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ON THE RELIABILITY OF STANDBY REDUNDANT SYSTEMS WITH REPAIR

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Abstract

The reliability of repairable standby systems is investigated using semi-Markov processes. The systems are composed of n units having general failure time distribution with mean $1/\lambda$ and of r repair facilities having exponential repair time distribution with mean $1/\mu$. For these systems, the Laplace-Stieltjes transform (LST) of the system failure time distribution is derived in the form of a generating function. For indefinitely large n, asymptotic values of the mean time to system failure (MTSF) and limiting distributions of system failure time are given. It is shown that under certain conditions, the system failure time distribution tends to an exponential distribution as n increases indefinitely. The results can be applied to the first passage time problem of the maximal queue size of the queuing system $G/M/r$[1]. Some numerical examples of the MTSF and stationary availability are shown in figures for gamma failure time distributions of units.

[1] Kendall's notation $A/B/s$ means that A and B stand for distributions of the time to unit failure and repair, respectively, and s denotes the number of repair facilities.

1. Introduction

The reliability of systems with repair has been summerized by Barlow [1] as "repairman problems." For models shown in his book, methods in queuing theory can be applied to the reliability analysis of systems, where failure and repair of units correspond to arrival of customers and service, respectively. On these problems, Gnedenko [4] has treated the repairable system of $M/G/1$ type, while Srinivasan [6] has discussed the system $G/M/n-1$. The results obtained by Srinivasan are LST of system failure time distribution $F_s(t)$, and MTSF, while it has been shown by Gnedenko that $F_s(t)$ approaches to exponential distribution under certain conditions. On the other hand, for the queue $G/M/1$, Vinogradov [7] has discussed the first passage time to the instant when the queue length becomes n from the arrival of the first customer, and proved that the distribution of this first passage time becomes exponential as n increases indefinitely under certain conditions. His results can be easily translated to the reliability problem of systems with repair. The above exponential property indicates that, under some conditions, repairable systems can be considered to have a constant failure rate, which is equal to a reciprocal of the MTSF. In this case, the reliability of a system can be easily estimated by obtaining only its MTSF.

In this paper, the reliability of $G/M/r$ systems is discussed, and LST of $F_s(t)$ is derived as an extension of the results obtained by Vinogradov and the author [5]. Moreover, asymptotic behavior of MTSF is analysed for various values of ρ, which is defined by $\rho=\lambda/r\mu$. The asymptotic values of MTSF are shown to be approximately proportional to $\exp(cn)$, n^2, and n in case $\rho<1$, $\rho=1$ and $\rho>1$, respectively, where c is a positive constant. Therefore, this result means that the increase of repair capacity, $r\mu$, is more effective than an increase in the number of spares, $n-1$, in order to improve the system reliability.

Moreover, if $\rho \geqq 1$, MTSF of systems with r repairmen and repair rate μ is asymptotically equal to that with one repairman and repair rate $r\mu$, while, if $\rho<1$, this relation does not hold.

Furthermore, the limiting distribution of system failure time as $n\rightarrow\infty$, is shown to be exponential when $\rho<1$, and unit distribution when $\rho>1$. As is seen in the former case, which is more important in practice, the exponential property of repairable systems seems to correspond, in a sense, to that in Drenick's theorem [2] for series systems.

2. System Model

A standby system with repair which is analysed in this paper, is the same model as in the previous paper [5] written by the author. The system is assumed to consist of n identical units, one of which is in operation while the other $n-1$ units are spares. Where an operating unit fails, one of the spares is substituted for the failed unit. It is assumed that there are r repairmen, each of whom can deal with one unit at a time, and that a repaired unit joins the spares. If all repairmen are busy, each newly failed unit joins a queue and waits until a repairman becomes free. The system failure is assumed to occur as soon as all n units are defective. Suppose that the distribution $F(t)$ of time from the start of operation to failure of a unit is general, and that the repair time distribution is a negative exponential distribution with repair rate μ. A unit whose repair is finished, is assumed to recover its function completely.

In this paper, the system model is described by using the terminology of reliability. However, the results of the system failure time can also be applied to the first passage time of the maximal queue size, which makes the same stochastic prosses as the maximal number of the failed units does. Let us consider the queuing system $G/M/r$, and denote by τ_n the time when the number of customers present in the system is $n+1$ for the first time under the condition that the first customer arrives at time 0. When the arrival distribution of customers is $F(t)$ and the service rate is μ, the distribution of τ_n is equal to the failure time distribution of the above standby system with repair.

In [5], solutions are obtained for $r=1, 2, n-1$ and n, while in this paper they are obtained for general r when $r<n-1$.

3. Fundamental Equations

Let us express the state of systems by the number of failed units at the instants just after the failure of a unit. The transition probability from state i to state j within time interval t is denoted by $Q_{ij}(t)$. The LS transform of $Q_{ij}(t)$ for systems with r repairmen is denoted by ${}_rq_{ij}(s)$. Furthermore, the LST of $F(t)$ is denoted by $f(s)$. From [5], the LST of failure time distribution of n-unit systems with r repairmen, $\psi_{nr}(s)$ is given by

$$(1) \qquad \psi_{nr}(s)=\prod_{j=0}^{r-1} f(s+j\mu)[f(s+r\mu)]^{n-r}/{}_rS_{n-1} ,$$

where ${}_rS_i$ is written by a determinant with respect to ${}_rq_{jk}(s)$, as follows:

$$(2) \qquad {}_rS_i=\begin{vmatrix} 1-{}_rq_{11} & -{}_rq_{12} & 0 & \cdots & & 0 \\ -{}_rq_{21} & 1-{}_rq_{22} & -{}_rq_{23} & 0 & \cdots & 0 \\ \vdots & -{}_rq_{32} & 1-{}_rq_{33} & \ddots & \ddots & \vdots \\ & \vdots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & -{}_rq_{i-1,i} \\ -{}_rq_{i1} & -{}_rq_{i2} & \cdots & & -{}_rq_{i,i-1} & 1-{}_rq_{ii} \end{vmatrix}$$

