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MINIMUM CONCAVE COST SERIES PRODUCTION SYSTEMS WITH DETERMINISTIC DEMANDS—A BACKLOGGING CASE*

HIROSHI KONNO**

Mathematics Research Center,
The University of Wisconsin-Madison
Madison, Wisconsin 53706

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Abstract

In this paper, we will consider a concave-cost series production and inventory system with deterministic demands in which

(i) backlogging at the final facility and/or

(ii) demands at an intermediate facility

is allowed. A dynamic programming algorithm for (i) will be developed, which is a direct extension of the results obtained by W. Zangwill [3], [4] for a similar system but without backlogging or intermediate demands. Our algorithm requires $0(Nn^4)$ additions and comparisons where N is the number of facilities and n is the number of periods. Also, an analogous algorithm for (ii) is suggested.

1. Introduction and Summary

Let us consider the general series system depicted in Fig. 1 where the market demands are known in advance. Let r_i^k, x_i^k, y_i^k, w_i^k be

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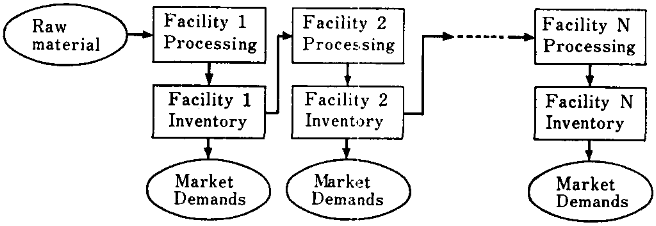

Fig. 1.

respectively the amount of market demand, production, stock on hand and backlogged demand in period i at facility k. We shall assume all these quantities to be non-negative. Let $c_i^k(x_i^k)$, $h_i^k(y_i^k)$, $p_i^k(w_i^k)$ be respectively the costs associated with producing x_i^k, holding y_i^k and backlogging w_i^k in period i at facility k, where $c_i^k(\cdot)$, $h_i^k(\cdot)$, $p_i^k(\cdot)$ are assumed to be concave, non-decreasing functions on the non-negative real line.

W. Zangwill [4] devised a very efficient dynamic programming algorithm for obtaining an optimal production schedule for a system in which there are no market demands for intermediate facilities 1 through $N-1$ and backlogging is not allowed at any facility. It requires $0(Nn^4)$ additions and comparisons where n is the number of periods. Later, S. Love [1] analyzed the same system with additional conditions on the cost structure and obtained a still more efficient algorithm which requires only $0(Nn^3)$ additions and comparisons.

In this paper we will extend these algorithms to a slightly more general system, *i.e.*, a series system with: (i) backlogging at facility N (an efficient algorithm for this problem has previously been obtained only when $N=1$, see [4]) and/or (ii) demands at some intermediate facility in addition to those at facility N. These two extensions enable us to apply the algorithm developed in [4] to more general production systems such as those found in steel mills and oil refineries, *etc.*

2. Series System with Backlogging at the Final Facility

In this section, we develop an algorithm for the system described in Fig. 1 in which $w_i^k = r_i^k = 0$, $i=1, \cdots, n$; $k=1, \cdots, N-1$. This represents a situation in which the only outside demands are for finished products (products of the final facility) and backlogging of demands for the finished products is allowed.

Our problem is now formulated as follows:

$$(1) \qquad \text{minimize} \quad \sum_{i=1}^{n} \left[\sum_{k=1}^{N} \{c_i^k(x_i^k) + h_i^k(y_i^k)\} + p_i^N(w_i^N) \right]$$

s.t.

$$(2) \qquad \begin{cases} y_i^k = x_i^k + y_{i-1}^k - x_i^{k+1}, \quad i=1, \cdots, n;\ k=1, \cdots, N-1 \\ y_i^N = x_i^N + y_{i-1}^N + w_i^N - w_{i-1}^N - r_i^N, \quad i=1, \cdots, n \\ x_i^k \geq 0,\ y_i^k \geq 0, \\ \qquad i=1, \cdots, n;\ k=1, \cdots, N;\ y_0^k = 0,\ k=1, \cdots, N; \\ w_i^N \geq 0,\ (r_i^N \geq 0), \quad i=1, \cdots, n;\ w_n^N = 0\ . \end{cases}$$

