ become inoperative and next component } i_2 \text{ becomes inoperative both in } (0, w]\} \leqslant \varepsilon_1(w)\varepsilon_2(w) .$$

From (5.23) and (5.24) we have

$$p_2(w) \leqslant \varepsilon_0(w)\varepsilon_1(w) .$$

By the same argument as above we obtain

$$p_i(w) \leqslant \varepsilon_0(w)\varepsilon_1^{i-1}(w) \qquad (i=1, 2, \cdots) .$$

Hence,

$$\pi_k(w) = \frac{1}{k!} P^{(k)}(1, w) \leqslant \varepsilon_0(w) \sum_{i=k}^{\infty} \binom{i}{k} \varepsilon_1^{i-1}(w) = \frac{\varepsilon_0(w)}{\{1-\varepsilon_1(w)\}^2} \left\{ \frac{\varepsilon_1(w)}{1-\varepsilon_1(w)} \right\}^{k-1}$$

for every $k=1, 2, \cdots$. Thus, the proof is completed.

On applying Theorem 5.3, we should notice that the functions $\varepsilon_0(w)$ and $\varepsilon_1(w)$ are monotone nondecreasing and continuous with $\varepsilon_0(0) = \varepsilon_1(0) = 0$, and the inequality in (5.18) is satisfied for every $0 \leqslant w < w_0$, where $\varepsilon_1(w_0) < 1/2$. This implies that Theorem 5.3 can be applied to the networks whose components are highly reliable. However, the right side of the formula in (5.19) does not include the effect of the repair-time distributions $G_i(x)$ $(i \in N)$. As this fact may limit the applicability of the theorem, we shall show another theorem under some restrictions on both functions $F_i(x)$ and $G_i(x)$ $(i \in N)$.

Define the functions $f_i(x|u_i)$, $\bar{G}_i(x|v_j)$ and $h(u_i, v_j, w)$ as

(5.25) $$f_i(x|u_i)=\begin{cases} f_i(x+u_i)/\bar{F}_i(u_i) & \text{if } \bar{F}_i(u_i)>0\,, \\ 0 & \text{otherwise;} \end{cases}$$

(5.26) $$\bar{G}_j(x|v_j)=\begin{cases} \bar{G}_j(x+v_j)/\bar{G}_j(v_j) & \text{if } \bar{G}_j(v_j)>0\,, \\ 0 & \text{otherwise;} \end{cases}$$

(5.27) $$h(u_i, v_j, w)=\begin{cases} \min\{w-u_i, w-v_j\} & \text{if } 0\leqslant u_i,\ v_j\leqslant w\,, \\ 0 & \text{otherwise;} \end{cases}$$

and put

(5.28) $$\max_{i\in N}\ \sup_{u_i\in[0,w]}\ \max_{t\in[0,w-u_i]}\int_0^t f_i(s|u_i)\bar{G}_i(t-s)ds=\eta_1(w)\,,$$

(5.29) $$\max_{i,j\in N}\ \sup_{u_i,v_j\in[0,w]}\ \max_{t\in[0,h(u_i,v_j,w)]}\int_0^t f_i(x|u_i)\bar{G}_j(x|v_j)dx=\eta_2(w)\,,$$

(5.30) $$\eta_0(w)=\max\{\eta_1(w),\ \eta_2(w)\}\,,$$

(5.31) $$H_0(w)=\max_{i\in N} H_i(w)\,,$$

(5.32) $$\nu_0=\max_{\alpha=1,\cdots,\nu} m_\alpha\,,$$

where m_α is the number of the critical triplets $[I_\beta, C_\beta, D_\beta]$ such that satisfy $C_\alpha I_\beta \neq \phi$, $D_\alpha D_\beta \neq \phi$ and $\beta\neq\alpha$ for the given triplet $[I_\alpha, C_\alpha, D_\alpha]$, for each $\alpha=1, 2, \cdots, \nu$. By the definition, it is easily verified that

(5.33) $$0\leqslant\nu_0<\nu\,.$$

Let us assume that

(5.34) $$(\nu+\nu_0)\eta_0(w)(1+H_0(w))^2<1$$

and assume also that

(5.35) $$\delta=\min_{\alpha=1,\cdots,\nu}|D_\alpha|\geqslant 1\,.$$

The latter condition may naturally be satisfied in most of networks, because $\delta=0$ implies that there exists a critical triplet $[I_\alpha, C_\alpha, D_\alpha]$ such that $D_\alpha=\phi$, *i.e.*, the component $i_\alpha \in I_\alpha$ forms a cut of the network, which is nothing but that the network is constructed by the series connection of the component i_α and the subnetwork $N-\{i_\alpha\}$ and, hence, the reliability of the network is the reliability of the subnetwork $N-\{i_\alpha\}$ times the reliability of the unique component i_α. Thus, we can assume (5.35) without loss of generality. On the other hand, the condition (5.34) is considerably restrictive, although there is always a positive constant w_1 such that (5.34) holds for every $w \in (0, w_1)$ owing to the continuity assumption in 2°. In order for (5.34) to hold it is needful that both $\eta_1(w)$ and $\eta_2(w)$ are sufficiently small. Roughly speaking, $0 \leqslant \eta_1(w) \ll 1$ holds when the repair-time of component i is sufficiently smaller than the life-time of the component for each $i \in N$, because the integral on the left side of (5.28) can be rewritten as

$$\int_0^t f_i(s|u_i)\bar{G}_i(t-s)ds = F_i(t|u_i) - G_i * F_i(t|u_i) \qquad \left(F_i(t|u_i) = \int_0^t f_i(s|u_i)ds\right). \tag{5.36}$$

And $0 \leqslant \eta_2(w) \ll 1$ holds when it rarely occurs that

(5.37) a component, i, operative at an instant $s \in (0, w)$, becomes inoperative before any other component, j, inoperative at the instant, becomes operative

for each $i, j \in N$ and for every $s \in (0, w)$, whatever the past histories, u_i and v_j, of the components i and j may be, because the integral in (5.29) is the probability of the event (5.37), where u_i [v_j] is the operative [inoperative] time length at the instant s since the last recovery [failure] of component i [j].

As for the condition (5.34), we further remark that $\eta_0(w)$ and $H_0(w)$ are both monotone non-decreasing functions with $\eta_0(0)=0$ and $H_0(0)=1$. Moreover, we have

$$\max_{t \in [0, w-u_i^-]} \int_0^t f_i(s|u_i)\bar{G}_i(t-s)ds \leqslant \{F_i(w) - F_i(u_i)\}/\bar{F}(u_i) \leqslant F_i(w)/\bar{F}_i(w)$$

$$\max_{t\in[0,h(u_i,v_j,w)]}\int_0^t f_i(x|u_i)\bar{G}_j(x|v_j)dx$$
$$\leqslant\{F_i(w)-F_i(u_i)\}/\bar{F}_i(u_i)\leqslant F_i(w)/\bar{F}_i(w) \qquad (\bar{F}_i(w)>0).$$

Therefore, just as in Theorem 5.3, there always exists a positive constant w_1 such that $w=w_1$ satisfies the condition (5.34) and that the condition holds for every $0\leqslant w\leqslant w_1$. We may say that the condition (5.34) holds for wide range of w in case of networks with highly reliable components and/or highly speedy repairs. Now we get the following theorem.