Elements ${}_rq_{jk}(s)$ are given by the following equations,

$$(3.1) \qquad {}_rq_{jk}(s)=0 , \qquad k>j+1$$

$$(3.2) \qquad {}_rq_{j,j+1}(s)=f(s+r\mu) , \qquad j\geq r$$

$$(3.3) \qquad {}_rq_{jk}(s)=\int_0^\infty e^{-st}\binom{j}{k-1}e^{-(k-1)\mu t}(1-e^{-\mu t})^{j-k+1}dF(t) , \qquad k-1\leq j\leq r$$

(3.4) $$ {}_rq_{jk}(s)=\frac{(-r\mu)^{j-k+1}}{(j-k+1)!}f^{(j-k+1)}(s+r\mu)\,, \qquad r<k\leq j $$

(3.5) $$ {}_rq_{jk}(s)=r^{j-r}\frac{r!}{(k-1)!}\mu^{j-k+1}\int_0^\infty e^{-st}\int_0^t\int_0^{t_{j-k+1}}\int_0^{t_{j-k}}\cdots\int_0^{t_{j-r+2}} \times\frac{t_{j-r+1}^{j-r}}{(j-r)!}\cdot\exp\left(-\mu\sum_{l=j-r+1}^{j-k+1}t_l-(k-1)\mu t\right) \times dt_{j-r+1}\cdots dt_{j-k}dt_{j-k+1}dF(t)\,, \qquad k\leq r<j\,, $$

where $f^{(k)}(s)$ denotes the kth derivative of $f(s)$. It is assumed that all the moments of the distribution $F(t)$ exist. By integrating (3.5), the following recurrence relation for ${}_rq_{jk}(s)$ holds.

(4) $$ \sum_{l=0}^{r-k}\binom{k-1+l}{l}{}_rq_{j,k+l}(s) =\begin{cases}\binom{j}{k-1}f(s+\overline{k-1}\mu)\,, & k-1\leq j\leq r\\ r^{j-r}\binom{r}{k-1}\left[\dfrac{f(s+\overline{k-1}\mu)}{(r-k+1)^{j-r}}-\displaystyle\sum_{l=0}^{j-r}\frac{(-\mu)^{j-r-l}}{(r-k+1)^l}\cdot\frac{1}{(j-r-l)!}\cdot\right. \\ \qquad\left.\times f^{(j-r-l)}(s+r\mu)\right]\,, & k\leq r<j\,.\end{cases} $$

By using (4), the determinant ${}_rS_{n-1}(s)$ is reduced to a simpler form, whose elements in the jth row, kth column are zeros when $k<j\leq r$ or $k>j+1$. Let us define ${}_rW_i(s)$ by

(5) $$ {}_rS_i(s)=(1-f(s))_rW_i(s)+\gamma_{r-1}(s)[f(s+r\mu)]^{i-r+1}\,, \qquad i\geq r\,, $$

where

(5.1) $$ \gamma_j(s)=\prod_{l=1}^{j}f(s+l\mu)\,. $$

Therefore, if we set

(6) $$ W(s,z)=\sum_{j=1}^{\infty}{}_rW_{r+j}(s)z^j+{}_rW_r^*(s)\,, $$

(6.1) $$ {}_rW_r^*(s)=Y(s,0)\,, $$

then, from Appendix 1, we have

(7) $$ W(s,z)=\frac{f(s+r\mu)Y(s,z)}{\{f[s+r\mu(1-zf(s+r\mu))]-zf(s+r\mu)\}(1-zf(s+r\mu))}\,. $$

Here $Y(s,z)$ is defined by

(8) $$Y(s, z)=\gamma_{r-1}(s)+(1-zf(s+r\mu))\sum_{j=1}^{r-1}\binom{r}{j}\theta_{j1}(s)y_j(s, z)$$

where

$$\theta_{jk}(s)=\prod_{l=k}^{j-1}(1-f(s+l\mu))\prod_{l=j+1}^{r-1}f(s+l\mu)$$

and

(9) $$y_j(s, z)=\frac{1}{1-\frac{rz}{r-j}f(s+r\mu)}\cdot\left\{f[s+r\mu(1-zf(s+r\mu))]-zf(s+r\mu)\right.$$
$$\left.+\frac{r-j}{r}\left(1-\frac{r}{r-j}f(s+j\mu)\right)\right\}.$$

Here we define

$$\sum_{k=l}^{m}g(k)=0,\ m<l, \quad \text{and} \quad \prod_{k=l}^{m}g(k)=1,\ m<l.$$

Hence ${}_rW_{n-1}(s)$ is obtained by

(10) $${}_rW_{n-1}(s)=\frac{1}{(n-r-1)!}\cdot\frac{d^{n-r-1}}{dz^{n-r-1}}W(s, z)\bigg|_{z=0}, \qquad r<n-1.$$

Since from (1) and (5)

(11) $$\psi_{nr}(s)=\frac{f(s)\gamma_{r-1}(s)[f(s+r\mu)]^{n-r}}{(1-f(s))_rW_{n-1}(s)+\gamma_{r-1}(s)[f(s+r\mu)]^{n-r}},$$

$\psi_{nr}(s)$ can be obtained by using (7), (10) and (11).

