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THE ORDER OF n ITEMS PROCESSED ON m MACHINES [III]

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Abstract

Sufficient conditions are presented for determining the definite order of adjacent two jobs (items) for min-makespan problem in flow shop where no passing is allowed, by using dynamic programming formulation. In relation to them, their applicabilities to the algorithms for obtaining the optimal sequence are mentioned.

Finally algorithms for obtaining the approximate sequence are presented together with numerical example. It is obtained satisfying approximate sequence.

1. Introduction

Recently many branch-and-bound type algorithms [1]-[5] for obtaining the optimal schedule for min-makespan problem in job shop have been presented. On the other hand, there exist special types of algorithms [6]-[12] for this problem in flow shop for each case according to whether passing is allowed or not, based on the structure of the flow shop. Especially the problem for three machines case in flow shop can be assumed that no passing is allowed which makes the analysis easier.

This paper presents certain analytic devices which may be efficient to use together with those algorithms in such flow shop.

That is the set of sufficient conditions for determining the definite order of adjacent two jobs regardless of their position in optimal sequence, which is derived by dynamic programming formulation. First sufficient conditions for three machines case are presented and its generalizations to m machines case follow. Finally efficient algorithms for obtaining the approximate sequence for the problem are presented together with numerical example.

2. Elapsed Times $T_k(i_q)$

Let $\omega=i_1i_2\cdots i_{n-1}i_n$ be any sequence of n jobs and $T_k(i_q)$ be the elapsed time of job i_q counted from the completion time of this job on first machine M_1 to that of this job on kth machine M_k $(q=1\sim n, k=2\sim m)$, that is, $T_k(i_q)$ is the time committed ahead on M_k to M_1 by the processing of job i_q. Then total elapsed time (makespan) $TE(\omega)$ is expressed by

$$TE(\omega)=\sum_{q=1}^{n} p_{i_q,1}+T_m(i_n) \tag{2.1}$$

where $p_{i,j}$ means the processing time of job i on machine M_j.

Since the first term of (2.1) is constant for any sequence ω, hence, in order to obtain the optimal sequence, it is sufficient to seek for the sequence which has minimum $T_m(i_n)$ among all sequences.

On the other hand, next recurrence relations hold for $T_k(i_q)$:

$$T_k(i_q)=p_{i_q,k}+\max[T_k(i_{q-1})-p_{i_q,1},\ T_{k-1}(i_q)] \quad (q=1\sim n,\ k=2\sim m) \tag{2.2}$$

where $T_k(i_0)\equiv 0$, $T_1(i_q)\equiv 0$, by the reason that the starting time of the processing of job i_q on M_k is not sooner than both completion time of job i_{q-1} on M_k and completion time of job i_q on M_{k-1} under taking the origin of time scale as the completion time of job i_q on M_1.

From these recurrence relations, we see that each $T_k(i_q)$ is an increasing function of $T_k(i_{q-1})$ and $T_{k-1}(i_q)$ ($q=1\sim n$, $k=2\sim m$).

Also, by using these relations successively, we can compute total elapsed time (makespan) of any sequence.

3. Order of Adjacent Two Jobs in Optimal Sequence

When optimal scheduling procedure is employed and after the processing of certain definite subsequence S with last job l, let two jobs i, j be processed after S.

If two jobs i, j are processed in this order after S, then next recurrence relations on $T_k(i)$ and $T_k(ij)\equiv T_k(j)$ hold from (2.2) where notation $T_k(ij)$ will be used to show the order of i, j and in general $T_k(S)$ means $T_k(l)$ of last job l of the sequence S:

(3.1) $$T_k(i)=p_{i,k}+\max[T_k(l)-p_{i,1},\ T_{k-1}(i)],$$

(3.2) $$T_k(ij)=p_{j,k}+\max[T_k(i)-p_{j,1},\ T_{k-1}(ij)]$$

$$(k=2\sim m)$$

where $T_1(i)\equiv 0$, $T_1(ij)\equiv 0$ and $T_k(l)\equiv 0$ when i is a first job.

If we interchange the order of i and j, then recurrence relations which can be easily obtained by exchanging i and j in (3.1) and (3.2) hold.

Then, since $T_k(i)$, $T_k(ij)$ say is an increasing function of $T_k(l)$ and $T_{k-1}(i)$, $T_k(i)$ and $T_{k-1}(ij)$ respectively ($k=2\sim m$) and optimal sequence has minimum $T_m(i_n)$ among all sequences with last job i_n, if $(m-1)$ inequalities $T_k(ij)\leqq T_k(ji)$ ($k=2\sim m$) hold, next theorem obviously holds: (cf. Theorem 1 in [17])

Theorem 1. When optimal scheduling procedure is employed and after the processing of definite subsequence S with last job l, let two jobs i, j be processed after S. Then if $(m-1)$ inequalities

(3.3) $$T_k(ij)\leqq T_k(ji) \qquad (k=2\sim m)$$

hold, we can regard that job i precedes job j after S in order to construct the optimal sequence.

