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OPTIMAL STOPPING IN SAMPLING FROM A BIVARIATE DISTRIBUTION

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Abstract

This paper studies an optimal stopping problem without recall in which the experimenter observes a sequential random sample from a specified bivariate probability distribution. The problem can be interpreted as deciding to buy a house which has the two-dimensional worth, for example, the values for a husband and for his wife. The concept of equilibrium neutral values is introduced, and by using it the explicit solutions are derived for the infinite-opportunity case and for the finite case. The examples are included to illustrate the computations required by the "optimal" strategy.

1. Introduction

Let (X_i, Y_i), $i=1, 2, \cdots$, be independent and identically distributed bivariate random variables that can be observed sequentially at a cost of c_1 and c_2 (both$\geqq 0$) per observation of X and Y, respectively. The common distribution function $H(x, y)$ of each of the observation (X_i, Y_i) is assumed to be known to the observer which we shall hereafter call the experimenter. We shall suppose that if the experimenter terminates the sampling process after having observed the values $(X_i=x_i, Y_i=y_i)$, $i=1, \cdots, m$, his gain is a pair of values x_m-mc_1 and

$y_m - mc_2$. We are now interested in finding a stopping rule which can be considered as optimal in some reasonable sense. This formulation of the problem provides a model for studying an immediate extension of optimal stopping problems, exhaustively discussed in [1].

For the sake of a concrete example, consider a man who desires to buy a house. There is a large population of houses available, and he proceeds by selecting one of these at random and going to see it. Having done this, he may either reject it immediately as being unsatisfactory and go on to look at another, or he may buy it on the spot, which has the eventual outcome of the two-dimensional reward (X, Y). We may consider, for example, X as representing monetary worth of the house, and Y as representing travelling expenses from the location of the house to his working place. We may also consider X and Y as the values of the house for him, and for his wife, respectively.

We shall suppose that there is a given upper bound n, $2 \leqq n \leqq \infty$, on the number of observations that can be taken. Since the sequential sampling is taken from a bivariate distribution we shall introduce the concept of equilibrium neutral values, and by using it the explicit solutions are derived for the case where $c_1, c_2 > 0$ and $n = \infty$ in Section 2, and the case where $c_1 = c_2 = 0$ and $n < \infty$ in Section 3. In the final section some examples are given in order to illustrate the computations required by the "optimal" strategy.

2. Optimal Strategy for the Case Where the Number of Observations is not Limited

We shall consider a class of stopping rules in which the experimenter has a pair of "neutral" values u and v such that, the sampling procedure is terminated at the first m such that $X_m \geqq u$ and $Y_m \geqq v$. Let $\tau = \tau(u, v)$ denote the random stopping time when the neutral values u and v are used. Let

$$(1)\qquad \begin{cases} M_1(u, v)=E[X_\tau-\tau c_1 \mid u, v], \\ M_2(u, v)=E[Y_\tau-\tau c_2 \mid u, v]. \end{cases}$$

Then $M_1(u, v)$ is the expected net gain from the observations of X when using the neutral values (u, v). $M_2(u, v)$ has the similar meaning.

Now the experimenter wants to maximize his expected net gains of *both* X and Y. In such cases the concept of the equilibrium points in the context of non-cooperative game theory [5] would be useful. Under this concept the neutral value u is chosen so as to try to maximize the net gain $M_1(u, v)$, and simultaneously the value of v to maximize $M_2(u, v)$. An equilibrium point (u^*, v^*) for the functions $M_1(u, v)$ and $M_2(u, v)$ is a pair of values u^* and v^* such that each value maximizes its own net gain if the other value is held fixed. More precisely,

$$(2)\qquad \begin{cases} M_1(u^*, v^*)=\max\limits_u M_1(u, v^*), \\ M_2(u^*, v^*)=\max\limits_v M_2(u^*, v). \end{cases}$$