4. Numerical Examples

Here the MTSF and stationary availability for n-unit systems with r repairmen are denoted by T_{nr} and A_{nr}, respectively. Numerical examples of T_{nr} and A_{nr} where $n=5$, and $F(t)$ is the Weibull distribution, are shown as follows. Since

(12) $$T_{nr}=-\psi'_{nr}(0)=-f'(0)\cdot\left(1+\frac{{}_rW_{n-1}(0)}{\gamma_{r-1}(0)[f(r\mu)]^{n-r}}\right),$$

then for $r=3$,

(12.1) $$T_{53}=-f'(0)(1+{}_3W_4(0)/\{f(\mu)f(2\mu)[f(3\mu)]^2\})$$

From (7) and (10),

$$(12.2)\qquad {}_3W_4(0)=(1-f(\mu))\left\{1-f(2\mu)+3f(3\mu)-\frac{11}{2}f(2\mu)f(3\mu)+9[f(3\mu)]^2 +3\mu f'(3\mu)(1-f(2\mu))\right\}+\frac{1}{2}f(2\mu)f(3\mu)(9f(3\mu)-1)\,.$$

Since $F(t)$ is the Weibull distribution, it is given by

$$F(t)=\exp(-t^m/a)\,,$$

where m and a are a shape, and a scale parameter.

Moreover, from [5], the stationary availability is given by

$$(13)\qquad A_{nr}=1-\frac{1}{r\mu}\cdot\frac{1}{\left.\dfrac{d}{ds}\left\{\dfrac{r\mu}{s+r\mu}\cdot\dfrac{{}_rS_{n-2}(s)}{{}_rS_{n-1}(s)}\cdot f(s+r\mu)\right\}\right|_{s=0}}\,,\qquad r<n\,.$$

When the stationary unavailability $\bar{A}_{nr}$ is defined by

$$\bar{A}_{nr}=1-A_{nr}\,,$$

then from (5) and (13) $\bar{A}_{53}$ is given by

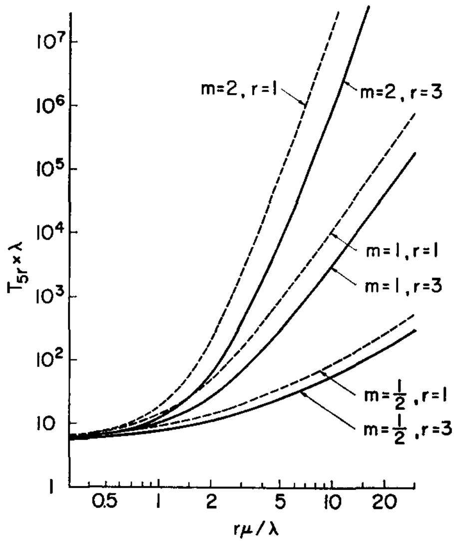

Fig. 1. The MTSF for 5-unit systems with r repairmen when failure time distributions of units are Weibull.

(13.1) $$\bar{A}_{53}=1\Big/\left\{1+\frac{3\mu}{\lambda}\cdot\frac{{}_3W_4(0)-{}_3W_3(0)f(3\mu)}{f(\mu)f(2\mu)[f(3\mu)]^2}\right\}$$

where, from (7) and (10)

(13.2) $${}_3W_3(0)=(1-f(\mu))(1-f(2\mu)+3f(3\mu))+3f(2\mu)f(3\mu)\ .$$

For $m=1/2$, 1 and 2, T_{53} and $\bar{A}_{53}$ are shown in Figures 1 and 2, compared with T_{51} and $\bar{A}_{51}$ respectively, where the horizontal axis is $r\mu/\lambda$. From these figures, it is seen that curves T_{5r} and $\bar{A}_{5r}$ are markedly dependent upon the shape parameter m. The figures also show that $T_{51}>T_{53}$ and $\bar{A}_{51}<\bar{A}_{53}$ in case $r\mu$ is fixed. This result can be easily explained by the fact that the total repair rate of the system is larger for $r=1$ than for $r=3$ when the number of failed units is less than 3.

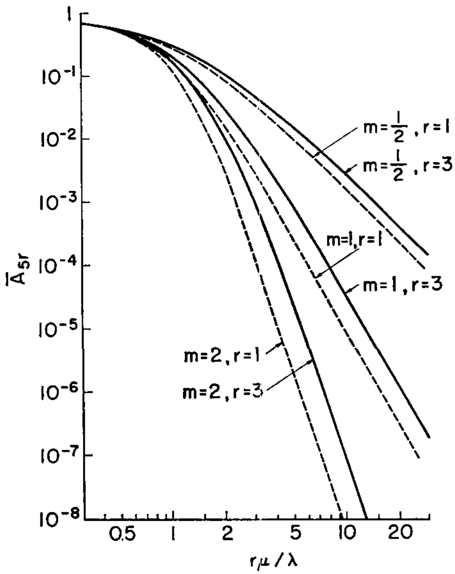

Fig. 2. The unavailability for 5-unit systems with r repairmen when failure time distributions of units are Weibull.

