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RELIABILITY CONSIDERATION ON A REPAIRABLE MULTICOMPONENT SYSTEM WITH REDUNDANCY IN PARALLEL

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1. Introduction

Kulshrestha [1], [2] has investigated a multicomponent system with two series subsystems: S_1 consists of M identical components in parallel (all must fail for S_1 to fail—major breakdown), while S_2 contains N different components connected in series (the failure of any one causes S_2 to fail—minor breakdown). In the above papers, the repair of any component is possible only when the system stops operating, *i.e.*, only when either major or minor breakdown occurs.

Against his repair policy, we propose the followings:

(1) Whenever any of S_1-component fails, it is sent to the repair station at once and repaired in turn by a single server.

(2) In a major breakdown, as soon as the S_1-component being served is repaired, it is reentered and then the system operates again.

(3) In a minor breakdown, as soon as the failed component of S_2 is repaired the system operates again. If a S_2-component fails when a S_1-component is being repaired, (i) it is repaired without delay

interrupting the repair of S_1-component—preemptive resume and preemptive repeat discipline, (ii) it waits till the S_1-component completes its repair—head-of-the-line discipline.

In special situations, which the system-down inflicts some vast loss, it may be required that the system is in operable state as long as possible, and that the system returns to operable state as soon as possible when it fails. Then we will study in this paper the behavior of the system under the repair policies mentioned above.

2. Definitions and Basic Equations

The following assumptions are made;

(1) The failure time for M ($\geqq 2$) identical components in S_1 obeys a negative exponential distribution with failure rate λ_0, and its repair time follows such a general type distribution that

$$f_0(t)=\mu_0(t)\exp\left[-\int_0^t \mu_0(t)dt\right].$$

(2) The failure time for ith component in S_2 obeys a negative exponential distribution with failure rate λ_i, and its repair time follows such a general type distribution that

$$f_i(t)=\mu_i(t)\exp\left[-\int_0^t \mu_i(t)dt\right],$$

where $i=1, 2, \cdots, N$, and we define $\lambda\equiv\sum_{i=1}^{N}\lambda_i$ and $f(t)\equiv\frac{1}{\lambda}\sum_{i=1}^{N}\lambda_i f_i(t)$.

(3) The failures of all components are statistically independent.

(4) Each component after repair is considered to be new again and it has the same failure time distribution function as initial one.

Preemptive resume case

So as to formulate the preemptive resume case let us introduce the following notations:

$P_0(t)$: probability that, at time t the system is operating and all components in S_1 are operating.

$P_m(t, x)dx$: probability that, at time t (1) the system is operating, and (2) $m(1 \leqq m \leqq M-1)$ components in S_1 have failed and the elapsed repair time lies between x and $x+dx$.

$P_M(t, x)dx$: probability that, at time t the system is down due to failure of all component in S_1 and the elapsed repair time lies between x and $x+dx$.

$Q_0^i(t, x)dx$: probability that, at time t (1) the system is down due to failure of the ith component of S_2 and the elapsed repair time lies between x and $x+dx$, and (2) all components in S_1 are operating. $(i=1, 2, \cdots, N)$

$Q_m^i(t, x, y)dxdy$: probability that, at time t (1) the system is down due to failure of the ith component in S_2 and the elapsed repair time lies between y and $y+dy$, and (2) m components of S_1 had failed at time $t-y$ and the elapsed time for a component in S_1 under repair lies between x and $x+dx$. $(m=1, 2, \cdots, M-1, i=1, 2, \cdots, N)$

By connecting the above state probabilities at time $t+h$ with those at time t and taking limits as $h \to 0$, we can easily set up the following basic equations for the preemptive resume case;

(1)
$$\left(\frac{d}{dt}+M\lambda_0+\lambda\right)P_0(t)=\int_0^t \mu_0(x)P_1(t, x)dx + \sum_{i=1}^{N}\int_0^t \mu_i(x)Q_0^i(t, x)dx,$$

(2)
$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+(M-m)\lambda_0+\lambda+\mu_0(x)\right)P_m(t, x) =(1-\delta_{m,1})(M-m-1)\lambda_0 P_{m-1}(t, x) + \sum_{i=1}^{N}\int_0^{t-x} \mu_i(y)Q_m^i(t, x, y)dy, \quad (1 \leqq m \leqq M-1)$$

(3)
$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_0(x)\right)P_M(t, x)=\lambda_0 P_{M-1}(t, x),$$

(4)
$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\mu_i(y)\right)Q_m^i(t, x, y)=0, \quad (1 \leqq m \leqq M-1, \ 1 \leqq i \leqq N)$$

$$(5)\qquad \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_i(x)\right)Q_0^i(t, x)=0, \quad (1\leqq i\leqq N)$$

where $\delta_{i,j}$ is the Kronecker's delta.