Each term appearing in the objective function is assumed to be a concave, non-decreasing function on the non-negative real line. Thus, its optimal solution must satisfy $y_n^k = 0$, $k=1, \cdots, N$. Zangwill [4] observed that this is a minimal cost flow problem on the rectangular network described in Fig. 2 where node (i, k) corresponds to facility k at period i. Since the cost associated with each arc is concave, a minimal cost flow must exist among extreme flows (*i.e.*, flows without cycles). It is not difficult to show that in an extreme flow on this network, every node can have at most one arc with positive input. (For details, see [2] or [3]).

Based upon this observation, we can prove the following theorem which is crucial to the development of our algorithm.

Theorem 1. If node $(i, k) \neq (0, 0)$ has S units of input in an extreme flow, then S must have the following representation

$$S = \sum_{l=\alpha}^{\beta} r_l^N$$

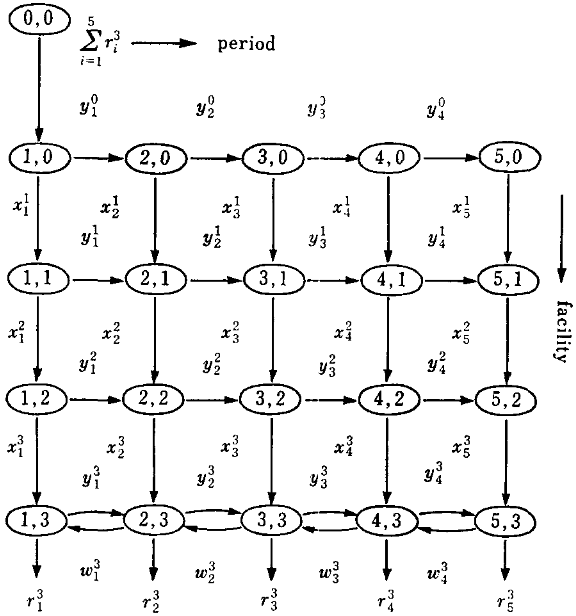

Fig. 2. Network representation of equation (2) when $N=3$, $n=5$.

for some $1 \leq \alpha \leq \beta \leq n$.

Proof.

(i) S must have the form $S=\sum_{l \in K} r_l^N$ where $K \subset \{1, 2, \cdots, n\}$ because if S cannot be expressed in this form, then some node must have more than two positive inputs to combine the split lot into a complete lot to satisfy the market demand at some node corresponding to facility N.

(ii) If S satisfies demands at nodes (α, N) and (β, N), $\alpha+1<\beta$, but not the demand at some intermediate node $\alpha<\gamma<\beta$, then two flows must cross somewhere—a violation of the properties of an extreme flow. (i) and (ii) establish the assertion of the theorem.

Now we develop a dynamic programming recursion to obtain an optimal schedule. From now on, we will use the notation r_l instead of r_l^N, suppressing the superscript N.

Algorithm. (Series System with Backlogging)

Let $c_i^k(\alpha, \beta)$ and $h_i^k(\alpha, \beta)$ be respectively the cost associated with producing and stocking $\sum_{l=\alpha}^{\beta} r_l$ units at facility k in period i, and let $p_i^N(\alpha, \beta)$ be the cost of backlogging $\sum_{l=\alpha}^{\beta} r_l$ units at facility N in period i. Then

$$c_i^k(\alpha, \beta)=c_i^k\left(\sum_{l=\alpha}^{\beta} r_l\right) \qquad i=1, \cdots, n;\ k=1, \cdots, N,$$

$$h_i^k(\alpha, \beta)=h_i^k\left(\sum_{l=\alpha}^{\beta} r_l\right) \qquad i=1, \cdots, n;\ k=1, \cdots, N,$$

$$p_i^N(\alpha, \beta)=p_i^N\left(\sum_{l=\alpha}^{\beta} r_l\right) \qquad i=1, \cdots, n.$$

We assume these quantities are equal to zero when $\alpha>\beta$. Let $f_i^k(\alpha, \beta)$ be the minimal cost of shipping $\sum_{l=\alpha}^{\beta} r_l$ units from node (i, k) to destinations $(\alpha, N), (\alpha+1, N), \cdots, (\beta, N)$. By convention, we assume $f_i^k(\alpha, \beta)=0$ if $\alpha>\beta$ and $p_0^k(\cdot, \cdot)=h_0^k(\cdot, \cdot)=c_0^k(\cdot, \cdot)=0$ for all k. We then have the following recurrence relations by the property of an extreme flow.