5. Asymptotic Formulae for the MTSF

It is not easy to calculate the $(n-r-1)$th derivative of $W(s, z)$ in (10) in order to obtain the MTSF, particularly when $n-r$ is a large value. In this section, asymptotic formulae of the MTSF T_{nr} are given for indefinitely large n. Now (11) is reduced to

$$(14) \qquad \phi_{nr}(s)=\frac{f(s)}{(1-f(s))_r\bar{W}_{n-1}(s)+1}$$

where

$$(15) \qquad {}_r\bar{W}_{n-1}(s)={}_rW_{n-1}(s)/\{\gamma_{r-1}(s)\cdot[f(s+r\mu)]^{n-r}\}\ , \qquad n>r+1\ ,$$
$${}_r\bar{W}_r(s)={}_rW_r^*(s)/\gamma_r(s)\ .$$

Let the generating function $\bar{W}(s, z)$ be defined by

$$(15.1) \qquad \bar{W}(s, z)=\sum_{j=0}^{\infty}{}_r\bar{W}_{r+j}(s)z^j$$

then, from (15)

$$(15.2) \qquad \bar{W}(s, z)=W\left(s, \frac{z}{f(s+r\mu)}\right)\Big/\gamma_r(s)\ .$$

Since T_{nr} is given by (12), it is also written by means of the Cauchy integral, from (7) and (15), as

$$(16) \qquad T_{nr}=\frac{1}{\lambda}+\frac{1}{2\pi i\lambda}\int_{|z|=r_1}\bar{W}(0, z)\frac{dz}{z^{n-r}}$$
$$=\frac{1}{\lambda}+\frac{1}{2\pi i\lambda}\int_{|z|=r_1}\frac{\bar{Y}(0, z)dz}{\{f[r\mu(1-z)]-z\}(1-z)z^{n-r}}\ ,$$

where $r_1>0$, $i=\sqrt{-1}$ and

$$(16.1) \qquad \bar{Y}(s, z)=Y\left(s, \frac{z}{f(s+r\mu)}\right)\Big/\gamma_{r-1}(s)\ .$$

5.1 When $\lambda/r\mu<1$

When $\lambda/r\mu<1$, it is known, from Rouche's theorem, that equation

$$(17) \qquad f[r\mu(1-z)]-z=0$$

has the unique root $z=\beta_0$, $0<\beta_0<1$, inside the unit circle [3]. Therefore, when r_1 in (16) is set as $r_1<\beta_0$, by means of the remainder theorem, we have, from (16),

(18) $$T_{nr}=\frac{\bar{Y}(0,\beta_0)}{\lambda(1-\beta_0)[1+r\mu f'(r\mu(1-\beta_0))]\beta_0^{n-r}}+\frac{1}{\lambda}+\frac{I_n}{\lambda}$$

where

(18.1) $$I_n=\frac{1}{2\pi i}\int_{|z|=r_2}\bar{W}(0,z)\frac{dz}{z^{n-r}}, \qquad \beta_0<r_2<1 .$$

Thus, if $\lambda/r\mu<1$ and all the moments of $F(t)$ are finite, as is proved in Appendix 2, T_{nr} is asymptotically given, when $n\to\infty$, by

(19) $$T_{nr}=\frac{\bar{Y}(0,\beta_0)}{\lambda(1-\beta_0)[1+r\mu f'(r\mu(1-\beta_0))]\beta_0^{n-r}}+\frac{1}{\lambda}+\frac{1}{\lambda}[B_1+B_2\cdot(n-r+C_0)]+\frac{R_n}{\lambda},$$

where

(19.1) $$B_1=-\frac{r^2\mu^2f''(0)}{2(1-r\mu/\lambda)^2}, \quad B_2=\frac{1}{1-r\mu/\lambda},$$

(19.2) $$C_0=\frac{1}{\gamma_{r-1}(0)}\sum_{j=1}^{r-1}\binom{r}{j}\theta_{j1}(0)\frac{r-j}{rj}[j-r(1-f(j\mu))],$$

(19.3) $$|R_n|<\frac{C_2}{2C_b}\cdot\frac{1+2C_u}{\sqrt{2\pi(n-r-3)}}+\frac{(r\mu)^{n-r}}{(n-r)!}\cdot(-B_1-r\mu B_2)\times\frac{C_f(1+r\mu)}{\gamma_{r-1}(0)}\cdot\exp(r^2\mu^2)\sum_{j=1}^{r-1}\binom{r}{j}\theta_{j1}(0)\frac{r-j}{r}, \qquad n>r+3 .$$

Here C_2, C_b, C_u and C_f are positive constants which will be defined in Appendix 2. It is easy to see that $R_n\to 0$ when $n\to\infty$.

5.2 When $\lambda/r\mu=1$

When $\lambda/r\mu=1$, from [7] equation (17) has only a double root $z=1$ for $|z|\leq 1$. By the same consideration as in the section 4.1, we have

(20) $$T_{nr}=\frac{1}{\lambda}-\operatorname*{res}_{z=1}\frac{\bar{Y}(0,z)}{\lambda(1-z)[f(r\mu(1-z))-z]z^{n-r}}+\frac{1}{2\pi i\lambda}\int_{|z|=1+\varepsilon}\frac{\bar{Y}(0,z)dz}{(1-z)[f(r\mu(1-z))-z]z^{n-r}}, \qquad \varepsilon>0 .$$

Since the integral in (20) tends to 0 as $n\to\infty$, T_{nr} is given, for sufficiently large n, approximately by

$$T_{nr}\sim\frac{1}{f''(0)\lambda^3}\left\{(n-r)^2+(n-r)\left[1-2\bar{Y}'(0,1)-\frac{2f'''(0)\lambda}{3f''(0)}\right]+C_1\right\}+\frac{1}{\lambda}\sim\frac{n^2}{f''(0)\lambda^3}, \tag{20.1}$$

where

$$C_1=\bar{Y}''(0,1)+\frac{2f'''(0)\lambda}{3f''(0)}\cdot\bar{Y}'(0,1)+\frac{2}{9}\left(\frac{f'''(0)\lambda}{f''(0)}\right)^2-\frac{f^{(4)}(0)\lambda^2}{6f''(0)},$$

and $\bar{Y}'$ and $\bar{Y}''$ denote 1st and 2nd derivatives of $\bar{Y}$ with respect to z.