The above equations are to be solved subject to the following boundary conditions and initial condition;

$$(6)\qquad P_m(t, 0)=\delta_{m,1}M\lambda_0 P_0(t)+\int_0^t \mu_0(x)P_{m+1}(t, x)dx, \quad (1\leqq m\leqq M-1)$$

$$(7)\qquad P_M(t, 0)=0,$$

$$(8)\qquad Q_0^i(t, 0)=\lambda_i P_0(t), \quad (1\leqq i\leqq N)$$

$$(9)\qquad Q_m^i(t, x, 0)=\lambda_i P_m(t, x), \quad (1\leqq m\leqq M-1,\ 1\leqq i\leqq N)$$

$$(10)\qquad P_0(0)=1.$$

3. Solutions of the Problem

Let the Laplace transform of function $P(t)$ be denoted by $\bar{P}(s)$.

In order to solve the equations described in the previous section, applying the Laplace transform to the set of equations (1)—(9) with respect to t under the initial condition (10), and introducing the following relation;

$$(11)\qquad A_j(s, x)=\sum_{m=j}^{M-1}\binom{m}{j}\bar{P}_{M-m}(s, x), \quad (1\leqq j\leqq M-1)$$

it can be shown that the following simultaneous equations hold

$$(12)\qquad [s+M\lambda_0+\lambda(1-\bar{f}(s))]\bar{P}_0(s)=1+\bar{f}_0(s, M-1)A_{M-1}(s, 0),$$

$$(13)\qquad \delta_{m,1}M\lambda_0\bar{P}_0(s)=\sum_{j=M-m-1}^{M-1}(-1)^{j+m-M}\left[\binom{j}{M-m}+\binom{j}{M-m-1}\times\bar{f}_0(s, j)\right]A_j(s, 0), \quad (1\leqq m\leqq M-2)$$

$$(14)\qquad \delta_{M,2}M\lambda_0\bar{P}_0(s)=\sum_{j=1}^{M-1}(-1)^{j-1}j\left[1-\lambda_0\frac{\bar{f}_0(s)-\bar{f}_0(s, j)}{j\lambda_0+\lambda(1-\bar{f}(s))}\right]A_j(s, 0),$$

where $\bar{f}_0(s, j)\equiv\bar{f}_0(s+j\lambda_0+\lambda(1-\bar{f}(s)))$ and $\binom{j}{i}=0$ for $j<i$.

Solving the above equations (12), (13) and (14), we obtain $\bar{P}_0(s)$ and $A_j(s, 0)$ $(j=1, 2, \cdots, M-1)$, and we can also calculate $\bar{P}_m(s, x)$ $(m=1, 2, \cdots, M)$, $\tilde{Q}_0^i(s, x)$ and $\tilde{Q}_m^i(s, x, y)$ $(m=1, 2, \cdots, M-1,\ i=1, 2, \cdots, N)$ using them.

Let

$P_m(t)$: probability that, at time t the system is operating and just m $(m=1, 2, \cdots, M-1)$ components in S_1 have failed,

$$=\int_0^t P_m(t, x)dx,$$

$P(t)$: Probability that, at time t the system is operating irrespective of the number of failed components in S_1,

$$=\sum_{m=0}^{M-1} P_m(t).$$

Then we have

$$\bar{P}_m(s)=\sum_{j=M-m}^{M-1}(-1)^{j-M+m}\binom{j}{M-m}\frac{1-\bar{f}_0(s, j)}{s+j\lambda_0+\lambda(1-\bar{f}(s))}A_j(s, 0), \quad (1\leqq m\leqq M-1) \tag{15}$$

$$\bar{P}(s)=\frac{1+\bar{f}_0(s, M-1)A_{M-1}(s, 0)}{s+M\lambda_0+\lambda(1-\bar{f}(s))} +\sum_{j=1}^{M-1}(-1)^{j-1}\frac{1-\bar{f}_0(s, j)}{s+j\lambda_0+\lambda(1-\bar{f}(s))}A_j(s, 0). \tag{16}$$