4. Constant Order of Definite Adjacent Two Jobs for Three Machines Case

In the following sections 4 and 5, they will be shown sufficient conditions followed from theorem 1 to decide the constant order of any two definite adjacent jobs regardless of their position in order to construct the optimal sequence for m machines ($m \geqq 2$) where they coincide with Johnson's criterion for two machines case ($m=2$) [15].

In order to make clear the content and process of the demonstration of these sufficient conditions, first we shall treat three machines case ($m=3$) in this section.

4.1 Proposed Sufficient Conditions

By using the recurrence relations (3.1), (3.2) for $k=2$, 3 and putting $T_2(i)$ into $T_2(ij)$ and $T_3(i)$, we have

(4.1) $$T_2(ij)=p_{j,2}+p_{i,2}-p_{i,1}-p_{j,1} +\max[T_2(l),\ p_{i,1},\ p_{i,1}+p_{j,1}-p_{i,2}]$$

(4.2) $$=p_{i,3}-p_{i,1}-p_{j,1}+\max[T_2(l)+p_{i,2}+p_{j,2}-p_{i,3},\ p_{i,1} +p_{i,2}+p_{j,2}-p_{i,3},\ p_{i,1}+p_{j,1}+p_{j,2}-p_{i,3}]\ ,$$

(4.3) $$T_3(i)=p_{i,3}-p_{i,1}+\max[T_3(l),\ T_2(l)+p_{i,2},\ p_{i,2}+p_{i,1}]\ .$$

Then, by putting (4.2) and (4.3) into $T_3(ij)$, we have

(4.4) $$T_3(ij)=p_{j,3}+p_{i,3}-p_{i,1}-p_{j,1}+\max[T_3(l),\ T_2(l)+p_{i,2},\ T_2(l)+p_{i,2}+p_{j,2}-p_{i,3},\ p_{i,2}+p_{i,1},\ p_{i,1}+p_{i,2}+p_{j,2}-p_{i,3},\ p_{i,1}+p_{j,1}+p_{j,2}-p_{i,3}]\ .$$

By the same way we have

(4.5) $$T_2(ji)=p_{i,2}+p_{j,2}-p_{j,1}-p_{i,1} +\max[T_2(l),\ p_{j,1},\ p_{j,1}+p_{i,1}-p_{j,2}]\ ,$$

(4.6) $$T_3(ji)=p_{i,3}+p_{j,3}-p_{j,1}-p_{i,1} +\max[T_3(l),\ T_2(l)+p_{j,2},\ T_2(l)+p_{j,2}+p_{i,2}-p_{j,3},\ p_{j,2}+p_{j,1},\ p_{j,1}+p_{j,2}+p_{i,2}-p_{j,3},\ p_{j,1}+p_{i,1}+p_{i,2}-p_{j,3}]\ .$$

Hence, if inequality

(4.7) $\max[p_{i,1}\ p_{i,1}+p_{j,1}-p_{i,2}]\leqq\max[p_{j,1},\ p_{j,1}+p_{i,1}-p_{j,2}]$

holds, inequality $T_2(ij)\leqq T_2(ji)$ follows.

Inequality (4.7) is equivalent to

(4.8) $\min[p_{i,1},\ p_{j,2}]\leqq \min[p_{j,1},\ p_{i,2}]$.

Next, it follows from $T_3(ij)\leqq T_3(ji)$

(4.9) $$\max[T_3(l),\ \max\{T_2(l)+p_{i,2}, T_2(l)+p_{i,2}+p_{j,2}-p_{i,3}\},$$
$$\max\{p_{i,2}+p_{i,1},\ p_{i,1}+p_{i,2}+p_{j,2}-p_{i,3},\ p_{i,1}+p_{j,1}+p_{j,2}-p_{i,3}\}]$$
$$\leqq\max[T_3(l),\ \max\{T_2(l)+p_{j,2},\ T_2(l)+p_{j,2}+p_{i,2}-p_{j,3}\},$$
$$\max\{p_{j,2}+p_{j,1},\ p_{j,1}+p_{j,2}+p_{i,2}-p_{j,3}\ p_{j,1}+p_{i,1}+p_{i,2}-p_{j,3}\}] .$$

So if both inequalities

(4.10) $$\max[T_2(l)+p_{i,2},\ T_2(l)+p_{i,2}+p_{j,2}-p_{i,3}]$$
$$\leqq\max[T_2(l)+p_{j,2},\ T_2(l)+p_{j,2}+p_{i,2}-p_{j,3}]$$

and

(4.11) $$\max[p_{i,2}+p_{i,1},\ p_{i,1}+p_{i,2}+p_{j,2}-p_{i,3},\ p_{i,1}+p_{j,1}+p_{j,2}-p_{i,3}]$$
$$\leqq\max[p_{j,2}+p_{j,1},\ p_{j,1}+p_{j,2}+p_{i,2}-p_{j,3},\ p_{j,1}+p_{i,1}+p_{i,2}-p_{j,3}]$$

hold, then (4.9), that is, $T_3(ij)\leqq T_3(ij)$ follows.