5.3 When $\lambda/r\mu>1$

From Appendix 3C, equation (17) has two roots, $z_1=1$ and $z_2=\beta_2>1$. In this case it is assumed that $f[r\mu(1-z)]$ is regular for Re $z<a$ and $\beta_2<a<\infty$. From (20), setting ε as $0<\varepsilon<\beta_2-1$, we have approximately, as n tends to infinity,

$$T_{nr}\sim\frac{1}{\lambda-r\mu}\left[n-r-\frac{\lambda}{\lambda-r\mu}\cdot\frac{r^2\mu^2}{2}f''(0)-\bar{Y}'(0,1)\right]+\frac{1}{\lambda}\sim\frac{n}{\lambda-r\mu}. \tag{21}$$

6. Limiting Distributions

The limiting distribution of the system failure time, as $n\to\infty$ and $T_{nr}\to\infty$, is discussed when T_{nr} is taken to be the unit of time. From (14), the LS transform of the system failure time distribution is given by

$$\psi_{nr}\left(\frac{s}{T_{nr}}\right)=f\left(\frac{s}{T_{nr}}\right)\Big/\left\{\left(1-f\left(\frac{s}{T_{nr}}\right)\right)_r\bar{W}_{n-1}\left(\frac{s}{T_{nr}}\right)+1\right\}. \tag{22}$$

6.1 When $\lambda/r\mu<1$

Analogously to (17), equation

$$f[s+r\mu(1-z)]-z=0 \tag{23}$$

has a unique root $z=\beta(s)$ inside the unit circle for $\lambda/r\mu<1$, where $0<\beta(s)<1$. As is shown in Appendix 3A,

(24) $${}_r\bar{W}_{n-1}(s/T_{nr})(1-f(s/T_{nr})) \longrightarrow s$$

as $n\to\infty$. Therefore

(25) $$\psi_{nr}(s/T_{nr}) \longrightarrow \frac{1}{1+s}, \qquad n\to\infty .$$

Thus, if $\lambda/r\mu<1$, the limiting distribution of system failure time as n tends to infinity, becomes exponential.

6.2 When $\lambda/r\mu=1$

From [7], equation (17) has two roots, $z_1=\beta_1(s)$ and $z_2=\beta_2(s)$, for $s>0$, where

$$0<\beta_1(s)<1<\beta_2(s) .$$

Here, and also in the next section, $f(z)$ is assumed to be regular for Re $z>-a$, $0<a$. Using the proof given in Appendix 3B, we have

(26) $$\lim_{n\to\infty}(1-f(s/T_{nr}))_r\bar{W}_{n-1}(s/T_{nr})=\cosh\sqrt{2s}-1 .$$

Then, from (22),

(27) $$\lim_{n\to\infty}\psi_{nr}(s/T_{nr})=1/\cosh\sqrt{2s} .$$

From the inverse transform of (27), we get

(28) $$\lim_{n\to\infty}P_r\{T/T_{nr}\leqq t\}=1-\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left[-\pi^2\left(k+\frac{1}{2}\right)^2\cdot\frac{t}{2}\right]$$

where T is the time to system failure.

6.3 When $\lambda/r\mu>1$

From Appendix 3C, we have, as $n\to\infty$,

(29) $$(1-f(s/T_{nr}))_r\bar{W}_{n-1}(s/T_{nr}) \longrightarrow e^s-1 .$$

Therefore, as $n\to\infty$,

(30) $$\psi_{nr}(s/T_{nr}) \longrightarrow e^{-s} .$$

As its inverse transform gives a delta function $\delta(t/T_{nr}-1)$, the limiting distribution is a unit distribution given by

(31) $$P_r\{T/T_{nr}\leq t\}=\begin{cases}1, & t\geq 1 ,\\ 0, & 0\leq t<1 .\end{cases}$$

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Appendix 1

The proof in Appendix 1, 2, and 3 is based upon the method used in [7]. In order to obtain ${}_rS_{n-1}(s)$ in (1), we define $h_j(s)$ by

$$h_j(s)={}_rS_j(s)-v_j(s),\ j>r,\ h_r(s)={}_rS_r^*(s)-v_r(s)\ , \tag{32}$$

where

$$v_j(\mathrm{s})=[f(s+r\mu)]^{j-r+1}\left\{\sum_{l=1}^{r-1}\theta_{l0}(s)\binom{r}{l}\left(\frac{r}{r-l}\right)^{j-r}+\gamma_{r-1}(s)\right\}\ ,\quad j\geq r, \tag{32.1}$$
$${}_rS_r^*(s)=(1-f(s))_rW_r^*(s)+\gamma_r(s)\ .$$

Expanding the simpler form of the determinant ${}_rS_j(s)$ by elements of the last row, we have the recurrence relation for $h_j(s)$,