Continuing to work within the framework of net-gains maximization, we recognize that if we partially differentiate expressions for $M_1(u, v)$ and $M_2(u, v)$ with respect to u and v respectively, equate the partial derivatives to zero and solve the resultant equations then we shall obtain the equilibrium pair of values. We prove[1]:

Theorem 1:

(i) Let S denote the event $\{X\geqq u,\ Y\geqq v\}$. Then we have

$$(3)\qquad \begin{cases} M_1(u, v)=E[X \mid S]-c_1/\Pr\{S\} \\ M_2(u, v)=E[Y \mid S]-c_2/\Pr\{S\}, \end{cases}$$

where $E[\cdot \mid S]$ denotes the conditional expectation under the condition that the event S has occurred.

(ii) The equilibrium point (u^*, v^*) satisfies the simultaneous

[1] This paper was motivated by an article by Matsuda and Sekiguchi [4]. Theorem 1 is, in fact, a restatement of the main result obtained in [4].

equations

$$(4)\qquad \begin{cases} E[X-u \mid S]=c_1/\Pr\{S\} \\ E[Y-v \mid S]=c_2/\Pr\{S\}. \end{cases}$$

Moreover we have

$$(5)\qquad \begin{cases} M_1(u^*, v^*)=u^* \\ M_2(u^*, v^*)=v^* . \end{cases}$$

Proof. (i): $M_1(u, v)=E[X_\tau-\tau c_1]=\sum_{m=1}^{\infty}(1-\Pr\{S\})^{m-1}\Pr\{S\}E[X-mc_1 \mid S]$
$=\Pr\{S\}\left[\frac{E[X \mid S]}{\Pr\{S\}}-\frac{c_1}{(\Pr\{S\})^2}\right]=E[X \mid S]-c_1/\Pr\{S\}$,
and similarly for $M_2(u, v)$.

(ii) Suppose the cumulative distribution function (ab. by cdf) $H(x, y)$ has the probability density function (abbr. by pdf) $h(x, y)$. Then from (3)

$$(3')\qquad \begin{cases} M_1(u, v)=\dfrac{\int_u^\infty x dx \int_v^\infty h(x, y) dy - c_1}{\int_u^\infty dx \int_v^\infty h(x, y) dy}, \\[2ex] M_2(u, v)=\dfrac{\int_v^\infty y dy \int_u^\infty h(x, y) dx - c_2}{\int_v^\infty dy \int_u^\infty h(x, y) dx}. \end{cases}$$

Carrying through the computations $\frac{\partial M_1}{\partial u}=\frac{\partial M_2}{\partial v}=0$ we obtain

$$(4')\qquad \int_u^\infty (x-u) dx \int_v^\infty h(x, y) dy = c_1, \quad \int_v^\infty (y-v) dy \int_u^\infty h(x, y) dx = c_2$$

which is equivalent to (4). (3), combined with (4) and taking $(u, v)=(u^*, v^*)$, give (5), completing the proof.

For later use we shall rewrite the expressions (4) or (4′) in two different ways as follows: Let

$F[Y\geqq v]$=conditional cdf of X given that $Y\geqq v$

$G[X\geqq u]$=conditional cdf of Y given that $X\geqq u$.

For any cdf $K(z)$ of the random variable Z with finite mean $\int_{-\infty}^{\infty} zdK(z)$, define

$$T_K(t)=\int_t^{\infty}(z-t)dK(z)=E[(Z-t)^+] .$$

This function is non-negative, convex and strictly decreasing on the set where it is positive and has been known to play an important role in optimal stopping problems (DeGroot [1]).