Here, (4.10) follows from inequality

$$\max[p_{i,2},\ p_{i,2}+p_{j,2}-p_{i,3}]\leqq\max[p_{j,2},\ p_{j,2}+p_{i,2}-p_{j,3}]$$

which is equivalent to

(4.12) $\min[p_{i,2},\ p_{j,3}]\leqq\min[p_{j,2},\ p_{i,3}]$.

And, by subtracting the quantity $p_{i,1}+p_{j,1}+p_{i,2}+p_{j,2}$ from both sides of (4.11), we have altimately

(4.13) $$\min[p_{i,1}+p_{i,2},\ p_{i,1}+p_{j,3},\ p_{j,2}+p_{j,3}]$$
$$\leqq\min[p_{j,1}+p_{j,2},\ p_{j,1}+p_{i,3},\ p_{i,2}+p_{i,3}] .$$

Hence next theorem holds from theorem 1:

Theorem 2. If three inequalities

(4.8) $\min[p_{i,1},\ p_{j,2}]\leqq\min[p_{j,1},\ p_{i,2}]$,

(4.12) $\min[p_{i,2},\ p_{j,3}]\leqq\min[p_{j,2},\ p_{i,3}]$,

(4.13) $$\min[p_{i,1}+p_{i,2},\ p_{i,1}+p_{j,3},\ p_{j,2}+p_{j,3}]$$
$$\leqq\min[p_{j,1}+p_{j,2},\ p_{j,1}+p_{i,3},\ p_{i,2}+p_{i,3}]$$

hold, then always job i must precede job j, for adjacent two jobs i, j, regardless of their position in order to construct the optimal sequence. In these three inequalities, transitive property does not hold for (4.13). The theorem with three inequalities as sufficient conditions, each of them has transitive property, is obtained by using next lemma which will be also applied to similar theorem for general m machines case in section 5.1.

Lemma. If three inequalities

(4.14) $\min[A_1, A_2] \leqq \min[A_3, A_4]$,

(4.15) $\min[B_1, B_2] \leqq \min[B_3, B_4]$,

(4.16) $\min[A_1+B_1, A_2+B_2] \leqq \min[A_3+B_3, A_4+B_4]$

hold, then inequality

(4.17) $\min[A_1+B_1, A_1+B_2, A_2+B_2]$
$\leqq \min[A_3+B_3, A_3+B_4, A_4+B_4]$

follows except the case (P) where four inequalities

$$A_2 < A_1,\ A_3 < \min[A_1, A_4],\ B_1 < B_2,\ B_4 < \min[B_2, B_3]$$

hold.

Proof. Let $\underline{A} = \min[A_1, A_2]$, $\bar{A} = \min[A_3, A_4]$, $\underline{B} = \min[B_1, B_2]$,
$\bar{B} = \min[B_3, B_4]$.

In case where $\underline{A} = A_1$, $\underline{B} = B_1$; or $\underline{A} = A_1$, $\underline{B} = B_2$; or $\underline{A} = A_2$, $\underline{B} = B_2$, $A_1 + B_1$ or $A_1 + B_2$ or $A_2 + B_2$ is a smallest term among six terms in (4.17) and then (4.17) holds. In each case, inequality (4.16) can be omitted from conditions.

Then we devide the remained case where $\underline{A} = A_2 < A_1$, $\underline{B} = B_1 < B_2$ into subsequent cases. If $A_3 \geqq A_1$ or $B_4 \geqq B_2$, then it holds $A_3 + B_4 \geqq A_1 + B_4 \geqq A_1 + B_1$ or $A_3 + B_4 \geqq A_3 + B_2 \geqq A_2 + B_2$ respectively which shows that (4.16) coincides with (4.17). In case where $\bar{A} = A_3$, $\bar{B} = B_3$, they hold $A_2 + B_2 < A_1 + B_2$, $A_3 + B_3 \leqq A_3 + B_4$ and then (4.16) coincides with (4.17). In case where $\bar{A} = A_4$, $\bar{B} = B_3$, they hold $A_2 + B_2 < A_1 + B_2$, $A_4 + B_4 \leqq A_3 + B_4$ and then (4.16) coincides with (4.17). In case where $\bar{A} = A_4$, $\bar{B} = B_4$, also they hold $A_2 + B_2 < A_1 + B_2$, $A_4 + B_4 \leqq A_3 + B_4$ and then

(4.16) coincides with (4.17).