(33) $$h_j(s)+h_{j-1}(s)a_1(s)+h_{j-2}(s)a_2(s)a_0(s)+\cdots +h_{r+1}(s)a_{j-r-1}(s)[a_0(s)]^{j-r-2}+h_r(s)a_{j-r}(s)[a_0(s)]^{j-r-1}=d_j(s)\ , \quad j\geq r$$

where

(34) $$a_k(s)=\frac{(-r\mu)^k}{k!}f^{(k)}(s+r\mu)-\delta_{k1}\ , \qquad k\geq 0\ ,$$

$$d_k(s)=(1-f(s))[f(s+r\mu)]^{k-r}\cdot\left\{\gamma_{r-1}(s) +\sum_{l=1}^{r-1}\binom{r}{l}\left(\frac{r}{r-l}\right)^{k-r-1}\cdot\left(1-\frac{r}{r-l}f(s+l\mu)\right)\theta_{l1}(s)\right\}\ , \quad k\geq r,$$

and δ_{jk} denotes Kronecker's delta. Furthermore generating functions $A(s, z)$, $H(s, z)$ and $D(s, z)$ are defined by

(35) $$A(s, z)=1+\sum_{j=1}^{\infty} a_j(s)[a_0(s)]^{j-1}z^j\ ,$$

$$H(s, z)=\sum_{j=0}^{\infty} h_{r+j}(s)z^j\ ,$$

$$D(s, z)=\sum_{j=0}^{\infty} d_{r+j}(s)z^j\ .$$

Then from (33) we have

(36) $$H(s, z)A(s, z)=D(s, z)\ .$$

From (34) and (35),

(37) $$A(s, z)=\{f[s+r\mu(1-zf(s+r\mu))]-zf(s+r\mu)\}/f(s+r\mu)\ ,$$

$$D(s, z)=(1-f(s))\left\{\frac{\gamma_{r-1}(s)}{1-zf(s+r\mu)}+\sum_{j=1}^{r-1}\binom{r}{j}\frac{r-j}{r} \times\frac{1}{1-\frac{rz}{r-j}f(s+r\mu)}\left(1-\frac{r}{r-j}f(s+j\mu)\right)\theta_{j1}(s)\right\}\ .$$

Moreover we define generating functions for $v_j(s)$ and ${}_rS_j(s)$ by

(38) $$V(s, z)=\sum_{j=0}^{\infty} v_{r+j}(s)z^j, \qquad S(s, z)=\sum_{j=1}^{\infty} {}_rS_{j+r}(s)z^j+{}_rS_r^*(s)\ .$$

Then, using (32.1), we obtain

(39) $$V(s, z)=\frac{\gamma_r(s)}{1-zf(s+r\mu)}+(1-f(s))\sum_{j=1}^{r-1}\theta_{j1}(s)\binom{r}{j} \times\frac{f(s+r\mu)}{1-\frac{rz}{r-j}f(s+r\mu)}\ .$$

Since, from (32)

(40) $$S(s, z)=H(s, z)+V(s, z)\ ,$$

using (35), (36), (37) and (39) we get

(41) $$S(s, z)=\frac{(1-f(s))f(s+r\mu)Y(s, z)}{\{f[s+r\mu(1-zf(s+r\mu))]-zf(s+r\mu)\}(1-zf(s+r\mu))}+\frac{\gamma_r(s)}{1-zf(s+r\mu)}\ .$$

Let us note that the root of equation

$$1-rzf(s+r\mu)/(r-j)=0$$

is not a singular point of $y_j(s, z)$ which is an element of $Y(s, z)$, as is defined by (8).

Appendix 2

When in (16) w and $p(w)$ are defined by

$$w=1-z, \qquad p(w)=\bar{Y}(0, 1-w)\ ,$$

then the integral I_n in (18) is given by

(42) $$I_n=\frac{-1}{2\pi i}\int_{|w-1|=r_2}\frac{p(w)dw}{w[f(r\mu w)+w-1](1-w)^{n-r}}\ .$$

It is easy to see that at $w=0$, the denominator of the integrand in I_n has the zero of order 2. Here we define a function $g(w)$ by

$$g(w)=\frac{1}{w[f(r\mu w)+w-1]}-\frac{B_1}{w}-\frac{B_2}{w^2}$$

and choose constants B_1 and B_2, as given by (19.1), in order that $g(w)$ becomes finite at $w=0$. Thus we have

(43)
$$I_n=-\frac{1}{2\pi i}\int_{|w-1|=r_2}\frac{p(w)g(w)}{(1-w)^{n-r}}dw-\frac{1}{2\pi i}\int_{|w-1|=r_2}p(w)\times\frac{\dfrac{B_1}{w}+\dfrac{B_2}{w^2}}{(1-w)^{n-r}}dw\ .$$

On applying the remainder theorem to the integral I_{n2} of the second term in the right-hand side of (43), then I_{n2} is given by

(44)
$$I_{n2}=\frac{(-1)^{n-r-1}}{(n-r-1)!}\frac{d^{n-r-1}}{dw^{n-r-1}}\left[p(w)\cdot\left(\frac{B_1}{w}+\frac{B_2}{w^2}\right)\right]\Big|_{w=1}$$

Considering that all the moments of $F(t)$ are less than a positive finite constant C_f, from the $(n-r-1)$th derivatives of (44), we obtain the asymptotic form of I_{n2}, when $n\to\infty$, as