Now dividing both sides of the first equation in (4′) by $\Pr\{Y\geqq v\}$ and those of the second equation by $\Pr\{X\geqq v\}$, we obtain

$$(4'') \qquad T_{F[Y\geqq v]}(u)=c_1/\Pr\{Y\geqq v\}, \qquad T_{G[X\geqq u]}(v)=c_2/\Pr\{X\geqq u\} .$$

Another way of expressions equivalent to (4′) are obtained by interchanging the order of integrations in (4′). Let

$F[y]$=conditional cdf of X given that $Y=y$,

$G[x]$=conditional cdf of Y given that $X=x$,

and let F and G be the marginal cdf's of X and Y, respectively. Then, from (4′) we obtain

$$(4''') \qquad \int_v^{\infty} T_{F[y]}(u)g(y)dy=c_1, \qquad \int_u^{\infty} T_{G[x]}(v)f(x)dx=c_2 ,$$

where f and g are the marginal pdf's of X and Y, respectively. If X and Y are independent, we have $F[Y\geqq v]=F[y]=F$, $G[X\geqq u]=G[x]=G$ and hence each of (4″) and (4‴) reduces to

$$(6) \qquad T_F(u)=c_1/(1-G(v)) , \qquad T_G(v)=c_2/(1-F(u)) .$$

This gives the following corollary.

(Corollary 2). Assume that X and Y are independent and let u^0 denote the optimal expected net gain from the observations on $\{X_i\}$, having no regard for Y_i's. Similarly, define v^0 as the optimal expected net gain from $\{Y_i\}$ only. Then we have $u^*<u^0$ and $v^*<v^0$.

Proof. From the well-known result in optimal stopping theory (see, for example [1; Sec. 13.4]) the neutral values u^0 and v^0 satisfies

$$(7) \qquad T_F(u^0)=c_1 \quad \text{and} \quad T_G(v^0)=c_2$$

respectively. These equations together with (6) and the strictly decreasing property of the functions T_F and T_G give $u^*<u^0$ and $v^*<v^0$.

3. Optimal Strategy for the Case Where the Number of Observations is Limited

We shall assume in this section that the experimenter knows n, the number of the total opportunities permitted to him. If he has not terminated the sampling until the final observation, then he is forced to choose this observed value. Also we set $c_1=c_2=0$.

We shall consider a class of stopping rules in which the experimenter has a set of neutral values $\{u_i\}_{i=1}^{n-1}$ and $\{v_i\}_{i=1}^{n-1}$, such that the sampling is terminated at the first m such that $X_m \geqq u_{n-m}$ and $Y_m \geqq v_{n-m}$. Here (X_m, Y_m) is the mth observed value from the beginning. If both of the above inequalities hold he stops sampling; and if not he continues it and observes (X_{m+1}, Y_{m+1}).

We use the abbreviated notations $u^{n-1}=\{u_i\}_{i=1}^{n-1}$ and so on. Let $\tau=\tau(u^{n-1}, v^{n-1})$ denote the stopping time when the set of neutral values u^{n-1} and v^{n-1} are used. Let

$$(8)\qquad \begin{cases} M_n^{(1)}(u^{n-1}, v^{n-1})=E[X_\tau \mid u^{n-1}, v^{n-1}] \\ M_n^{(2)}(u^{n-1}, v^{n-1})=E[Y_\tau \mid u^{n-1}, v^{n-1}]\,. \end{cases}$$

Then $M_n^{(1)}(u^{n-1}, v^{n-1})$ is the expected gain from the observations of X when using the set of neutral values u^{n-1} and v^{n-1}. $M_n^{(2)}(u^{n-1}, v^{n-1})$ has the similar meaning for the observations of Y.

We shall determine the set of the equilibrium neutral values $\{(u_i^*, v_i^*)\}_{i=1}^{n-1}$ as follows: First set $u_1^*=\mu_1\equiv E[X]$ and $v_1^*=\nu_1\equiv E[Y]$. After having determined the sequence of values $\{u_i^*\}_{i=1}^{m-1}$ and $\{v_i^*\}_{i=1}^{m-1}$ (abbreviated by u^{*m-1} and v^{*m-1}, respectively), let (u_m^*, v_m^*) be an equilibrium point of the pair of the functions $M_{m+1}^{(i)}(u^{*m-1}, u_m; v^{*m-1}, v_m)$, $i=1, 2$. More precisely, (u_m^*, v_m^*) satisfies

$$(9)\qquad \begin{cases} M^{(1)}_{m+1}(u^{*m},\ v^{*m})=\max\limits_{u_m} M^{(1)}_{m+1}(u^{*m-1},\ u_m;\ v^{*m}) \\ M^{(2)}_{m+1}(u^{*m},\ v^{*m})=\max\limits_{v_m} M^{(2)}_{m+1}(u^{*m};\ v^{*m-1},\ v_m)\,. \end{cases}$$