So that, remaining case is the case (P) which does not always derive (4.17). Q.E.D.

Example of case (P) not leading to (4.17)

$$\left.\begin{array}{r} \min[6,\ 2]<\min[3,\ 5] \\ \min[2,\ 6]<\min[7,\ 4] \\ \min[6+2,\ 2+6]<\min[3+7,\ 5+4] \end{array}\right\} \begin{array}{l} \rightarrow \min[6+2,\ 6+6,\ 2+6] \\ \not< \min[3+7,\ 3+4,\ 5+4]. \end{array}$$

By this lemma, we obtain next theorem from theorem 2:

Theorem 3. If three inequalities

(4.8) $\min[p_{i,1},\ p_{j,2}]\leqq\min[p_{j,1},\ p_{i,2}]$,

(4.12) $\min[p_{i,2},\ p_{j,3}]\leqq\min[p_{j,2},\ p_{i,3}]$,

(4.18) $\min[p_{i,1}+p_{i,2},\ p_{j,2}+p_{j,3}]\leqq\min[p_{j,1}+p_{j,2},\ p_{i,2}+p_{i,3}]$

hold, then the same conclusion as that in theorem 2 follows.

When there is equality in each of these inequalities, either ordering is optimal. This working rule is called as *general working rule.* (cf. 4.2)

Proof. In lemma they hold $A_2=B_3$ and $A_4=B_1$. If the case (P) holds, then they hold $A_2\leqq A_3<A_4$ from (4.14) and $B_1\leqq B_4<B_3$ from (4.15). The latter inequality becomes $A_4\leqq B_4<A_2$ by $A_2=B_3$, $A_4=B_1$ which contradicts with the former. Then the case (P) does not occur.

Q.E.D.

4.2 Remarks on Johnson Working Rule

(1) Johnson has stated, in his first paper about optimal sequencing on two and three machines [15], that a sequence which is simultaneously optimal both for first two machines M_1, M_2 and for last two machines M_2, M_3 in three machines case becomes optimal for three machines M_1, M_2, M_3. This fact is true under his working rule [15] for deciding only one definite optimal sequence for two machines in case tie. That is, next theorem 4 holds which can be proved easily by showing that (4.13) holds for each case where $\underline{A}$ and $\underline{B}$

take possible values. (cf. proof of theorem 5 in next section).

Theorem 4. If job i simultaneously precedes jobs j both on M_1, M_2 and on M_2, M_3 under *Johnson working rule* for deciding one definite optimal sequence for two machines by using

(4.8) $\min[p_{i,1}, p_{j,2}] \leqq \min[p_{j,1}, p_{i,2}]$,

(4.12) $\min[p_{i,2}, p_{j,3}] \leqq \min[p_{j,2}, p_{i,3}]$

respectively, then job i must precede job j under transitive property in order to construct the optimal sequence for three machines M_1, M_2, M_3.

(2) There exist the cases where only one of the above two theorems 3, 4 can derive the optimal sequence.

Then general theorem with detailed sufficient conditions, which includes both theorem 3 and theorem 4 as special cases will be shown in next section.

4.3 General Theorem

Theorem 5. I. In case where $p_{j,2} \geqq p_{i,1}$ or $p_{i,2} \geqq p_{j,3}$ holds, if two inequalities

(4.8) $\min[p_{i,1}, p_{j,2}] \leqq \min[p_{j,1}, p_{i,2}]$,

(4.12) $\min[p_{i,2}, p_{j,3}] \leqq \min[p_{j,2}, p_{i,3}]$

hold, then job i must precede job j under transitive property in order to construct the optimal sequence for three machines. When all equalities hold, either ordering is optimal.

II. In case where both $p_{j,2} < p_{i,1}$ and $p_{i,2} < p_{j,3}$ hold, if three inequalities (4.8), (4.12) (they yield $p_{i,2} = p_{j,2}$) and

(4.18) $\min[p_{i,1}+p_{i,2}, p_{j,2}+p_{j,3}] \leqq \min[p_{j,1}+p_{j,2}, p_{i,2}+p_{i,3}]$

hold, then the same conclusions as that in case I hold.

Proof. By theorem 3, it is necessary only to prove the case I. In case I, they hold $\underline{A} = p_{i,1}$ and $\underline{B} = p_{i,2}$, or $\underline{A} = p_{i,1}$ and $\underline{B} = p_{j,3}$, or $\underline{A} = p_{j,2}$ and $\underline{B} = p_{j,3}$. Hence (4.13) in theorem 2 follows. Q.E.D.