(i) $k=N$.

$$f_i^N(\alpha, \beta)=\sum_{l=\alpha}^{i-1} p_l^N(\alpha, l)+\sum_{l=i+1}^{\beta} h_l^N(l, \beta),$$
$$1\leq\alpha\leq i\leq\beta\leq n;\ 1\leq i\leq n.$$

(ii) $k=N-1$.

$$f_i^{N-1}(\alpha,\beta)=\min_{\{\gamma: i\leqq\gamma\leqq\beta, \gamma=\alpha-1\}}[c_i^N(\alpha, \gamma)+f_i^N(\alpha, \gamma)+h_i^{N-1}(\gamma+1, \beta)$$
$$+f_{i+1}^{N-1}(\gamma+1, \beta)], \qquad \alpha\leq\beta;\ 1\leq\alpha\leq i;\ 1\leq i\leq n-1$$
$$f_i^{N-1}(\alpha, \beta)=h_i^{N-1}(\alpha, \beta)+f_{i+1}^{N-1}(\alpha, \beta) \qquad i<\alpha\leq\beta;\ 1\leq i\leq n-1,$$
$$f_n^{N-1}(\alpha, n)=c_n^N(\alpha, n)+f_n^N(\alpha, n), \qquad 1\leq\alpha\leq n.$$

(iii) $0\leq k\leq N-2$.

$$f_i^k(\alpha, \beta)=\min_{\alpha-1\leqq\gamma\leqq\beta}[c_i^{k+1}(\alpha, \gamma)+f_i^{k+1}(\alpha, \gamma)+h_i^k(\gamma+1, \beta)$$
$$+f_{i+1}^k(\gamma+1, \beta)], \qquad 1\leq\alpha\leq\beta\leq n;\ 1\leq i\leq n-1.$$
$$f_n^k(\alpha, n)=c_n^k(\alpha, n)+f_n^{k+1}(\alpha, n), \qquad 1\leq\alpha\leq n.$$

Obviously, $f_1^0(1, n)$ gives us the cost associated with an optimal production schedule. This algorithm requires $(N-1)n^4/2+0(Nn^3)$ additions (excluding function evaluation) and $(N-1)n^4/6+0(Nn^3)$ comparisons. Zangwill's algorithm [4] for the no-backlogging case requires $(N-1)n^4/8+0(Nn^3)$ additions and $(N-1)n^4/24+0(Nn^3)$ comparisons. Thus our algorithm requires approximately four times as much computation as that required by Zangwill's algorithm.

3. Series System with Market Demands at Intermediate Facility

In this section, we consider the series system with market demands for the products at facility k_0 in addition to the demands for the products at facility N. For simplicity, we do not allow backlogging, but the algorithm we suggest here can be combined with the