For any infinite sequences of numbers $\{u_i\}_1^\infty$, and $\{v_i\}_1^\infty$, we simply write $M_n^{(i)}(u^{n-1},\ v^{n-1})$ $(i=1,\ 2;\ n=1,\ 2,\ \cdots)$ as $M_n^{(i)}$. Also $M_n^{(i)*}$ means $M_n^{(i)}(u^{*n-1},\ v^{*n-1})$. Then we obtain:

Theorem 3:

(i) The expected gains $M_n^{(i)}$ $(i=1,\ 2)$ when using the set of neutral values u^{n-1} and v^{n-1} satisfy the recurrence relations

$$(10)\qquad \begin{cases} M^{(1)}_{n+1}=M_n^{(1)}+\displaystyle\int_{u_n}^\infty (x-M_n^{(1)})dx\int_{v_n}^\infty h(x,\ y)dy\ , \\ M^{(2)}_{n+1}=M_n^{(2)}+\displaystyle\int_{v_n}^\infty (y-M_n^{(2)})dy\int_{u_n}^\infty h(x,\ y)dx\ , \\ \qquad (n=1,\ 2,\ \cdots,\ M_1^{(1)}=\mu_1,\ M_1^{(2)}=\nu_1)\ . \end{cases}$$

(ii) Let $\{\mu_n\}_{n=1}^\infty$ and $\{\nu_n\}_{n=1}^\infty$ be the infinite sequences of numbers defined by the simultaneous recurrence relations

$$(11)\qquad \begin{cases} \mu_{n+1}=\mu_n+\displaystyle\int_{\mu_n}^\infty (x-\mu_n)dx\int_{\nu_n}^\infty h(x,\ y)dy\ , \\ \nu_{n+1}=\nu_n+\displaystyle\int_{\nu_n}^\infty (y-\nu_n)dy\int_{\mu_n}^\infty h(x,\ y)dx\ , \\ \qquad (n=1,\ 2,\ \cdots;\ \mu_1\equiv E[X],\ \nu_1\equiv E[Y])\ . \end{cases}$$

Then the successive equilibrium points $(u_n^*,\ v_n^*)$ satisfying (9) are given by

$$(12)\qquad u_n^*=\mu_n,\ v_n^*=\nu_n \qquad (n=1,\ 2,\ \cdots)$$

and moreover

$$(13)\qquad M^{(1)*}_{n+1}=\mu_{n+1},\ M^{(2)*}_{n+1}=\nu_{n+1}\ .$$

Proof. (i) We have

$$M^{(1)}_{n+1}=(1-\Pr\{X_1\geqq u_n,\ Y_1\geqq v_n\})M_n^{(1)}+\int_{u_n}^\infty x_1dx_1\int_{v_n}^\infty h(x_1,\ y_1)dy_1$$

and the similar expression for $M_{n+1}^{(2)}$.