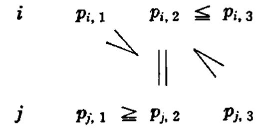

Fig. 1. Case II.

Conditions in case II is schematically shown in Fig. 1. Under the conditions in theorem 4 where *Johnson working rule* is applied, the case II in theorem 5 does not occur because equalities hold in (4.8) and (4.12) and job j precedes job i for M_1, M_2 when i is smaller than j and job j precedes job i for M_2, M_3 when j is smaller than i. Hence theorem 5 includes theorem 4. And obviously theorem 5 includes theorem 3. Theorem 5 is used as a criterion in algorithm for determining the optimal sequence for three machines.

5. Constant Order of Definite Adjacent Two Jobs for General *m* Machines Case

In this section, generalization of each theorem at former section 4 to m machines case ($m \geqq 2$) will be presented.

5.1. Generalization of Theorem 3

By using mathematical induction and lemma at section 4.1, we can generalize theorem 3 which shows sufficient conditions with transitive property for deciding constant order of adjacent two jobs i, j for three machines, to m machines case ($m \geqq 2$) in order to construct the optimal sequence. The generalized theorem is stated as below:

Theorem 6. If $m(m-1)/2$ inequalities

$$\min[p_{i,k},\ p_{j,k+1}] \leqq \min[p_{j,k},\ p_{i,k+1}], \quad (k=1 \sim m-1) \tag{5.1}$$

$$\min\left[\sum_{k=u}^{v} p_{i,k},\ \sum_{k=u+1}^{v+1} p_{j,k}\right] \leqq \min\left[\sum_{k=u}^{v} p_{j,k},\ \sum_{k=u+1}^{v+1} p_{i,k}\right] \quad (1 \leqq u < v \leqq m-1) \tag{5.2}$$

hold, then $T_k(ij) \leqq T_k(ji)$ $(k=2 \sim m)$ hold regardless of the value of $T_k(l)$ $(k=2 \sim m)$ and so always job i must precede job j for adjacent two jobs i, j regardless of their position in order to construct the optimal sequence. When there exist equalities in all inequalities, either ordering is optimal.

Proof.

I. Sufficient conditions to lead $T_k(ij) \leqq T_k(ji)$, $(k=2 \sim m)$.

When optimal scheduling procedure is employed and after the processing of certain definite subsequence S with last job l, let two jobs i and j be processed after S.

In case where these two jobs are processed by the order ij, they hold recurrence relations (3.1), (3.2). By successively putting $T_{q-1}(i)$ into $T_q(i)$ and putting $T_q(i)$, $T_{q-1}(ij)$ into $T_q(ij)$ $(q=2 \sim k)$, finally it holds

$$\begin{aligned} T_k(ij) = \max\Bigg[T_k(l),\ \max_{r=0\sim k-2} \Bigg\{ & T_{k-1-r}(l) + \sum_{q=k-1-r}^{k-1} p_{i,q},\ T_{k-1-r}(l) \\ & + p_{i,k-1} + p_{j,k-1} - p_{i,k} + \sum_{q=k-1-r}^{k-2} p_{i,q},\ T_{k-1-r}(l) \\ & + p_{i,k-2} + \sum_{q=k-2}^{k-1} p_{j,q} - p_{i,k} + \sum_{q=k-1-r}^{k-3} p_{i,q},\ T_{k-1-r}(l) \\ & + p_{i,k-3} + \sum_{q=k-3}^{k-1} p_{j,q} - p_{i,k} + \sum_{q=k-1-r}^{k-4} p_{i,q},\ \cdots,\ T_{k-1-r}(l) \\ & + p_{i,k-r} + \sum_{q=k-r}^{k-1} p_{j,q} - p_{i,k} + p_{i,k-1-r},\ T_{k-1-r}(l) \\ & + p_{i,k-1-r} + \sum_{q=k-1-r}^{k-1} p_{j,q} - p_{i,k} \Bigg\} \Bigg] \end{aligned} \tag{5.3}$$

$$= \max\Bigg[T_k(l),\ \max_{r=0\sim k-2}\ \max_{t=-1\sim r} \big(T_{k-r-1}(l) + p_{i,k-t-1}$$

$$+\sum_{q=k-t-1}^{k-1} p_{j,q}-p_{i,k}+\sum_{q=k-r-1}^{k-t-2} p_{i,p})\Big]$$

$$\equiv \max\Big[T_k(l), \max_{r=0\sim k-2} A(ij, t, r, k)\Big]$$

$$(k=2\sim m)$$

where $T_1(l)=0$, $\sum_{q=u}^{v}=0 \quad (u>v)$.