(44.1)
$$I_{n2}=B_1+B_2(n-r+C_0)+R_{n2}\ ,$$

where C_0 is a constant defined by (19.2), and

$$|R_{n2}|<\frac{(r\mu)^{n-r}}{(n-r)!}\cdot(-B_1-r\mu B_2)\cdot\frac{C_f(1+r\mu)}{\gamma_{r-1}(0)}\cdot\exp(r^2\mu^2)\times\sum_{j=1}^{r-1}\binom{r}{j}\theta_{j1}(0)\frac{r-j}{r}\ ,\qquad n>r+1\ .$$

Moreover, when we set $p(w)=1+wu(w)$ in the first term integral, denoted by I_{n1}, of the right-hand side of (43), then we get

(45)
$$I_{n1}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{g(ix)dx}{(1-ix)^{n-r}}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{u(ix)g(ix)}{(1-ix)^{n-r}}dx-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{u(ix)g(ix)dx}{(1-ix)^{n-r-1}}\ ,$$

where u is defined by

$$u(ix)=\frac{1}{\gamma_{r-1}(0)}\sum_{j=1}^{r-1}\binom{r}{j}\theta_{j1}(0)\left[\frac{r-j}{r}+\frac{r-j}{irx-j}\times\left(\int_0^{\infty}e^{-ir\mu xt}dF(t)-f(j\mu)\right)\right]\ .$$

Therefore, for real x

(46)
$$|u(ix)|<\frac{1}{\gamma_{r-1}(0)}\sum_{j=1}^{r-1}\binom{r}{j}\theta_{j1}(0)\left(\frac{r-j}{r}+\frac{r-j}{j}\cdot(1+f(j\mu))\right)=C_u\ .$$

Then using the inequality [7]

$$|g(ix)|\leq\frac{C_2}{2C_b}\cdot(x^2+1)^{1/2}$$

where C_2 and C_b are positive finite constants defined by

$$C_2=\frac{1}{\left(\frac{r\mu}{\lambda}-1\right)^2}\cdot\left\{\frac{(r^2\mu^2f''(0))^2}{2}-\frac{r^3\mu^3}{6}f^{(3)}(0)\right.$$

$$\left.\times\max\left[r^2\mu^2f''(0),\ 2\left(\frac{r\mu}{\lambda}-1\right)\right]\right\},$$

$$C_b=\left(\frac{r\mu}{\lambda}-1\right)\Big/\left\{1+\frac{1}{2}f''(0)r^2\mu^2[\sqrt{4(r\mu/\lambda-1)/f''(0)r^2\mu^2+1}\right.$$

$$\left.+4[(r\mu/\lambda-1)/f''(0)r^2\mu^2]^2+1]\right\},$$

we obtain

$$\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{g(ix)}{(1-ix)^m}dx\right|\leq\frac{C_2}{4\pi C_b}\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^{(m-1)/2}}. \tag{47}$$

Thus, from (45), (46) and (47), by means of the relation

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^{(m-1)/2}}=\frac{\Gamma(m/2-1)}{\Gamma(m/2-1/2)}<\sqrt{\frac{2}{m-3}},$$

it is shown that

$$|I_{n1}|<\frac{C_2}{2C_b}\cdot(1+2C_u)/\sqrt{2\pi(n-r-3)},\qquad n>r+3. \tag{47.1}$$

Thus from (18), (44.1) and (47.1), we obtain (19).

Appendix 3

A. When $\lambda/r\mu<1$

Since

$${}_r\bar{W}_{n-1}(s/T_{nr})=\frac{1}{2\pi i}\int_{|z|=r_2}\bar{W}(s/T_{nr},\ z)\frac{dz}{z^{n-r}},$$

applying the remainder theorem to the above Cauchy integral, and using (7) and (15.2), we obtain

(48)
$$ {}_r\bar{W}_{n-1}(s/T_{nr})(1-f(s/T_{nr})) = \frac{(1-f(s/T_{nr}))\bar{Y}(s/T_{nr}, \beta(s/T_{nr}))}{(1-\beta(s/T_{nr}))\{f'[s/T_{nr}+r\mu(1-\beta(s/T_{nr}))]r\mu+1\}\beta(s/T_{nr})^{n-r}} + (1-f(s/T_{nr}))I_n^* $$

where

$$ I_n^* = \frac{1}{2\pi i}\int_{|z|=r_2} \frac{\bar{Y}(s/T_{nr}, z)dz}{\{f[s/T_{nr}+r\mu(1-z)]-z\}(1-z)z^{n-r}}, \quad \beta(0)<r_2<1 , $$

and $\beta(s)$ is a unique root of (23) inside the unit circle. Let us note that by virtue of the property of $\beta(s)$,

$$ [\beta(0)+\beta'(0)s/T_{nr}]^n \leqq [\beta(s/T_{nr})]^n \leqq [\beta(0)]^n = \beta_0^n . $$

Thus since $n/T_{nr}\to 0$ as $n\to\infty$, then $[\beta(s/T_{nr})/\beta_0]^n\to 1$. Using (19), for the first term of the right-hand side in (48), we have

$$ \lim_{n\to\infty} (1-f(s/T_{nr}))[{}_r\bar{W}_{n-1}(s/T_{nr})-I_n^*] = \lim_{n\to\infty} \frac{(1-f(s/T_{nr}))\bar{Y}(0, \beta_0)}{(1-\beta_0)\beta_0^{n-r}[f'(r\mu(1-\beta_0))r\mu+1]} = \lim_{n\to\infty} \frac{s\bar{Y}(0, \beta_0)}{\lambda T_{nr}(1-\beta_0\beta_0^{n-r}\cdot[f'(r\mu(1-\beta_0))r\mu+1]} = s $$

It is also easy to see that when $n\to\infty$,

$$ |(1-f(s/T_{nr}))I_n^*| \leqq \frac{s}{\lambda T_{nr}}\cdot\frac{1}{r_2^{n-r}} \times \max_{|z|=r_2}\left\{\frac{|\bar{Y}(s/T_{nr}, z)|}{|1-z|\cdot|f(s/T_{nr}+r\mu(1-z))-z|}\right\} < sC_1\left(\frac{\beta_0}{r_2}\right)^{n-r} \longrightarrow 0 $$

where C_1 is a finite positive constant. Thus it is shown that when $n\to\infty$, then $(1-f(s/T_{nr}))I_n^*\to 0$.