(ii) We use the induction arguments. Since $u_1{}^*=\mu_1$ and $v_1{}^*=\nu_1$ by definition, (10) and (11) with $n=1$ give

$$M_2{}^{(1)*}=\mu_1+\int_{\mu_1}^{\infty}(x-\mu_1)dx\int_{\nu_1}^{\infty}h(x,\ y)dy=\mu_2$$

$$M_2{}^{(2)*}=\nu_1+\int_{\nu_1}^{\infty}(y-\nu_1)dy\int_{\mu_1}^{\infty}h(x,\ y)dx=\nu_2\ .$$

Hence (12) and (13) are valid for $n=1$. Assume that they are valid up to $n-1$. Then, by (10)

$$M_{n+1}^{(1)}(u^{*n-1},\ u_n;\ v^{*n})=M_n{}^{(1)*}+\int_{u_n}^{\infty}(x-M_n{}^{(1)*})dx\int_{v_n{}^*}^{\infty}h(x,\ y)dy$$

$$=\mu_n+\int_{u_n}^{\infty}(x-\mu_n)dx\int_{v_n{}^*}^{\infty}h(x,\ y)dy\ ,$$

$$M_{n+1}^{(2)}(u^{*n};\ v^{*n-1},\ v_n)=M_n{}^{(2)*}+\int_{v_n}^{\infty}(y-M_n{}^{(2)*})dy\int_{u_n{}^*}^{\infty}h(x,\ y)dx$$

$$=\nu_n+\int_{v_n}^{\infty}(y-\nu_n)dy\int_{u_n{}^*}^{\infty}h(x,\ y)dx\ .$$

Each of these attains its maximum at $u_n=\mu_n$ and $v_n=\nu_n$, respectively. Thus $u_n{}^*=\mu_n$ and $v_n{}^*=\nu_n$, showing that (12) is true for n. Therefore (10) with $u^n=u^{*n}$ and $v^n=v^{*n}$ gives

$$M_{n+1}^{(1)*}=\mu_n+\int_{\mu_n}^{\infty}(x-\mu_n)dx\int_{\nu_n}^{\infty}h(x,\ y)dy=\mu_{n+1}$$

$$M_{n+1}^{(2)*}=\nu_n+\int_{\nu_n}^{\infty}(y-\nu_n)dy\int_{\mu_n}^{\infty}h(x,\ y)dx=\nu_{n+1}$$

by (11). These show that (13) is true for n, thus completing the proof.

Part (ii) of the above theorem implies the following fact: the μ_n's and ν_n's defined by the simultaneous recursion (11) play two roles. First, they are the equilibrium neutral values for the $(n+1)$st observation from the end, and secondly, $(\mu_n,\ \nu_n)$ is the equilibrium expected gains for a play of the permitted length n. Calculating μ_n and ν_n recursively is relatively easy for some simple bivariate distributions, and we have carried it out for $n=1(1)10$. Table 1 at the end of this paper shows the numerical values.

As was done for the equations in (4′) in the previous section, it will be convenient to rewrite (11) as follows:

$$(11')\qquad \begin{cases}\mu_{n+1}=\mu_n+\Pr\{Y\geqq\nu_n\}T_{F[Y\geqq\nu_n]}(\mu_n)\\ \nu_{n+1}=\nu_n+\Pr\{X\geqq\mu_n\}T_{G[x\geqq\mu_n]}(\nu_n)\end{cases}$$

or

$$(11'')\qquad \mu_{n+1}=\mu_n+\int_{\nu_n}^{\infty}T_{F[y]}(\mu_n)g(y)dy,\quad \nu_{n+1}=\nu_n+\int_{\mu_n}^{\infty}T_{G[x]}(\nu_n)f(x)dx\ .$$

If X and Y are independent each other, each of (11′) and (11″) reduces to

$$(14)\qquad \mu_{n+1}=\mu_n+(1-G(\nu_n))T_F(\mu_n)\ ,\qquad \nu_{n+1}=\nu_n+(1-F(\mu_n))T_G(\nu_n)\ .$$

We prove the following two corollaries.

(Corollary 4). μ_n is non-decreasing and concave in n, and so is ν_n.

Proof. The non-decreasing property is evident from (11″). Also (11″) gives

$$\mu_{n+1}-2\mu_n+\mu_{n-1}=-\int_{\nu_{n-1}}^{\nu_n}T_{F[y]}(\mu_{n-1})g(y)dy$$
$$-\int_{\nu_n}^{\infty}\{T_{F[y]}(\mu_{n-1})-T_{F[y]}(\mu_n)\}g(y)dy.$$

Since $T_{F[y]}(z)$ is non-negative and non-increasing in z for every y, the righthand side of the above equation is non-positive. This proves the concavity of μ_n's.