By the same way it holds for the order ji

$$T_k(ji)=\max\Big[T_k(l), \max_{r=0\sim k-2}\ \max_{t=-1\sim r}(T_{k-r-1}(l)+p_{j,k-t-1} \tag{5.4}$$

$$+\sum_{q=k-t-1}^{k-1} p_{i,q}-p_{j,k}+\sum_{q=k-r-1}^{k-t-2} p_{j,q})\Big]$$

$$\equiv \max\Big[T_k(l), \max_{r=0\sim k-2} A(ji, t, r, k)\Big].$$

$$(k=2\sim m)$$

Hence if inequality

$$\max_{r=0\sim k-2} A(ij, t, r, k) \leqq \max_{r=0\sim k-2} A(ji, t, r, k) \tag{5.5}$$

holds, then it becomes

$$T_k(ij) \leqq T_k(ji), \qquad (k=2\sim m). \tag{5.6}$$

Also in order to show the existence of (5.5), it is sufficient to show that next $(k-1)$ inequalities hold for all r $(r=0\sim k-2)$:

$$A(ij, t, r, k) \leqq A(ji, t, r, k). \tag{5.7}$$

Since

$$A(ij, t, r, k) = T_{k-r-1}(l)+\max_{t=-1\sim r}(p_{i,k-t-1}+\sum_{q=k-t-1}^{k-1} p_{j,q}-p_{i,k}$$

$$+\sum_{q=k-r-1}^{k-t-2} p_{i,q}) \equiv T_{k-r-1}(l)+B(ij, t, r, k)$$

say, if they hold

$$B(ij, t, r, k) \leqq B(ji, t, r, k);$$

that is, if they hold

$$(5.8)\qquad \max\Bigg[\sum_{q=k-r-1}^{k-1} p_{i,q},\ \max_{t=0\sim r-1}\Bigg(p_{i,k-t-1}+\sum_{q=k-t-1}^{k-1} p_{j,q} -p_{i,k}+\sum_{q=k-r-1}^{k-t-2} p_{i,q}\Bigg),\ p_{i,k-r-1}+\sum_{q=k-r-1}^{k-1} p_{j,q}-p_{i,k}\Bigg]$$
$$\leqq\max\Bigg[\sum_{q=k-r-1}^{k-1} p_{j,q},\ \max_{t=0\sim r-1}\Bigg(p_{j,k-t-1}+\sum_{q=k-t-1}^{k-1} p_{i,q} -p_{j,k}+\sum_{q=k-r-1}^{k-t-2} p_{j,q}\Bigg),\ p_{j,k-r-1}+\sum_{q=k-r-1}^{k-1} p_{i,q}-p_{j,k}\Bigg]$$

for all r ($r=0\sim k-2$), then (5.7) hold.

By subtracting the quantity $\sum_{q=k-r-1}^{k-1} p_{i,q}+\sum_{q=k-r-1}^{k-1} p_{j,q}$ from both sides of (5.8), we have next inequalities by using the relation

$$\max[-A,\ -B,\ -C]=-\min[A,\ B,\ C]:$$

$$(5.9)\qquad \min\Bigg[\sum_{q=k-r-1}^{k-1} p_{i,q},\ \min_{t=0\sim r-1}\Bigg(\sum_{q=k-r-1}^{k-t-2} p_{i,q}+\sum_{q=k-t}^{k} p_{i,q}\Bigg),\ \sum_{q=k-r}^{k} p_{j,q}\Bigg]$$
$$\leqq\min\Bigg[\sum_{q=k-r-1}^{k-1} p_{j,q},\ \min_{t=0\sim r-1}\Bigg(\sum_{q=k-r-1}^{k-t-2} p_{j,q}+\sum_{q=k-t}^{k} p_{i,q}\Bigg),\ \sum_{q=k-r}^{k} p_{i,q}\Bigg].$$
$$(r=0\sim k-2,\ k=2\sim m)$$

So that, if (5.9) hold for all r ($r=0\sim k-2$) and k ($k=2\sim m$), then (5.6) hold and job i precedes job j.

II. Mathematical induction to prove the existence of (5.9) *for all* m ($m\geqq 2$).

For $m=2$, it becomes $r=0$ and both conditions (5.1), (5.2) and (5.9) reduce to the same inequality:

$$\min[p_{i,1},\ p_{j,2}]\leqq\min[p_{j,1},\ p_{i,2}]$$

which shows that (5.9) holds for $m=2$. Also it holds for $m=3$ by theorem 3.