B. When $\lambda/r\mu=1$

From (7) and (15.2), by means of Cauchy's formula for power series coefficients,

$$ r\bar{W}_{n-1}(s/T_{nr}) = \frac{1}{2\pi i}\int_{|z|=r_1} \bar{W}(s/T_{nr}, z)\frac{dz}{z^{n-r}} $$

where $0<r_1<\beta_1(s/T_{nr})$, and

$$\bar{W}(s/T_{nr}, z)=\frac{\bar{Y}(s/T_{nr}, z)}{(1-z)\{f[s/T_{nr}+r\mu(1-z)]-z\}} .$$

Using the remainder theorem, we have

$$_r\bar{W}_{n-1}(s/T_{nr})=I_{n1}+I_{n2}+I_{n3}+R_n ,$$

where

$$I_{n1}=\underset{z=\beta_1(s/T_{nr})}{-\mathrm{res}}\ \bar{W}(s/T_{nr}, z)/z^{n-r} ,$$

$$I_{n2}=\underset{z=\beta_2(s/T_{nr})}{-\mathrm{res}}\ \bar{W}(s/T_{nr}, z)/z^{n-r} ,$$

$$I_{n3}=-\underset{z=1}{\mathrm{res}}\ \bar{W}(s/T_{nr}, z)/z^{n-r}$$

$$R_n=\frac{1}{2\pi i}\int_{|z|=\beta_2(s/T_{nr})+\varepsilon}\bar{W}(s/T_{nr}, z)\frac{dz}{z^{n-r}} , \qquad \varepsilon>0 .$$

From (23), we have, for sufficiently small positive s,

(49) $$1-\beta_j(s)\sim(-1)^{j-1}\cdot\sqrt{\frac{2s}{\lambda^3 f''(0)}} , \qquad j=1, 2 .$$

Then, from (20.1) and (49)

(50) $$\lim_{n\to\infty}[\beta_j(s/T_{nr})]^{n-r}=\lim_{n\to\infty}\exp[(n-r)(\beta_j(s/T_{nr})-1)]$$

$$=\lim_{n\to\infty}\exp\left[(-1)^j\cdot\frac{n-r}{n}\sqrt{2s}\right]=\exp[(-1)^j\sqrt{2s}] , \qquad j=1, 2 .$$

Thus, using (50), we obtain

$$\lim_{n\to\infty}[1-f(s/T_{nr})]I_{n1}$$

$$=\lim_{n\to\infty}\frac{(1-f(s/T_{nr}))\bar{Y}(s/T_{nr}, \beta_1(s/T_{nr}))}{[1-\beta_1(s/T_{nr})]\{1+r\mu f'[s/T_{nr}+r\mu(1-\beta_1(s/T_{nr}))]\}[\beta_1(s/T_{nr})]^{n-r}}$$

$$=\frac{1}{2e^{-\sqrt{2s}}} ,$$

and analogously

$$\lim_{n\to\infty}[1-f(s/T_{nr})]I_{n2}=\frac{1}{2e^{-\sqrt{2s}}} .$$

It is easy to see that as $n\to\infty$,

$$(1-f(s/T_{nr}))I_{n3}\longrightarrow-1, \quad \text{and} \quad R_n\longrightarrow 0 .$$

Thus (26) is obtained.

C. When $\lambda/r\mu>1$

It is easy to find that (23) has, for $s\geqq 0$, two roots $z_1=\beta_1(s)$ and $z_2=\beta_2(s)$ if z is real, where $0<\beta_1(s)\leqq 1<\beta_2(s)<1+s+a$. When z is complex, by means of Rouche's theorem, it can be shown that for $|z|<\beta_2(s)$, (23) has exactly one root, which coincides with $\beta_1(s)$. In the same way as in Section B.

$$(51)\qquad {}_r\bar{W}_{n-1}(s/T_{nr})=-\operatorname*{res}_{z=1}\bar{W}(s/T_{nr},z)/z^{n-r}-\operatorname*{res}_{z=\beta_1(s/T_{nr})}\bar{W}(s/T_{nr},z)/z^{n-r}+\frac{1}{2\pi i}\int_{|z|=1+\varepsilon}\bar{W}(s/T_{nr},z)\frac{dz}{z^{n-r}},$$

where $0<\varepsilon<\beta_2(s/T_{nr})-1$. It is not difficult to see that the integral in (51) tends to 0 as $n\to\infty$, and the residue at $z=1$ is equal to $\bar{Y}(s/T_{nr},1)/(1-f(s/T_{nr}))$. Considering that as $n\to\infty$,

$$1-\beta_1(s/T_{nr})\longrightarrow\frac{1}{\lambda-r\mu}\cdot\frac{s}{T_{nr}},$$

and $[\beta_1(s/T_{nr})]^{n-r}\to e^{-s}$, we can prove (29).