Theorem 5.4. In addition to Assumptions 1°—5°, if the conditions (5.34) and (5.35) are assumed, then the evaluation

$$0\leqslant\pi_k(w)\leqslant\nu H_0^{\delta+1}(w)\eta_0^{\delta}(w)\{(\nu+\nu_0)(1+H_0(w))^2\eta_0(w)\}^{k-1}$$
$$\leqslant\{(\nu+\nu_0)(1+H_0(w))^2\eta_0(w)\}^k$$
$$<1 \tag{5.38}$$

holds for every $k=1, 2, \cdots$.

Proof. Let us prove the theorem by induction. The first step is to prove (5.38) for $k=1$. In the formula (4.11) we obviously have

$$\prod_{i\in C_\alpha}\int_0^t \bar{F}_i(t-u)dH_i(u)\leqslant 1, \tag{5.39}$$

since this is the probability that all of the components $i\in C_\alpha$ satisfy $i\in C(t)$. We easily get

$$\prod_{j\in D_\alpha}\int_0^t dH_j(u)\int_u^t f_j(v-u)\bar{G}_j(t-v)dv \tag{5.40}$$
$$\leqslant\prod_{j\in D_\alpha}\int_0^t dH_j(u)\int_0^{t'} f_j(x)\bar{G}_j(t'-x)dx \qquad (t'\equiv t-u\leqslant t)$$
$$\leqslant\prod_{j\in D_\alpha}\{H_j(t)\eta_0(t)\}\leqslant\{H_0(w)\eta_0(w)\}^\delta,$$

$$\int_0^w\int_0^t f_i(t-u)dH_i(u)dt \tag{5.41}$$
$$\leqslant\int_0^w dH_i(u)\int_0^w f_i(t-u)dt\leqslant H_0(w).$$

The inequalities (5.39), (5.40) and (5.41) with (4.11) yield the statement of the theorem for $k=1$. The second step is, assuming the statement of the theorem to be valid for $\pi_{k-1}(w)$, to prove it for $\pi_k(w)$. For simplicity of notation let us put in Lemma 4.5

$$\alpha=\alpha_{k-1}\,,\quad \beta=\alpha_k\,,\quad s=s_{k-1}\,,\quad t=s_k\,, \tag{5.42}$$

$$\begin{aligned}\int_s^w Q_{\beta|\alpha}(t|s)dt &= \int_{t=s}^w \prod_{i=1}^n [P\{i\in I(t;\,dt)|i\in I(s;\,ds)\}^{\delta_i(I_\alpha I_\beta)} \\ &\quad\times P\{i\in I(t;\,dt)|i\in C(s;\,u_i)\}^{\delta_i(C_\alpha I_\beta)} \\ &\quad\times P\{i\in I(t;\,dt)|i\in D(s;\,u_i)\}^{\delta_i(D_\alpha I_\beta)} \\ &\quad\times P\{i\in C(t)|i\in I(s;\,ds)\}^{\delta_i(I_\alpha C_\beta)} \\ &\quad\times P\{i\in C(t)|i\in C(s;\,u_i)\}^{\delta_i(C_\alpha C_\beta)} \\ &\quad\times P\{i\in C(t)|i\in D(s;\,u_i)\}^{\delta_i(D_\alpha C_\beta)} \\ &\quad\times P\{i\in D(t)|i\in I(s;\,ds)\}^{\delta_i(I_\alpha D_\beta)} \\ &\quad\times P\{i\in D(t)|i\in C(s;\,u_i)\}^{\delta_i(C_\alpha D_\beta)} \\ &\quad\times P\{i\in D(t)|i\in D(s;\,u_i)\}^{\delta_i(D_\alpha D_\beta)}]\end{aligned} \tag{5.43}$$

for every $0\leqslant s<t\leqslant w$, where the notations on the right side of the above are defined in Section 4. Let us note that

$$\delta_i(I_\alpha I_\beta)+\delta_i(C_\alpha I_\beta)+\delta_i(D_\alpha I_\beta)=\begin{cases}1 & \text{if } i\in I_\beta\,,\\ 0 & \text{otherwise,}\end{cases} \tag{5.44}$$

$$P\{i\in C(t)|i\in E\}^{\delta_i(E_\alpha C_\beta)}\leqslant 1\,,\quad P\{i\in D(t)|i\in E\}^{\delta_i(E_\alpha D_\beta)}\leqslant 1\,,$$

for each $i\in N$, where $\{E,\,E_\alpha\}$ denotes $\{I(s;\,ds),\,I_\alpha\}$, $\{C(s;\,u),\,C_\alpha\}$ and $\{D(s;\,u),\,D_\alpha\}$, respectively. Then, we obviously have

$$\begin{aligned}\int_s^w Q_{\beta|\alpha}(t|s)dt &\leqslant \int_{t=0}^w P\{i_\beta\in I(t;\,dt)|i_\beta\in I(s;\,ds)\}^{\delta_{i_\beta}(I_\alpha I_\beta)} \\ &\quad\times P\{i_\beta\in I(t;\,dt)|i_\beta\in C(s;\,u)\}^{\delta_{i_\beta}(C_\alpha I_\beta)} \\ &\quad\times P\{i_\beta\in I(t;\,dt)|i_\beta\in D(s;\,u)\}^{\delta_{i_\beta}(D_\alpha I_\beta)} \\ &\quad\times P\{j_\beta\in D(t)|j_\beta\in I(s;\,ds)\}^{\delta_{j_\beta}(I_\alpha D_\beta)} \\ &\quad\times P\{j_\beta\in D(t)|j_\beta\in C(s;\,v)\}^{\delta_{j_\beta}(C_\alpha D_\beta)} \\ &\quad\times P\{j_\beta\in D(t)|j_\beta\in D(s;\,v)\}^{\delta_{j_\beta}(D_\alpha D_\beta)}\,,\end{aligned} \tag{5.45}$$

where

(5.46) $i_\beta \in I_\beta$ and $j_\beta \in D_\beta$,

which satisfy

(5.47)
$$\delta_{i_\beta}(I_\alpha I_\beta)+\delta_{i_\beta}(C_\alpha I_\beta)+\delta_{i_\beta}(D_\alpha I_\beta)=1,$$
$$\delta_{j_\beta}(I_\alpha D_\beta)+\delta_{j_\beta}(C_\alpha D_\beta)+\delta_{j_\beta}(D_\alpha D_\beta)=1.$$

From these, on evaluating the right side of (5.45), we can obtain the following nine cases and eight inequalities, which are proved in Appendix.