Next we assume that (5.9) hold for all k ($k=2\sim K$) under conditions (5.1), (5.2) for $m=K$:

$$\min[p_{i,k},\ p_{j,k+1}] \leqq \min[p_{j,k},\ p_{i,k+1}]\ , \quad (k=1\sim K-1) \tag{5.10}$$

$$\min\left[\sum_{k=u}^{v} p_{i,k},\ \sum_{k=u+1}^{v+1} p_{j,k}\right] \leqq \min\left[\sum_{k=u}^{v} p_{j,k},\ \sum_{k=u+1}^{v+1} p_{i,k}\right]. \quad (1\leqq u<v\leqq K-1) \tag{5.11}$$

Then we must show that (5.9) hold for $m=K+1$ under conditions (5.1), (5.2) for $m=K+1$; that is, under conditions (5.10), (5.11) and K additional conditions

$$\min\left[\sum_{k=K-r}^{K} p_{i,k},\ \sum_{k=K-r+1}^{K+1} p_{j,k}\right] \leqq \min\left[\sum_{k=K-r}^{K} p_{j,k},\ \sum_{k=K-r+1}^{K+1} p_{i,q}\right]. \quad (r=0\sim K-1) \tag{5.12}$$

Also they hold, by assumption, next inequalities for $k=K$ under conditions (5.10), (5.11):

$$\begin{aligned} &\min\left[\sum_{q=K-r-1}^{K-1} p_{i,q},\ \min_{t=0\sim r-1}\left(\sum_{q=K-r-1}^{K-t-2} p_{i,q}+\sum_{q=K-t}^{K} p_{j,q}\right),\ \sum_{q=K-r}^{K} p_{j,q}\right] \\ \leqq &\min\left[\sum_{q=K-r-1}^{K-1} p_{j,q}\ \min_{t=0\sim r-1}\left(\sum_{q=K-r-1}^{K-t-2} p_{j,q}+\sum_{q=K-t}^{K} p_{i,q}\right),\ \sum_{q=K-r}^{K} p_{i,q}\right]. \\ &(r=0\sim K-2) \end{aligned} \tag{5.13}$$

On the other hand, inequalities (5.9) for $k=K+1$ are

$$\begin{aligned} &\min\left[\sum_{q=K-r}^{K} p_{i,q},\ \min_{t=0\sim r-1}\left(\sum_{q=K-r}^{K-t-1} p_{i,q}+\sum_{q=K-t+1}^{K+1} p_{j,q}\right),\ \sum_{q=K-r+1}^{K+1} p_{j,q}\right] \\ \leqq &\min\left[\sum_{q=K-r}^{K} p_{j,q},\ \min_{t=0\sim r-1}\left(\sum_{q=K-r}^{K-t-1} p_{j,q}+\sum_{q=K-t+1}^{K+1} p_{i,q}\right),\ \sum_{q=K-r+1}^{K+1} p_{i,q}\right]. \\ &(r=0\sim K-1) \end{aligned} \tag{5.14}$$

These hold for all r $(r=0\sim K-2)$ since (5.14) for all r $(r=0\sim K-2)$ can be derived by taking $K+1$ for K in (5.10), (5.11) and (5.13) for r $(r=0\sim K-2)$ under conditions (5.12).

Hence it is remained to prove the existence of (5.14) for $r=K-1$; that is, next inequality:

$$\min\left[\sum_{q=1}^{K} p_{i,q},\ \min_{t=0\sim K-2}\left(\sum_{q=1}^{K-t-1} p_{i,q}+\sum_{q=K-t+1}^{K+1} p_{j,q}\right),\ \sum_{q=2}^{K+1} p_{j,q}\right]$$
$$\leqq\min\left[\sum_{q=1}^{K} p_{j,q},\ \min_{t=0\sim K-2}\left(\sum_{q=1}^{K-t-1} p_{j,q}+\sum_{q=K-t+1}^{K+1} p_{i,q}\right),\ \sum_{q=2}^{K+1} p_{i,q}\right]$$

which is expressed as next form:

$$\min\left[p_{i,1}+\min\left\{\sum_{q=2}^{K} p_{i,q},\ \min_{t=0\sim K-3}\left(\sum_{q=2}^{K-t-1} p_{i,q}+\sum_{q=K-t+1}^{K+1} p_{j,q}\right)\right\},\ p_{i,1}+\sum_{q=3}^{K+1} p_{j,q},\ \sum_{q=2}^{K+1} p_{j,q}\right]$$
$$\leqq\min\left[p_{j,1}+\min\left\{\sum_{q=2}^{K} p_{j,q},\ \min_{t=0\sim K-3}\left(\sum_{q=2}^{K-t-1} p_{j,q}+\sum_{q=K-t+1}^{K+1} p_{i,q}\right)\right\},\ p_{j,1}+\sum_{q=3}^{K+1} p_{i,q},\ \sum_{q=2}^{K+1} p_{i,q}\right]. \tag{5.15}$$