(Corollary 5). Assume that X and Y are independent. Let u_n^0 denote the expected gain in the optimal play of the permitted length n based on the observations of X_i's, having no regard for Y_i's. Define v_n^0 similarly for the observations of Y_i's, having no regard for X_i's. Then we have $\mu_n\leqq\mu_n^0$ and $\nu_n\leqq\nu_n^0$ for $n\geqq1$.

Proof. From the well-known result in optimal stopping theory (see, for example, [2; Section 5]) the optimal expected gains $\mu_n{}^0$ and $\nu_n{}^0$ satisfy

$$\mu_{n+1}^0=\mu_n{}^0+T_F(\mu_n{}^0)\,,\qquad \nu_{n+1}^0=\nu_n{}^0+T_G(\nu_n{}^0) \tag{15}$$
$$(n=1,2,\cdots;\ \mu_1{}^0=E[X],\ \nu_1{}^0=E[Y])$$

respectively. Obviously $\mu_1=\mu_1{}^0=E[X]$ and $\nu_1=\nu_1{}^0=E[Y]$. Assume that $\mu_n\leqq\mu_n{}^0$ and $\nu_n\leqq\nu_n{}^0$. Then we get by (14) and (15)

$$\mu_{n+1}\leqq\mu_n+T_F(\mu_n)\leqq\mu_n{}^0+T_F(\mu_n{}^0)=\mu_{n+1}^0$$
$$\nu_{n+1}\leqq\nu_n+T_G(\nu_n)\leqq\nu_n{}^0+T_G(\nu_n{}^0)=\nu_{n+1}^0\,,$$

since both of the functions $z+T_F(z)$ and $z+T_G(z)$ are non-decreasing in z. Thus we have completed the proof.

4. Examples

The behaviors of u^* and v^* in Section 2 and μ_n and ν_n in Section 3 depend on the distribution of the observations in two respects, namely, the shapes of upper tails and the degree of dependence between the two component variables. Consequently, to give variety for tails of distributions and degree of dependence, the following examples are worked out by three kinds of bivariate distributions: the uniform, the normal and the mixed-type.

Example 1. Bivariate uniform distribution: There exist infinitely many bivariate distributions with a given pair of component distributions. Let $f(x)$ and $g(y)$ be two given pdf's. A class of bivariate densities with given marginal densities $f(x)$ and $g(y)$ is given by

$$h(x,y)=f(x)g(y)\{1+\gamma(1-2F(x))(1-2G(y))\} \tag{16}$$

where F and G are the corresponding cdf's and γ is an arbitrary constant satisfying $-1\leqq\gamma\leqq1$ (Gumbel [3]). It is easy to check that the bivariate cdf is given by

$$H(x,y)=F(x)G(y)\{1+\gamma(1-F(x))(1-G(y))\}$$

and that x and y are independent if and only if $\gamma=0$. This class of bivariate distributions is theoretically important because of its simple structure and the fact that the constant γ actually measures the degree of dependency between the component variables, independently of $f(\cdot)$ and $g(\cdot)$ (Sakaguchi [7]).

Now for this bivariate distribution we have

$$h(x \mid y)=h(x, y)/g(y)=f(x)\{1+\gamma(1-2F(x))(1-2G(y))\}$$

$$T_{F[y]}(u)=\int_u^{\infty}(x-u)h(x \mid y)dx$$

$$=T_F(u)+\gamma(1-2G(y))\{T_F(u)-T_{F^2}(u)\}$$

where $F^2(x)=\{F(x)\}^2$, *i.e.* F^2 is the cdf of the maximum of the two independent random variables each having the common cdf F. Hence we obtain

(17) $$\int_v^{\infty}T_{F[y]}(u)g(y)dy=(1-G(v))[T_F(u)-\gamma G(v)\{T_F(u)-T_{F^2}(u)\}]$$

and similar expressions for $T_{G[x]}(v)$ and $\int_u^{\infty}T_{G[x]}(v)f(x)dx$.