Case (i): $\delta_{i_\beta}(I_\alpha I_\beta)=\delta_{j_\beta}(I_\alpha D_\beta)=1$. This implies that $j_\beta \in I_\alpha = I_\beta$, which contradicts to (5.46) since $I_\beta D_\beta$ should be vacant. Hence, this case can never occur.

Case (ii): $\delta_{i_\beta}(I_\alpha I_\beta)=\delta_{j_\beta}(C_\alpha D_\beta)=1$. In this case we have

(5.48)
$$\int_{t=0}^{w} P\{i_\beta \in I(t;\ dt)|i_\beta \in I(s;\ ds)\}P\{j_\beta \in D(t)|j_\beta \in C(s;\ v)\}$$
$$\leqslant H_0(w)(1+H_0(w))\eta_0(w)\leqslant(1+H_0(w))^2\eta_0(w).$$

Case (iii): $\delta_{i_\beta}(I_\alpha I_\beta)=\delta_{j_\beta}(D_\alpha D_\beta)=1$. In this case we have

(5.49)
$$\int_{t=0}^{w} P\{i_\beta \in I(t;\ dt)|i_\beta \in I(s;\ ds)\}P\{j_\beta \in D(t)|j_\beta \in D(s;\ v)\}$$
$$\leqslant(1+H_0(w))^2\eta_0(w).$$

Case (iv): $\delta_{i_\beta}(C_\alpha I_\beta)=\delta_{j_\beta}(I_\alpha D_\beta)=1$. In this case we have

(5.50)
$$\int_{t=0}^{w} P\{i_\beta \in I(t;\ dt)|i_\beta \in C(s;\ u)\}P\{j_\beta \in D(t)|j_\beta \in I(s;\ ds)\}$$
$$\leqslant(1+H_0(w))^2\eta_0(w).$$

Case (v): $\delta_{i_\beta}(C_\alpha I_\beta)=\delta_{j_\beta}(C_\alpha D_\beta)=1$. In this case we have

(5.51)
$$\int_{t=0}^{w} P\{i_\beta \in I(t;\ dt)|i_\beta \in C(s;\ u)\}P\{j_\beta \in D(t)|j_\beta \in C(s;\ v)\}$$
$$\leqslant(1+H_0(w))^2\eta_0(w).$$

Case (vi): $\delta_{i_\beta}(C_\alpha I_\beta)=\delta_{j_\beta}(D_\alpha D_\beta)=1$. In this case we get

(5.52)
$$\int_{t=0}^{w} P\{i_\beta \in I(t;\ dt)|i_\beta \in C(s;\ u)\}P\{j_\beta \in D(t)|j_\beta \in D(s;\ v)\}$$
$$\leqslant 2(1+H_0(w))^2\eta_0(w).$$

Case (vii): $\delta_{i_\beta}(D_\alpha I_\beta)=\delta_{j_\beta}(I_\alpha D_\beta)=1$. In this case we get

(5.53)
$$\int_{t=0}^{w} P\{i_\beta \in I(t;\ dt)|i_\beta \in D(s;\ u)\}P\{j_\beta \in D(t)|j_\beta \in I(s;\ ds)\}$$

$$\leqslant H_0(w)(1+H_0(w))\eta_0(w)\leqslant(1+H_0(w))^2\eta_0(w)\,.$$

Case (viii): $\delta_{i_\beta}(D_\alpha I_\beta)=\delta_{j_\beta}(C_\alpha D_\beta)=1$. In this case we get

$$\int_{t=0}^{w} P\{i_\beta\in I(t;\ dt)|i_\beta\in D(s;\ u)\}P\{j_\beta\in D(t)|j_\beta\in C(s;\ v)\} \tag{5.54}$$

$$\leqslant H_0(w)(1+H_0(w))\eta_0(w)\leqslant(1+H_0(w))^2\eta_0(w)\,.$$

Case (ix): $\delta_{i_\beta}(D_\alpha I_\beta)=\delta_{j_\beta}(D_\alpha D_\beta)=1$. In this case we get

$$\int_{t=0}^{w} P\{i_\beta\in I(t;\ dt)|i_\beta\in D(s;\ u)\}P\{j_\beta\in D(t)|j_\beta\in D(s;\ v)\} \tag{5.55}$$

$$\leqslant(1+H_0(w))^2\eta_0(w)\,.$$

Here, given the critical triplet $[I_\alpha,\ C_\alpha,\ D_\alpha]$, each critical triplet $[I_\beta,\ C_\beta,\ D_\beta]$ corresponds to one of the eight cases (ii)—(ix) mentioned above. And the right side of (5.45) can be dominated by $(1+H_0(w))^2\eta_0(w)$ for all critical triplets $[I_\beta,\ C_\beta,\ D_\beta]$, except for at most the ν_0 ones corresponding to Case (vi). For these ν_0 exceptional critical triplets the right side of (5.45) is bounded from above by $2(1+H_0(w))^2\eta_0(w)$. Thus, we obtain

$$\sum_{\beta=1}^{\nu}\int_{s}^{w} Q_{\beta|\alpha}(t|s)dt\leqslant(\nu-\nu_0)(1+H_0(w))^2\eta_0(w)+2\nu_0(1+H_0(w))^2\eta_0(w)$$

$$=(\nu+\nu_0)(1+H_0(w))^2\eta_0(w)\,.$$

Therefore, referring to Lemma 4.5 and noting the nonnegativity of the integrant of (4.32), we finally get

$$\pi_k(w)\leqslant\pi_{k-1}(w)\{(\nu+\nu_0)(1+H_0(w))^2\eta_0(w)\}$$

for $k\geqslant2$. This completes the inductive proof of the theorem, because assumption (5.34) and $H_0(w)\geqslant1$ imply that

$$H_0^{1+\delta}(w)\eta_0^{\delta}(w)\leqslant\{(1+H_0(w))^2\eta_0(w)\}^{\delta}\leqslant(1+H_0(w))^2\eta_0(w)$$

for every $\delta\geqslant1$.