If the steady-state probability $P_I(\infty)=\lim_{t\to\infty}P(t)$ exists, applying the well-known result in the Laplace transform, we can obtain the long-run availability for the preemptive resume case as follows;

$$\begin{aligned} P_I(\infty)&=\lim_{s\to 0} s\bar{P}(s) \\ &=\frac{1}{M\lambda_0}\bar{f}_0((M-1)\lambda_0)\tilde{A}_{M-1}(0, 0) \\ &\quad+\sum_{j=1}^{M-1}(-1)^{j-1}\frac{1}{j\lambda_0}(1-\bar{f}_0(j\lambda_0))\tilde{A}_j(0, 0), \end{aligned} \tag{17}$$

where $\tilde{A}_j(0, 0)=\lim_{s\to 0} sA_j(s, 0)$.

By the similar way we can derive the long-run availabilities for the other cases;

Preemptive repeat case

$$P_{II}(\infty)=\frac{1}{M\lambda_0}\bar{f}_0(\lambda+(M-1)\lambda_0)\tilde{A}_{M-1}(0,0)+\sum_{j=1}^{M-1}(-1)^{j-1}\frac{1-\bar{f}_0(\lambda+j\lambda_0)}{\lambda+j\lambda_0}\tilde{A}_j(0,0). \tag{18}$$

Head-of-the-line case

$$P_{III}(\infty)=\frac{1}{M\lambda_0}\{\bar{f}_0(\lambda+(M-1)\lambda_0)+\lambda g(0,M-1)\}\tilde{A}_{M-1}(0,0)+\sum_{j=1}^{M-1}(-1)^{j-1}\frac{1-\bar{f}_0(\lambda+j\lambda_0)}{\lambda+j\lambda_0}\tilde{A}_j(0,0), \tag{19}$$

where $g(0, M-1)\equiv\int_0^\infty\left\{\int_0^\infty\mu_0(x+y)\exp\left[-\int_0^y\mu_0(x+y)dy\right]dy\right\}\times\exp\left[-(\lambda+(M-1)\lambda_0)x-\int_0^x\mu_0(x)dx\right]dx.$

$A_j(s, 0)$ in (18) and (19) are obtained from (30)—(32) and (33)—(35) in Appendix, respectively (see Appendix).

4. Example

Let $M=3$, then in the preemptive resume case we have the following equations from (12), (13) and (14);

$$\begin{pmatrix} s+3\lambda_0+\lambda(1-\bar{f}(s)) & 0 & -\bar{f}_0(s,2) \\ -3\lambda_0 & -\bar{f}_0(s,1) & 1+2\bar{f}_0(s,2) \\ 0 & 1-\lambda_0\dfrac{\bar{f}_0(s)-\bar{f}_0(s,1)}{\lambda_0+\lambda(1-\bar{f}(s))} & 2\left(\lambda_0\dfrac{\bar{f}_0(s)-\bar{f}_0(s,2)}{2\lambda_0+\lambda(1-\bar{f}(s))}-1\right) \end{pmatrix}\times\begin{pmatrix}\bar{P}_0(s)\\ A_1(s,0)\\ A_2(s,0)\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}. \tag{20}$$

Here let $D(s)$ denotes the determinant of coefficient matrix of (20), then we have

(21) $$A_1(s, 0)=\frac{6\lambda_0}{D(s)}\left\{\lambda_0\frac{\bar{f}_0(s)-\bar{f}_0(s, 2)}{2\lambda_0+\lambda(1-\bar{f}(s))}-1\right\},$$

(22) $$A_2(s, 0)=\frac{3\lambda_0}{D(s)}\left\{\lambda_0\frac{\bar{f}_0(s)-\bar{f}_0(s, 1)}{\lambda_0+\lambda(1-\bar{f}(s))}-1\right\}.$$

Especially, assuming that all the repair time distributions are exponential;

$$f_i(x)=\mu_i \exp(-\mu_i x) \quad \text{for } i=0, 1, 2, \cdots, N,$$

We have

(23) $$\tilde{A}_1(0, 0)=6\lambda_0(1+\rho_0)^2/K_1,$$

(24) $$\tilde{A}_2(0, 0)=3\lambda_0(1+2\rho_0)/K_1,$$

where $\rho_0=\lambda_0/\mu_0$, $\rho_i=\lambda_i/\mu_i$ for $i=1, 2, \cdots, N$, and

$$K_1=(6\rho_0^2+3\rho_0+1)\left(1+\sum_{i=1}^{N}\rho_i\right)+6\rho_0^3.$$

Since the preemptive repeat case becomes the same with the preemptive resume case under the above assumptions, therefore we get from (17)