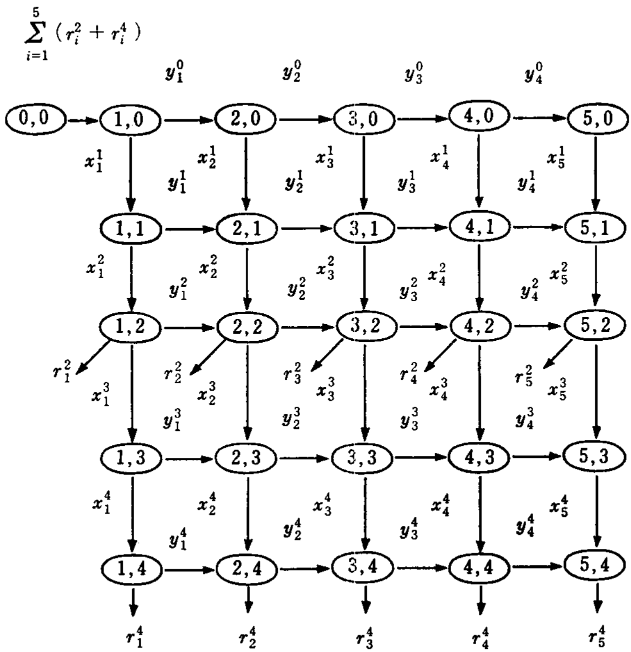

Fig. 3. Network representation of equation (4) when $N=4$, $n=5$, $k_0=2$.

algorithm of the preceding section to allow backlogging. Our problem is stated as follows:

$$(3)\qquad \text{minimize} \quad \sum_{i=1}^{n}\sum_{k=1}^{N}\{c_i^k(x_i^k)+h_i^k(y_i^k)\}$$

s.t.

$$(4)\qquad \begin{cases} y_i^k=x_i^k+y_{i-1}^k-x_i^{k+1}\,, & i=1,\cdots,n;\ k\neq k_0,\ N \\ y_i^{k_0}=x_i^{k_0}+y_{i-1}^{k_0}-x_i^{k_0+1}-r_i^{k_0}\,, & (r_i^{k_0}\geq 0,\ i=1,\cdots,n) \\ y_i^N=x_i^N+y_{i-1}^N-r_i^N\,, & (r_i^N\geq 0,\ i=1,\cdots,n) \\ x_i^k\geq 0,\ y_i^k\geq 0,\ i=1,\cdots,n;\ k=1,\cdots,N;\ y_0^k=0, & k=1,\cdots,N\,. \end{cases}$$

This problem can again be considered as a minimal cost flow problem on the rectangular network described in Fig. 3. By the same reasoning as above, there exists an optimal flow among extreme flows. In an extreme flow, every node can have at most one positive input [2]. Using this property, we can prove the following analogue of Theorem 1.

Theorem 2. If node $(i, k)\neq(0, 0)$ has S units of input in an extreme flow, then S must have the following representation:

$$S=\begin{cases} \sum_{l_1=\alpha_1}^{\beta_1} r_{l_1}^{k_0}+\sum_{l_2=\alpha_2}^{\beta_2} r_{l_2}^N\,, & i\leq\alpha_1\leq\beta_1\leq n;\ i\leq\alpha_2\leq\beta_2\leq n \\ & \alpha_1\leq\alpha_2;\ \beta_1\leq\beta_2\ ; \\ & 0\leq k\leq k_0\,, \\ \sum_{l_2=\alpha_2}^{\beta_2} r_{l_2}^N;\ i\leq\alpha_2\leq\beta_2\leq n;\ k_0<k\leq N\,. & \end{cases}$$

Proof.

(i) $k_0<k\leq N$. In this case, input to any node has to satisfy the demands at facility N, so the assertion is nothing but the one proved in Theorem 1 except $i\leq\alpha_2$. But this follows from the fact that any input at node (i, k) can flow out to nodes (p, q) where $i\leq p$, $k\leq q$.

(ii) $0\leq k\leq k_0$. In this case, input into (i, k) can be the combination of demands at facilities k_0 and N. It follows from Theorem 1 and (i) above that S has the representation of the theorem except $\alpha_1 \leq \alpha_2$; $\beta_1 \leq \beta_2$. If $\alpha_1 > \alpha_2$, then there exists at least one node (j, k_0), $i \leq j \leq \alpha_2$ which has more than two positive inputs, whence we must have $\alpha_1 \leq \alpha_2$. $\beta_1 \leq \beta_2$ follows analogously.

Although we do not go into detail here, we can develop (by virtue of this theorem) an algorithm which is analogous to the one given in the preceding section. The recursion formula, however, is much more complicated and it requires $(k_0-1)n^7/840+0(k_0 n^6)+(N-k_0)n^4/24+0((N-k_0)n^3)$ additions and comparisons for $k_0>1$ and $n^6/120+0(n^5)+(N-1)n^4/24+0(Nn^3)$ additions and comparisons when $k_0=1$.

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