6. Examples of Reliability Evaluations

Example 6.1. Two-component-paralleled redundant system with $N=\{1,\ 2\}$. Let us consider again the simplest network given in Example 3.1. As shown in the example we have the two critical triplets

$$[1;\ \phi;\ 2]\quad\text{and}\quad[2;\ \phi;\ 1]$$

i.e., $\nu=2$; $i_1=1$, $C_1=\phi$, $D_1=\{2\}$; $i_2=2$, $C_2=\phi$, $D_2=\{1\}$. Therefore, applying Lemma 4.3, we have

$$\text{(6.1)}\qquad \pi_1(w)=\int_0^w dt\int_0^t dH_1(u_1)f_1(t-u_1)\int_0^t dH_2(u_2)\int_{u_2}^t f_2(v-u_2)\bar{G}_2(t-v)dv$$
$$+\int_0^w dt\int_0^t dH_2(u_2)f_2(t-u_2)\int_0^t dH_1(u_1)\int_{u_1}^t f_1(v-u_1)\bar{G}_1(t-v)dv$$

and applying Lemma 4.4, we get

$$\text{(6.2)}\qquad \pi_2(w)=\int_{s=0}^w\int_{t=s}^w [P\{1\in I(s;\ ds)I(t;\ dt)\}P\{2\in D(s)D(t)\}$$
$$+P\{2\in I(s;\ ds)I(t;\ dt)\}P\{1\in D(s)D(t)\}$$
$$+P\{1\in I(s;\ ds)D(t)\}P\{2\in D(s)I(t;\ dt)\}$$
$$+P\{2\in I(s;\ ds)D(t)\}P\{1\in D(s)I(t;\ dt)\}]\,.$$

Specifically, if we assume that

$$\text{(6.3)}\qquad \begin{aligned}F_1(t)=F_2(t)=1-e^{-\lambda t}\qquad (t\geqslant 0)\,,\\ G_1(t)=G_2(t)=1-e^{-\mu t}\qquad (t\geqslant 0)\,,\end{aligned}$$

then, the system reliability will be given by [1, p. 146]

$$\text{(6.4)}\qquad R(w)=\frac{2\rho^2}{s_2-s_1}\left\{\frac{e^{-s_1w}}{s_1}-\frac{e^{-s_2w}}{s_1}\right\}\,,$$

where without loss of generality we put

$$\text{(6.5)}\qquad \mu=1\,,\quad \rho=\lambda/\mu=\lambda\,,$$
$$s_1=(3\rho+1+\sqrt{\rho^2+6\rho+1}\,)/2\,,\qquad s_2=(3\rho+1-\sqrt{\rho^2+6\rho+1}\,)/2\,.$$

Remark that although $m=2$ has been assumed in [1, p. 146], (6.4) is obtained also for $m=1$ owing to our Lemma 4.1. Now, in order to exemplify the evaluations $R_\kappa(w)$ $(k=1,\ 2)$ and to compare them with (6.4), let us obtain the explicit formulae of $R_\kappa(w)$ $(k=1,\ 2)$ under the additional assumptions (6.3) and (6.5). Let us put

$$\text{(6.6)}\qquad f(t)=f_1(t)=f_2(t)=\begin{cases}\rho e^{-\rho t} & \text{if } t\geqslant 0\,,\\ 0 & \text{otherwise,}\end{cases}$$
$$g(t)=g_1(t)=g_2(t)=\begin{cases}e^{-t} & \text{if } t\geqslant 0\,,\\ 0 & \text{otherwise,}\end{cases}$$
$$h^{(1)}(t)=f*g(t)\,.$$

Let $h^{(k)}(t)$ be the kth convolution of $h^{(1)}(t)$ for every $k=2,\ 3,\ \cdots$. Note that the Laplace transform, $\phi(s)$, of $h^{(1)}(t)$ is $\rho/\{(s+1)(s+\rho)\}$. Then, we obtain

$$\sum_{k=1}^{\infty}\int_0^{\infty} h^{(k)}(t)e^{-st}dt=\frac{\phi(s)}{1-\phi(s)}=\frac{\rho}{\sigma}\left\{\frac{1}{s}-\frac{1}{s+\sigma}\right\}$$

and, hence,

$$\sum_{k=1}^{\infty} h^{(k)}(t)=\frac{\rho}{\sigma}(1-e^{-\sigma t}) \qquad (t\geqslant 0),$$

where

$$\sigma=1+\rho .$$

From this we get

$$H(t)=H_1(t)=H_2(t)=\begin{cases}1+\int_0^t \frac{\rho}{\sigma}(1-e^{-\sigma x})dx & \text{if } t\geqslant 0, \\ 0 & \text{otherwise.}\end{cases}$$

In this case we have

$$\text{(6.7)} \qquad \int_0^t f(t-u)dH(u)=\frac{\rho}{\sigma}(1+\rho e^{-\sigma t}) \qquad (t\geqslant 0),$$

$$\text{(6.8)} \qquad \int_s^t d_yH(y-x)\int_y^t f(z-y)\bar{G}(t-z)dz$$

$$=\frac{\rho}{\sigma}(1-e^{-\sigma(t-x)}) \qquad (0\leqslant x\leqslant s\leqslant t)$$

and, hence,

$$\text{(6.9)} \qquad \pi_1(w)=2\left(\frac{\rho}{\sigma}\right)^2\int_0^w (1-e^{-\sigma t})(1+\rho e^{-\sigma t})dt$$

$$=2\left(\frac{\rho}{\sigma}\right)^2\left\{w-\frac{1-\rho}{\sigma}(1-e^{-\sigma w})-\frac{\rho}{2\sigma}(1-e^{-2\sigma w})\right\}$$

for every $w\geqslant 0$.

We also have

$$\text{(6.10)} \qquad \int_s^t d_xG(x-u)\int_x^t d_yH(y-x)f(t-y)$$

$$=\frac{\rho}{\sigma}e^{-(s-u)}(1-e^{-\sigma(t-s)}) \qquad (0\leqslant u\leqslant s\leqslant t),$$

$$\text{(6.11)} \qquad \int_s^t d_xH(x-s)\bar{G}(t-x)=\frac{\rho}{\sigma}\left(1+\frac{1}{\rho}e^{-\sigma(t-s)}\right) .$$

From (6.7), (6.8), (6.10) and (6.11) we obtain

$$P\{i \in I(s;\, ds)I(t;\, dt)\}=\left(\frac{\rho}{\sigma}\right)^2(1+\rho e^{-\sigma s})(1-e^{-\sigma(t-s)})ds\,dt\,,$$

$$P\{i \in D(s)D(t)\}=\left(\frac{\rho}{\sigma}\right)^2(1-e^{-\sigma s})\left(1+\frac{1}{\rho}e^{-\sigma(t-s)}\right)\,,$$

$$P\{i \in I(t;\, ds)D(t)\}=\left(\frac{\rho}{\sigma}\right)^2(1+\rho e^{-\sigma s})\left(1+\frac{1}{\rho}e^{-\sigma(t-s)}\right)ds\,,$$

$$P\{i \in D(s)I(t;\, dt)\}=\left(\frac{\rho}{\sigma}\right)^2(1-e^{-\sigma s})(1-e^{-\sigma(t-s)})dt$$

and, hence,

(6.12)
$$\begin{aligned}\pi_2(w)&=4\left(\frac{\rho}{\sigma}\right)^4\int_0^w ds\int_s^w (1-e^{-\sigma s})(1+\rho e^{-\sigma s})\left(1+\frac{1}{\rho}e^{-\sigma(t-s)}\right)\\&\qquad\times(1-e^{-\sigma(t-s)})dt\\&=4\left(\frac{\rho}{\sigma}\right)^4\left\{\frac{w^2}{2}+\left(\frac{1-4\rho+\rho^2}{2\sigma\rho}\right)w+\frac{(1-\rho)^2}{\sigma\rho}we^{-\sigma w}\right.\\&\qquad\left.-\frac{1}{2\sigma}we^{-2\sigma w}-\frac{(1-\rho)^2}{\sigma^2\rho}(1-e^{-\sigma w})-\frac{1-5\rho+\rho^2}{4\sigma^2\rho}(1-e^{-2\sigma w})\right\}.\end{aligned}$$