(25) $$P_{\text{I}}(\infty)=P_{\text{II}}(\infty)=(6\rho_0^2+3\rho_0+1)/K_1.$$

Also, in a similar way we have the following for the head-of-the-line case,

(26) $$P_{\text{III}}(\infty)=\{(\rho^*+1)(\rho^*+3\rho_0+1)+6\rho_0^2\}/K_2,$$

where $\rho^*=\lambda/\mu_0$ and $K_2=\{(\rho^*+1)(\rho^*+3\rho_0+1)+6\rho_0^2\}$

$$\times\left(1+\sum_{i=1}^{N}\rho_i\right)+3\rho_0\{\rho^*(\rho^*+2\rho_0+1)+2\rho_0^2\}.$$

While, according to Kulshrestha [2], in which the repair policy is adopted as (1) in major breakdown, all the failed components in S_1 are repaired and (2) in minor breakdown, only the failed component in S_2 is repaired, his result for the same system have been given as follows:

(27) $$P_K(\infty)=1\Big/\left(1+\sum_{i=1}^{N}\rho_i+\frac{6}{11}\lambda_0 K_0\right),$$

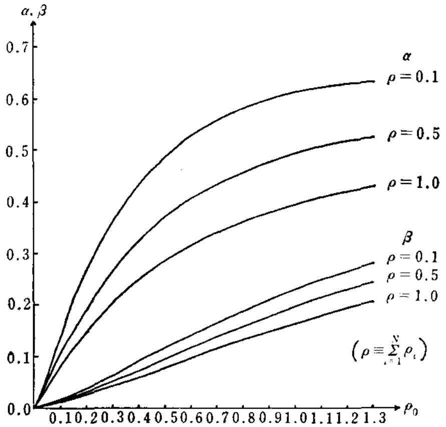

Fig. 1. Degree for advantage of $P_{\mathrm{I}}(\infty)$, $P_{\mathrm{III}}(\infty)$ over $P_K(\infty)$.

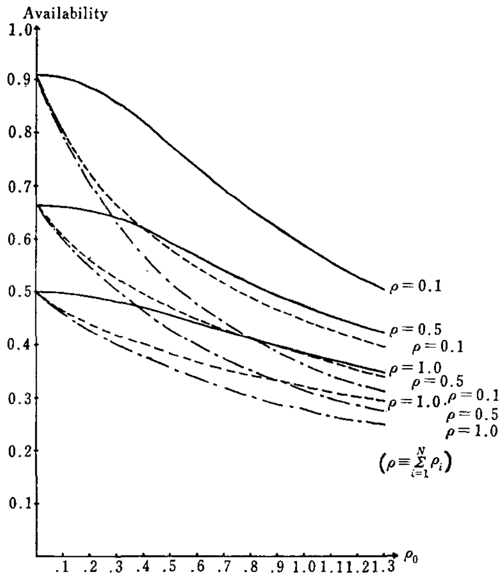

Fig. 2. Comparison among $P_{\mathrm{I}}(\infty)$, $P_{\mathrm{III}}(\infty)$ and $P_K(\infty)$.
——: $P_{\mathrm{I}}(\infty)$, ------: $P_{\mathrm{III}}(\infty)$ (when $\rho^*=1$),
–·–·: $P_K(\infty)$.

where K_0 denotes the mean repair time for major repair.