For bivariate uniform distribution put $F(x)=G(x)=x$ for $0\leqq x\leqq 1$. Substituting

$$T_F(u)=\frac{1}{2}(1-u)^2,$$

and

$$T_{F^2}(u)\equiv\int_u^1(x-u)d(x^2)=2\left(\frac{1}{3}-\frac{u}{2}+\frac{u^3}{6}\right)$$

$(0\leqq u\leqq 1)$ into (17) we get

$$\int_v^1 T_{F[y]}(u)dy=(1-v)(1-u)^2\left\{\frac{1}{2}+\frac{\gamma}{6}v(2u+1)\right\}.$$

If $c_1=c_2=c$ we should have $u^*=v^*$ because of symmetry and hence (4′′′) becomes to

(18) $$(1-u)^3\left\{\frac{1}{2}+\frac{\gamma}{6}u(2u+1)\right\}=c\,.$$

The lefthand side of this equation is strictly decreasing over $0\leqq u\leqq 1$. Therefore for any $0<c<1/2$ the equation (18) has a unique root u^* in

$0<u<1$.

Similarly, (11′), (17) and consideration of symmetry give $\mu_n=\nu_n$ $(n=1, 2, \cdots)$ and

$$\mu_{n+1}=\mu_n+(1-\mu_n)^3\left\{\frac{1}{2}+\frac{r}{6}\mu_n(2\mu_n+1)\right\}, \tag{19}$$

$$(n=1, 2, \cdots;\ \mu_1=1/2)\ .$$

Table 1 shows some computed values of μ_n given by (19) compared with those of $\mu_n{}^0$ determined by (15), *i.e.*,

$$\mu_{n+1}^0=\mu_n{}^0+T_F(\mu_n{}^0)=\frac{1}{2}(1+\mu_n{}^{02})\ , \qquad (n=1, 2, \cdots;\ \mu_1{}^0=1/2)\ .$$

Example 2. Bivariate normal distribution:

$$h(x, y)=\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right\}$$

where ρ, $-1\leqq\rho\leqq1$, is the correlation coefficient. Let

$$\phi(x)\equiv(2\pi)^{-1/2}e^{-x^2/2}, \quad \Phi(x)\equiv\int_x^\infty\phi(t)dt, \quad \Psi(x)\equiv\phi(x)-x\Phi(x).$$

Then since

$$h(x, y)=\phi(x)\cdot\frac{1}{\sqrt{1-\rho^2}}\phi\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)=\phi(y)\cdot\frac{1}{\sqrt{1-\rho^2}}\phi\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right),$$

we have

$$\begin{cases} T_{F[y]}(u)=\sqrt{1-\rho^2}\ \Psi\left(\dfrac{u-\rho y}{\sqrt{1-\rho^2}}\right), \\ T_{G[x]}(v)=\sqrt{1-\rho^2}\ \Psi\left(\dfrac{v-\rho x}{\sqrt{1-\rho^2}}\right), \end{cases} \tag{20}$$

so that, from (4‴)

$$\sqrt{1-\rho^2}\int_v^\infty\Psi\left(\frac{u-\rho y}{\sqrt{1-\rho^2}}\right)\phi(y)dy=c_1\ ,$$

$$\sqrt{1-\rho^2}\int_u^\infty\Psi\left(\frac{v-\rho x}{\sqrt{1-\rho^2}}\right)\phi(x)dx=c_2\ .$$

If $c_1=c_2=c$, it follows by symmetry that $u^*=v^*$ and u^* satisfies the equation

(21) $$\sqrt{1-\rho^2}\int_u^\infty \Psi\left(\frac{u-\rho t}{\sqrt{1-\rho^2}}\right)\phi(t)dt=c\ .$$

We easily find that this equation has a unique root for any $c>0$. In the case of independence, *i.e.*, $\rho=0$, (21) becomes to

(22) $$\Phi(u)\Psi(u)=c\ .$$

We note here that $u^*>0$ if and only if $0<c<(8\pi)^{-1/2}$. Hence the process is advantageous to the experimenter, in the sense that his expected gain is positive in both of its components, if and only if $c<(8\pi)^{-1/2}$.