Using (6.9) and (6.12), we finally obtain

(6.13) $$R_1(w)=1-\pi_1(w)\,,$$

(6.14) $$R_2(w)=1-\pi_1(w)+\pi_2(w)\,.$$

Theorem 5.1 asserts that

$$R_1(w)\leqslant R(w)\leqslant R_2(w)\,,$$

$$R(w)-R_1(w)\leqslant\pi_2(w)\,.$$

Under the present assumptions we can easily verify that

$$\nu_0=0\,,\quad \eta_0(w)\leqslant\frac{\rho}{1-\rho}\,,\quad H_0(w)\leqslant 1+\frac{\rho}{\sigma}w$$

for every $w\geqslant 0$. Therefore, Theorem 5.4 implies that if $2\rho(1-\rho)^{-1}(2+\sigma^{-1}\rho w)^2<1$, then,

$$\pi_k(w)\leqslant\left\{\frac{2\rho}{1-\rho}\left(2+\frac{\rho w}{1+\rho}\right)^2\right\}^k$$

holds for each $k=1,\,2,\,\cdots$. Moreover, putting

$$R(w)=\sum_{i=0}^{\infty}r_i w^i\,,$$

(6.15) $$R(w)-R_k(w)=\sum_{i=0}^{\infty} r_{k,i}w^i \qquad (k=1,\ 2,\ \cdots),$$

a simple calculation yields

$$r_0=1\ ,\quad r_1=0\ ,\quad r_2=-\rho^2\ ,\quad r_3=(1+3\rho)\rho^2/3\ ,$$
$$r_4=-(1+6\rho+7\rho^2)\rho^2/12\ ,\quad r_5=(1+\rho)(1+3\rho)(1+5\rho)\rho^2/60;$$

Table 6.1. Computational result of Example 6.1 with $\rho=0.01$.

w	$R(w)$	$R_1(w)$	$R_2(w)$	$R(w)-R_1(w)$	$R(w)-R_2(w)$	$\pi_2(w)$
0.0	1.0	1.0	1.0	0.0	0.0	0.0
10.0	0.998248	0.998233	0.998248	0.000015	−0.000000	0.000016
20.0	0.996311	0.996272	0.996312	0.000039	−0.000000	0.000040
30.0	0.994378	0.994311	0.994379	0.000067	−0.000001	0.000068
40.0	0.992449	0.992351	0.992450	0.000098	−0.000001	0.000099
50.0	0.990523	0.990390	0.990525	0.000133	−0.000002	0.000135
60.0	0.988601	0.988430	0.988604	0.000172	−0.000002	0.000174
70.0	0.986683	0.986469	0.986686	0.000214	−0.000003	0.000217
80.0	0.984769	0.984508	0.984773	0.000260	−0.000004	0.000265
90.0	0.982858	0.982548	0.982863	0.000310	−0.000005	0.000316
100.0	0.980951	0.980587	0.980958	0.000364	−0.000006	0.000370
110.0	0.979048	0.978627	0.979056	0.000421	−0.000008	0.000429
120.0	0.977148	0.976666	0.977157	0.000482	−0.000009	0.000491
130.0	0.975253	0.974705	0.975263	0.000547	−0.000011	0.000558
140.0	0.973360	0.972745	0.973373	0.000615	−0.000012	0.000628
150.0	0.971472	0.970784	0.971486	0.000687	−0.000014	0.000702
160.0	0.969587	0.968824	0.969603	0.000763	−0.000017	0.000780
170.0	0.967706	0.966863	0.967725	0.000843	−0.000019	0.000861
180.0	0.965828	0.964902	0.965849	0.000926	−0.000021	0.000947
190.0	0.963954	0.962942	0.963978	0.001012	−0.000024	0.001036
200.0	0.962084	0.960981	0.962111	0.001103	−0.000027	0.001130
210.0	0.960217	0.959021	0.960247	0.001197	−0.000030	0.001227
220.0	0.958354	0.957060	0.958388	0.001294	−0.000034	0.001328
230.0	0.956495	0.955100	0.956532	0.001395	−0.000037	0.001432
240.0	0.954639	0.953139	0.954680	0.001500	−0.000041	0.001541
250.0	0.952787	0.951178	0.952832	0.001608	−0.000045	0.001653
260.0	0.950938	0.949218	0.950987	0.001720	−0.000049	0.001770
270.0	0.949093	0.947257	0.949147	0.001836	−0.000054	0.001890
280.0	0.947252	0.945297	0.947310	0.001955	−0.000059	0.002014
290.0	0.945414	0.943336	0.945478	0.002078	−0.000064	0.002142
300.0	0.943579	0.941375	0.943649	0.002204	−0.000069	0.002273
310.0	0.941749	0.939415	0.941824	0.002334	−0.000075	0.002409
320.0	0.939921	0.937454	0.940002	0.002467	−0.000081	0.002548
330.0	0.938098	0.935494	0.938185	0.002604	−0.000087	0.002691
340.0	0.936278	0.933533	0.936371	0.002745	−0.000094	0.002838
350.0	0.934461	0.931572	0.934562	0.002889	−0.000101	0.002989
360.0	0.932648	0.929612	0.932756	0.003036	−0.000108	0.003144
370.0	0.930839	0.927651	0.930954	0.003187	−0.000115	0.003303
380.0	0.929032	0.925691	0.929156	0.003342	−0.000123	0.003465
390.0	0.927230	0.923730	0.927361	0.003500	−0.000131	0.003631
400.0	0.925431	0.921769	0.925571	0.003661	−0.000140	0.003801

$$r_{1,0}=r_{1,1}=r_{1,2}=r_{1,3}=0\,, \quad r_{1,4}=\rho^3/6\,, \quad r_{1,5}=-2\sigma\rho^3/15;$$
$$r_{2,0}=r_{2,1}=r_{2,2}=r_{2,3}=r_{2,4}=r_{2,5}=0\,.$$

These facts suggest that our $R_2(w)$ is a fairly good approximation of $R(w)$ as far as ρ is small and ρw is not large. This can be clearly seen in Table 6.1, where we have tabulated $R(w)$, $R_1(w)$, $R_2(w)$, $R(w)-R_1(w)$, $R(w)-R_2(w)$, and $\pi_3(w)$, given by (6.4), (6.13), (6.14), (6.15) ($k=1$), (6.15) ($k=2$), and (6.12), respectively, putting $\mu=1$ and $\rho=\lambda/\mu=0.01$.