Hence, assuming $K_0=3/\mu_0$ by means of his policy for major repair, we can compare our results (25) and (26) with Kulshrestha's (27), that is,

$$\frac{P_I(\infty)}{P_K(\infty)}=1+6\rho_0(7\rho_0^2+9\rho_0+3)/11K_1, \tag{28}$$

and

$$\frac{P_{\mathrm{III}}(\infty)}{P_K(\infty)}=1+3\rho_0\{14\rho_0^2+2(9-2\rho^*)\rho_0-(5\rho^*-6)(\rho^*+1)\}/11K_2. \tag{29}$$

Let us denote by α and β the second term of the RHS of (28) and (29), respectively, then α is always nonnegative for any ρ_0 and ρ_i, and β is also nonnegative if $\rho^*<\bar{\rho}^*$ or as long as $\rho_0>\max\left[0, \frac{-(9-2\rho^*)+\sqrt{74\rho^{*2}-50\rho^*-3}}{14}\right]$ if $\rho^*\geqq\bar{\rho}^*$, where $\bar{\rho}^*\equiv(25+\sqrt{847})/74$. Therefore α and β show the degree for advantage of our repair policies over Kulshrestha's on the operational behavior of this system.

The values of α and β (when $\rho^*=1$), and the availabilities $P_{\mathrm{I}}(\infty)$, $P_{\mathrm{III}}(\infty)$ and $P_K(\infty)$, computed for several values of ρ_0 and $\sum_{i=1}^{N}\rho_i$ are shown in Figure 1 and Figure 2, respectively.

Appendix

The simultaneous equations correspond to (12)—(14) in section 3 for preemptive repeat and head-of-the-line cases are as follows;

Preemptive repeat case

$$[s+M\lambda_0+\lambda(1-\bar{f}(s)]\bar{P}_0(s)=1+\bar{f}_0(s+\lambda+(M-1)\lambda_0)A_{M-1}(s, 0), \tag{30}$$

$$\delta_{m,1}M\lambda_0\bar{P}_0(s)=\sum_{j=M-m-1}^{M-1}(-1)^{j+m-M}\left[\binom{j}{M-m}\times\left(1-\frac{\lambda\bar{f}(s)(1-\bar{f}_0(s+\lambda+\lambda_0))}{s+\lambda+j\lambda_0}\right)+\binom{j}{M-m-1}\times\bar{f}_0(s+\lambda+j\lambda_0)\right]A_j(s, 0), \quad (1\leqq m\leqq M-2) \tag{31}$$

(32)
$$\delta_{M,2} M\lambda_0 \bar{P}_0(s) = \sum_{j=1}^{M-1} (-1)^{j-1} j \left[1 - \frac{\lambda_0(\bar{f}_0(s) - \bar{f}_0(s+\lambda+\lambda_0))}{\lambda + j\lambda_0} - \frac{\lambda \bar{f}(s)(1-\bar{f}_0(s+\lambda+j\lambda_0))}{s+\lambda+j\lambda_0} \right] A_j(s, 0),$$

where $\binom{j}{i} = 0$ for $i < 0$.

Head-of-the-line case

(33)
$$[s + M\lambda_0 + \lambda(1-\bar{f}(s))]\bar{P}_0(s) = 1 + [\bar{f}_0(s+\lambda+(M-1)\lambda_0) + \lambda \bar{f}(s) g(s, M-1)] A_{M-1}(s, 0),$$

(34)
$$\delta_{m,1} M\lambda_0 \bar{P}_0(s) = \sum_{j=M-m-1}^{M-1} (-1)^{j+m-M} \left[\binom{j}{M-m} + \binom{j}{M-m-1} \right] \times (\bar{f}_0(s+\lambda+j\lambda_0) + \lambda \bar{f}(s) g(s, j)) A_j(s, 0), \quad (1 \leqq m \leqq M-2)$$

(35)
$$\delta_{M,2} M\lambda_0 \bar{P}_0(s) = \sum_{j=1}^{M-1} (-1)^{j-1} j \left[1 - \frac{\lambda_0(\bar{f}_0(s) - \bar{f}_0(s+\lambda+j\lambda_0))}{\lambda + j\lambda_0} \right] \times A_j(s, 0),$$

where
$$g(s, j) \equiv \int_0^\infty \left\{ \int_0^\infty \mu_0(x+y) \exp\left[-sy - \int_0^y \mu_0(x+y) dy \right] dy \right\} \times \exp\left[-(s+\lambda+j\lambda_0)x - \int_0^x \mu_0(x) dx \right] dx. \quad (1 \leqq j \leqq M-1)$$

References

[1] Kulshrestha, D.K., "Reliability of a Parallel Redundant Complex system," *Opns. Res.*, **16**, No. 1 (1968).

[2] Kulshrestha, D.K., "Reliability of a Repairable Multicomponent System with Redundancy in Parallel," *IEEE Trans. on Reliability.*, **19**, No. 2 (1970).