On the other hand, in the case of the finite length of the process, (11''), (20) and symmetry give $\mu_n=\nu_n$ $(n=1, 2, \cdots)$ and

$$\mu_{n+1}=\mu_n+\sqrt{1-\rho^2}\int_{\mu_n}^\infty \Psi\left(\frac{\mu_n-\rho y}{\sqrt{1-\rho^2}}\right)\phi(y)dy \qquad (n=1, 2, \cdots;\ \mu_1=0)\ .$$

This reduces to, if $\rho=0$,

(23) $$\mu_{n+1}=\mu_n+\Phi(\mu_n)\Psi(\mu_n)\ .$$

Table 1 shows some computed values of μ_n given by (23) compared with those of $\mu_n{}^0$ determined by (15), *i.e.*,

$$\mu_{n+1}^0=\Psi(\mu_n{}^0)\ . \qquad (n=1, 2, \cdots;\ \mu_1{}^0=0)\ .$$

Example 3. Mixed-type bivariate distribution:

$$h(x, y)=\begin{cases}\phi(y)\ , & \text{if } 0\leqq x\leqq 1\\ 0\ , & \text{if otherwise}\ ,\end{cases}$$

i.e., X is uniformly distributed over the unit interval and Y is standard-normally distributed, both of them being independent. We have from (6) and (14)

$$\begin{cases}\frac{1}{2}(1-u)^2\Phi(v)=c_1\\ (1-u)\Psi(v)=c_2\ ,\end{cases}$$

and

$$(24)\quad \begin{cases} \mu_{n+1}=\mu_n+\frac{1}{2}(1-\mu_n)^2\Phi(\nu_n) \\ \nu_{n+1}=\nu_n+(1-\mu_n)\Psi(\nu_n), \end{cases} \quad \left(n=1, 2, \cdots;\ \mu_1=\frac{1}{2},\ \nu_1=0\right).$$

Table 1 shows some computed values of μ_n and ν_n determined by (24).

Table 1. Optimum neutral values for uniform, normal and mixed-type bivariate distributions.

n	Uniform				Normal		Mixed	
	μ_n^0	μ_n			μ_n^0	μ_n	μ_n	ν_n
		$\gamma=-\frac{1}{2}$	$\gamma=0$	$\gamma=\frac{1}{2}$				
1	.5000	.5000	.5000	.5000	.0000	.0000	.5000	.0000
2	.6250	.5520	.5625	.5729	.3989	.1995	.5625	.1995
3	.6953	.5901	.6044	.6198	.6297	.3295	.6028	.3338
4	.7417	.6172	.6353	.6536	.7904	.4242	.6319	.4346
5	.7751	.6388	.6596	.6795	.9127	.4987	.6544	.5151
6	.8004	.6566	.6793	.7000	1.0108	.5597	.6725	.5819
7	.8203	.6718	.6958	.7173	1.0924	.6115	.6875	.6388
8	.8364	.6848	.7099	.7320	1.1621	.6564	.7003	.6882
9	.8498	.6962	.7221	.7445	1.2227	.6959	.7113	.7319
10	.8611	.7063	.7328	.7553	1.2762	.7310	.7210	.7710

(Values of μ_n^0 for uniform and normal distributions were reproduced from Table 13 of [2]. Computations of values of μ_n for normal distribution and ν_n for mixed-type distribution, were performed by exploiting Table II of Raiffa and Schlaifer [6].)

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