Example 6.2. Series system of two two-component-paralleled redundant subsystems, $N^1=\{1, 2\}$ and $N^2=\{3, 4\}$. As the second example we consider the model which has been treated in Example 3.3 and assume here that $m\geqslant m_0=2$. The network $N=\{1, 2, 3, 4\}$ has $\nu=12$ m.e.c.t. and Lemmas 4.3—4.5 can certainly be applied directly to the model to calculate $\pi_k(w)$ and $R_k(w)$ for $k=1, 2, \cdots$. However, since the model is constructed by the series connection of the two subnetworks N^1 and N^2, each of which is analyzed just as in the previous example. Therefore, the reliability $R(w)$ of the network N can be obtained as $R^{(1)}(w)R^{(2)}(w)$, and it is approximated by $R_k^{(1)}(w)R_k^{(2)}(w)$, where

$R^{(i)}(w)=$the reliability of subnetwork N^i ($i=1, 2$),

$R_k^{(i)}(w)=$the kth approximation for $R^{(i)}(w)$, given by Theorem 5.1 with respect to N^i ($i=1, 2$; $k=1, 2, \cdots$).

The procedure just mentioned is much simpler than the direct evaluation in use of 12 m.e.c.t. of the network.

Remark 6.1. Example 6.2 can trivially be extended to more general networks that are composed of the series connection of a number of subnetworks. That is, the evaluation of the reliability of the network can be greately simplified if it is "in series" decomposed to subnetworks, N^i ($i=1, 2, \cdots, d$)[3]. In this case the kth approximation of

[3] Although one may conceive other decompositions than the series one of the network, they do not necessarily facilitate the reliability evaluation of the network, since any subnetwork with two or more components behaves no longer as a renewal process. Therefore, we investigate no more decompositions here.

the network reliability $R(w)$ is given by

$$\prod_{i=1}^{d} R_k^{(i)}(w) ,$$

where $R_k^{(i)}(w)$ is the kth approximation of the reliability, $R^{(i)}(w)$, of the subnetwork N^i, obtained from Theorem 5.1. Remark also that Assumption 4° should be satisfied for each subnetwork N^i $(i=1, 2, \cdots, d)$.

7. Remarks on Some Extensions

In this section we shall sketch some extensions of the model and method described in the previous sections.

(1) First, we extend the model so as to include rank-2 and/or rank-3 components, as stated in Section 1. Here we denote the rank-1 components by $i=1, \cdots, n_1$; rank-2 components by $i=n_1+1, \cdots, n_1+n_2$; and rank-3 components by $i=n_1+n_2+1, \cdots, n_1+n_2+n_3$; and let N_1, N_2, N_3, N be

$$N_1=\{1, \cdots, n_1\}, \quad N_2=\{n_1+1, \cdots, n_1+n_2\}, \quad N_3=\{n_1+n_2+1, \cdots, n_1+n_2+n_3\}, \quad N=N_1+N_2+N_3 . \tag{7.1}$$

If $N_2+N_3\neq\phi$, then, we should modify the condition (2.1) in Assumption 4° as follows:

$$m \geqslant m_0 = \max_{j=1,\cdots,r} \{n_1-|A_jN_1|+n_2\} , \tag{7.2}$$

where $\{A_j\}$ is defined on $N=N_1+N_2+N_3$. Let us denote the modified assumption as Assumption 4*. Now we put two additional Assumptions as follows:

6° Each component $i\in N_2$ is repairable, and has a repair station and a cold stand-by component of its own. Component i and its stand-by one have the common life-time distribution $F_i(t)$ and the common repair-time distribution $G_i(t)$, both of which are continuous and $F_i(0)=G_i(0)=0$ for each $i\in N_2$.

7° Each component $i\in N_3$ is not repairable but has $b_i\geqslant 0$ additional cold stand-by components of its own, which are used successively and sequentially, one by one, in the place of the failed one as far as the stand-by are provided. The life-time distributions are $F_{ij}(t)$ $(j=0, 1, \cdots, b_i)$, where $j=0$ stands for the component i that is initially installed in the network and $j=1, 2, \cdots, b_i$ denotes each of the b_i cold stand-by components, and they are assumed to be continuous with $F_{ij}(0)=0$ $(j=0, 1, \cdots, b_i)$ for each $i \in N_3$.

Assumption 6° states that each component i with its stand-by forms a two-component stand-by redundant subnetwork and it has a repair station of its own for each $i \in N_2$. Let us denote this subnetwork simply as i and say that the subnetwork i fails if and only if its operative component fails before the other inoperative component recovers, for each $i \in N_2$.

Remark here that as soon as the maintenance of the inoperative component starts, the other component bigins to operate. In order to analyze the procedure let us define the notations:

$J_i^{(0)}(t)$ = the unit distribution whose probability is concentrated at the origin $t=0$,

$J_i^{(k)}(t)$ = the kth convolution of $F_i(t)G_i(t)$,

$$J_i(t)=\sum_{k=0}^{\infty} J_i^{(k)}(t)$$

for each $i \in N_2$. Then, using the same notations as in the case of the activities of component i, one obtains the formulae on the activities of subnetwork i as follows:

$$P\{i \in I(s;\ ds)\}=\int_0^s d_uF_i(u)\int_u^s d_vJ_i(v-u)f_i(s-v)\bar{G}_i(s-v)\cdot ds\,, \tag{7.3}$$

$$P\{i \in C(s)\}=\int_0^s d_uF_i(u)\int_u^s d_vJ_i(v-u)\bar{F}_i(s-v)+\bar{F}_i(s)\,, \tag{7.4}$$

$$P\{i \in D(s)\}=\int_0^s d_uF_i(u)\int_u^s d_vJ_i(v-u)F_i(s-v)\bar{G}_i(s-v)\,, \tag{7.5}$$

which correspond to (4.8), (4.9), (4.10), respectively.

Let $i \in I(s;\ ds;\ v)$ denote the event that the failure of subnetwork i occurs in the interval $(s, s+ds)$ and its inoperative component at time s has been in repair since the repair of the component has started at a time in $(v, v+dv)$, where $0 \leqslant v \leqslant s$. Similarly, let $i \in C'(s; v)$ [$D'(s; v)$] denote the event that subnetwork i is operative [inoperative] at time s and its component in use [repair] at the instant has been in use [repair] since the usage [repair] of the component has started at a time in $(v, v+dv)$. Then, one can obtain the following formulae:

(7.6a) $$P\{i \in I(t; dt) | i \in I(s; ds; v)\} = \int_s^t d_x G_i(x-s|s-v) \int_x^t d_y J_i(y-x) f_i(t-y) \bar{G}_i(t-y) \cdot dt,$$

(7.6b) $$P\{i \in C(t) | i \in I(s; ds; v)\} = \int_s^t d_x G_i(x-s|s-v) \int_x^t d_y J_i(y-x) \bar{F}_i(t-y),$$

(7.6c) $$P\{i \in D(t) | i \in I(s; ds; v)\} = \int_s^t d_x G_i(x-s|s-v) \int_x^t d_y J_i(y-x) F_i(t-y) \bar{G}_i(t-y) + \bar{G}_i(t-s|s-v),$$

(7.7a) $$P\{i \in I(t; dt) | i \in C'(s; v)\} = \int_s^t d_x F_i(x-s|s-v) \int_x^t d_y J_i(y-x) f_i(t-y) \bar{G}_i(t-y) \cdot dt,$$

(7.7b) $$P\{i \in C(t) | i \in C'(s; v)\} = \int_s^t d_x F_i(x-s|s-v) \int_x^t d_y J_i(y-x) \bar{F}_i(t-y) + \bar{F}_i(t-s|s-v),$$

(7.7c) $$P\{i \in D(t) | i \in C'(s; v)\} = \int_s^t d_x F_i(x-s|s-v) \int_x^t d_y J_i(y-x) F_i(t-y) \bar{G}_i(t-y),$$

(7.8a) $$P\{i \in I(t; dt) | i \in D'(s; v)\} = \int_s^t d_x G_i(x-s|s-v) \int_x^t d_y J_i(y-x) f_i(t-y) \bar{G}_i(t-y) \cdot dt,$$

(7.8b) $$P\{i \in C(t) | i \in D'(s; v)\} = \int_s^t d_x G_i(x-s|s-v) \int_x^t d_y J_i(y-x) \bar{F}_i(t-y),$$

(7.8c) $$P\{i \in D(t) | i \in D'(s; v)\}$$

$$=\int_{s}^{t} d_x G_i(x-s|s-v)\int_{x}^{t} d_y J_i(y-x)F_i(t-y)\bar{G}_i(t-y)$$
$$+\bar{G}_i(t-s|s-v)\,,$$

for each $i \in N_2$ and for every $0 \leqslant v \leqslant s \leqslant t$. These formulae correspond to (4.26a)—(4.28c). They certainly enable us to develop the same theory for the extended network N as for the primal one except the minor changes of the notations and restrictions on the number of repair stations shown in Assumption 4*.

For component $i \in N_3$, we only remark that under Assumption 7° component i with the b_i stand-by ones has the total life-time distribution function

(7.9) $\qquad F_i(t)=F_{i0}*F_{i1}*\cdots*F_{ib_i}(t)$

and the hypothetical repair-time distribution $G_i(t)$ such that

(7.10) $\qquad G_i(t)=0$ for every $t \leqslant T$, where T is a sufficiently large positive const.

By the conventional definitions of $F_i(t)$ and $G_i(t)$ given in (7.9) and (7.10) the theory in Sections 4 and 5 holds for the network with rank-3 components.

(**2**) Second, we consider the so-called intermittently used networks. Conventionally, let us assume that the lengths of use and non-use periods are independently distributed random variables with distribution functions, $G_{n+1}(t)$ and $F_{n+1}(t)$, respectively. Let us assume also that we can neglect the failure of the network during the non-use periods. Note that under the assumptions the sequence of use and non-use periods forms an alternating renewal process. Therefore, we can suppose the process to be that of the hypothetical rank-1 component denoted here by $n+1$, where use and non-use periods correspond to non-operative and operative periods, respectively, of the hypothetical component. Now, let the network N' be the paralleled connection of the original network N and the rank-1 component $n+1$. Then, the reliability of N' is nothing but that of the intermittently used network N.

Thus, we can easily get the reliabilities of intermittently used networks by the simple modification of the networks.

(3) Third, we remark on the generalization of the initial condition in Assumption 5°. As the assumption is not essential, we can obviously modify the results of the paper so as to fit other initial condition of the components.

Moreover, if $N_3=\phi$ and $m=n$, then, enumerating all of the mutually exclusive states of the system failures, we may evaluate the point availability of the network, whose asymptotic value independent of the initial condition may sometimes be interesting. Furthermore, we can obviously solve the dual problem of evaluating the probability that the failed network recovers its function up to time w within the scheme of the present paper.

(4) Finally, we may extend, if necessary, the model and method of the paper to the case where each component of the network has more states of operation and inoperation, introducing, for instance, an aggregate of Markov renewal processes.

In any way it may perhaps be in case of the highly reliable networks that need the delicate and rigorous analysis of the reliability. For such networks our evaluations $R_1(w)$ and $R_2(w)$ may fortunately be the excellent approximations of the reliability $R(w)$, as suggested by Theorems 5.3 and 5.4, and as exemplified by Example 6.1.

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Appendix

Proofs of the inequalities (5.48)—(5.55).

First, let us note that under the assumptions of Theorem 5.4 we obviously have

$$\text{(A.1)} \qquad \int_y^t f_i(z-y)\bar{G}_i(t-z)dz \leqslant \eta_0(w),$$

$$\text{(A.2)} \qquad \int_y^w f_i(t-z)\bar{G}_j(t-s|s-u)dt \leqslant \int_y^w f_i(t-z)\bar{G}_j(t-z|s-u)dt \leqslant \eta_0(w),$$

for every $0\leqslant u\leqslant s\leqslant y\leqslant z\leqslant t\leqslant w$. From (4.26c) and (A.1) we get

$$\begin{aligned}\text{(A.3)} \qquad & P\{i\in D(t)|i\in I(s;\ ds)\} \\ &= \int_s^t d_x H_i(x-s)\bar{G}_i(t-x) \\ &= \bar{G}_i(t-s)+\int_s^t d_x H_i(x-s)\int_x^t d_y G_i(y-x)\int_y^t f_i(z-y)\bar{G}_i(t-z)dz \\ &\leqslant \bar{G}_i(t-s)+H_0(w)\eta_0(w) \qquad (0\leqslant s\leqslant t\leqslant w).\end{aligned}$$

From (4.27c) and (A.3) we get

$$\begin{aligned}\text{(A.4)} \qquad & P\{i\in D(t)|i\in C(s;\ u)\} \\ &= \int_s^t d_x F_i(x-s|s-u)\int_s^t d_z H_i(z-x)\bar{G}_i(t-z) \\ &\leqslant \int_s^t f_i(x-s|s-u)\{\bar{G}_i(t-x)+H_0(w)\eta_0(w)\}dx \\ &\leqslant (1+H_0(w))\eta_0(w) \qquad (0\leqslant u\leqslant s\leqslant t\leqslant w).\end{aligned}$$

From (4.28c) and (A.3) we get

$$\begin{aligned}\text{(A.5)} \qquad & P\{i\in D(t)|i\in D(s;\ u)\} \\ &\leqslant \bar{G}_i(t-s|s-u)+\int_s^t d_x G_i(x-s|s-u)\int_x^t d_y F_i(y-x) \\ &\quad \times\{\bar{G}_i(t-y)+H_0(w)\eta_0(w)\} \\ &\leqslant \bar{G}_i(t-s|s-u)+(1+H_0(w))\eta_0(w) \qquad (0\leqslant u\leqslant s\leqslant t\leqslant w).\end{aligned}$$

For simplicity of notation let us put $i=i_\beta$ and $j=j_\beta$, and let us prove (5.48)—(5.55) as follows.

Proof of (5.48). From (4.26a) and (A.4) we have

$$\text{(A.6)}\qquad \int_{t=0}^{w} P\{i\in I(t;\,dt)|i\in I(s;\,ds)\}P\{j\in D(t)|j\in C(s;\,v)\}$$
$$\leqslant\int_0^w d_xG_i(x-s)\int_0^w d_yH_i(y-x)\int_0^w f_i(t-y)dt(1+H_0(w))\eta_0(w)$$
$$\leqslant H_0(w)(1+H_0(w))\eta_0(w)\leqslant(1+H_0(w))^2\eta_0(w)\,.$$

Proof of (5.49). From (4.26a) and (A.5) we have

$$\text{(A.7)}\qquad \int_{t=0}^{w} P\{i\in I(t;\,dt)|i\in I(s;\,ds)\}P\{j\in D(t)|j\in D(s;\,v)\}$$
$$\leqslant\int_0^w d_xG_i(x-s)\int_0^w d_yH_i(y-x)\int_0^w f_i(t-y)\{\bar G_j(t-s|s-v)$$
$$+(1+H_0(w))\eta_0(w)\}dt$$
$$\leqslant H_0(w)(2+H_0(w))\eta_0(w)\leqslant(1+H_0(w))^2\eta_0(w)\,.$$

Proof of (5.50). From (4.27a) and (A.3) we have

$$\text{(A.8)}\qquad \int_{t=0}^{w} P\{i\in I(t;\,dt)|i\in C(s;\,u)\}P\{j\in D(t)|j\in I(s;\,ds)\}$$
$$\leqslant\int_0^w dt\Big\{f_i(t-s|s-u)+\int_0^w d_xF_i(x-s|s-u)\int_0^w d_yG_i(y-x)$$
$$\times\int_0^w d_zH_i(z-y)f_i(t-z)\Big\}\{\bar G_j(t-s)+H_0(w)\eta_0(w)\}$$
$$\leqslant(1+H_0(w))\eta_0(w)+H_0(w)(1+H_0(w))\eta_0(w)$$
$$=(1+H_0(w))^2\eta_0(w)\,.$$

Proof of (5.51). From (4.27a) and (A.4) we have

$$\text{(A.9)}\qquad \int_{0}^{w} P\{i\in I(t;\,dt)|i\in C(s;\,u)\}P\{j\in D(t)|j\in C(s;\,v)\}$$
$$\leqslant\int_0^w dt\Big\{f_i(t-s|s-u)+\int_0^w d_xF_i(x-s|s-u)\int_0^w d_yG_i(y-x)$$
$$\times\int_0^w d_zH_i(z-y)f_i(t-z)\Big\}(1+H_0(w))\eta_0(w)$$
$$\leqslant(1+H_0(w))\eta_0(w)+H_0(w)(1+H_0(w))\eta_0(w)$$
$$=(1+H_0(w))^2\eta_0(w)\,.$$

Proof of (5.52). From (4.27a) and (A.5) we have

$$\text{(A.10)}\qquad \int_{t=0}^{w} P\{i \in I(t;\, dt) | i \in C(s;\, u)\} P\{j \in D(t) | j \in D(s;\, v)\}$$

$$\leqslant \int_0^w dt \Big\{ f_i(t-s|s-u) + \int_0^w d_x F_i(x-s|s-u) \int_0^w d_y G_i(y-x)$$

$$\times \int_0^w d_z H_i(z-y) f_i(t-z) \Big\} \{\bar{G}_j(t-s|s-v) + (1+H_0(w))\eta_0(w)\}$$

$$\leqslant (2+H_0(w))\eta_0(w) + H_0(w)(2+H_0(w))\eta_0(w)$$

$$\leqslant 2(1+H_0(w))^2 \eta_0(w)\,.$$

Proof of (5.53). From (4.28a) and (A.3) we have

$$\text{(A.11)}\qquad \int_{t=0}^{w} P\{i \in I(t;\, dt) | i \in D(s;\, u)\} P\{j \in D(t) | j \in I(s;\, ds)\}$$

$$\leqslant \int_0^w d_x G_i(x-s|s-u) \int_0^w d_z H_i(z-x) \int_0^w dt f_i(t-z)$$

$$\times \{\bar{G}_j(t-s) + H_0(w)\eta_0(w)\}$$

$$\leqslant H_0(w)(1+H_0(w))\eta_0(w) \leqslant (1+H_0(w))^2 \eta_0(w)\,.$$

Proof of (5.54). From (4.28a) and (A.4) we have

$$\text{(A.12)}\qquad \int_{t=0}^{w} P\{i \in I(t;\, dt) | i \in D(s;\, u)\} P\{j \in D(t) | j \in C(s;\, v)\}$$

$$\leqslant \int_0^w d_x G_i(x-s|s-u) \int_0^w d_z H_i(z-x) \int_0^w f_i(t-z) dt$$

$$\times (1+H_0(w))\eta_0(w)$$

$$\leqslant H_0(w)(1+H_0(w))\eta_0(w) \leqslant (1+H_0(w))^2 \eta_0(w)\,.$$

Proof of (5.55). From (4.28a) and (A.5) we have

$$\text{(A.13)}\qquad \int_{t=0}^{w} P\{i \in I(t;\, dt) | i \in D(s;\, u)\} P\{j \in D(t) | j \in D(s;\, v)\}$$

$$\leqslant \int_0^w d_x G_i(x-s|s-u) \int_0^w d_z H_i(z-x) \int_0^w f_i(t-z)$$

$$\times \{\bar{G}_j(t-s|s-v) + (1+H_0(w))\eta_0(w)\} dt$$

$$\leqslant H_0(w)(2+H_0(w))\eta_0(w) \leqslant (1+H_0(w))^2 \eta_0(w)\,.$$

(A.6)—(A.13) prove surely (5.48)—(5.55), respectively.