By putting each term, min{ }, in left and right side of (5.15) as A, B respectively, (5.15) is expressed as

$$\min\left[p_{i,1}+A,\ p_{i,1}+\sum_{q=3}^{K+1} p_{j,q},\ \sum_{q=2}^{K+1} p_{j,q}\right]\leqq\min\left[p_{j,1}+B,\ p_{j,1}+\sum_{q=3}^{K+1} p_{i,q},\ \sum_{q=2}^{K+1} p_{i,q}\right]. \tag{5.16}$$

So, in order to prove (5.15), first it is shown that inequality

$$\min\left[p_{i,1}+A,\ \sum_{q=2}^{K+1} p_{j,q}\right]\leqq\min\left[p_{j,1}+B,\ \sum_{q=2}^{K+1} p_{i,q}\right] \tag{5.17}$$

holds under conditions (5.10), (5.11) and (5.12) because existing inequality for $r=K-2$ in (5.14) is

$$\min\left[A,\ \sum_{q=3}^{K+1} p_{j,q}\right]\leqq\min\left[B,\ \sum_{q=3}^{K+1} p_{i,q}\right] \tag{5.18}$$

and then we can have (5.17) under conditions (5.10), (5.11), (5.12) by substituting $\sum_{q=1}^{2} p_{i,q}$, $\sum_{q=2}^{3} p_{j,q}$; $\sum_{q=1}^{2} p_{j,q}$, $\sum_{q=2}^{3} p_{i,q}$ for $p_{i,2}$, $p_{j,3}$; $p_{j,2}$, $p_{i,3}$ in (5.18) respectively.

Next, inequality (5.16) is proved by using lemma at section 4.1 under conditions (5.17), (5.18) and

(5.19) $\min[p_{i,1}, p_{j,2}] \leqq \min[p_{j,1}, p_{i,2}]$

as below:

The case (P) where all next inequalities:

(5.20) $p_{j,2}<p_{i,1}$,

(5.21) $p_{j,1}<p_{i,2}$,

$$p_{j,1}<p_{i,1},\ A<\sum_{q=3}^{K+1} p_{j,q},\ \sum_{q=3}^{K+1} p_{i,q}<B,$$

(5.22) $\sum_{q=3}^{K+1} p_{i,q}<\sum_{q=3}^{K+1} p_{j,q}$,

hold does not occur because if all these inequalities hold, they become $p_{j,2}<p_{i,2}$ from (5.19), (5.20), (5.21) and then $p_{j,3} \leqq p_{i,3}$ from condition (5.10) which lead to $\sum_{q=3}^{K+1} p_{j,q} \leqq \sum_{q=3}^{K+1} p_{i,q}$ by successively applying (5.11) and (5.12), with $p_{j,2}<p_{i,2}$, which is a contradiction to (5.22). Hence (5.14) hold for all r ($r=0 \sim K-1$) and then (5.9) hold for all k ($k=2 \sim K+1$) and theorem holds for all m ($m \geqq 2$). Q.E.D.

For $m=2$, this theorem coincides with well known Johnson's theorem for two machines [15] and in case where $m=3$ in this theorem we obtain theorem 3.

Theorem 6 is efficiently applied to the reduction of the number of sequences in branch-and-bound algorithms and other algorithms for obtaining the optimal sequence for m machines ($m \geqq 3$) in flow shop where no passing is allowed. cf. [11].

Also this theorem gives some ways of obtaining the approximate sequence similar as Johnson approximation [14] for any m machines ($m \geqq 3$) in such flow shop. (cf. section 6.3)

5.2 Generalization of General Theorem 5

Next theorem 7 is a generalization of theorem 5 to m machines ($m \geqq 2$), which includes the theorem 6 and generalization of theorem

4 presented in next section 5.3 as its special cases.

Theorem 7. Let job i and j be adjacent two jobs.

I. In case where relation (A): $p_{j,k_0+1}<p_{i,k_0},\ p_{i,k_0+n_0}<p_{j,k_0+n_0+1}$; does not hold for any k_0 ($k_0=1\sim m-2$) and for any n_0 ($n_0=1\sim m-k_0-1$) for each k_0, if next $(m-1)$ inequalities:

$$\text{(5.23)}\qquad \min[p_{i,k},\ p_{j,k+1}]\leqq\min[p_{j,k},\ p_{i,k+1}]\qquad (k=1\sim m-1)$$

hold, then job i must precedes job j under transitive property in order to construct the optimal sequence for m machines ($m\geqq 2$).

When equalities hold for all inequalities, either ordering is optimal.

II. In case where above relation (A) holds for a certain k_0 ($k_0=1\sim m-2$) and a certain n_0 ($n_0=1\sim m-k_0-1$) for this k_0, if additional inequalities to (5.23):

$$\text{(5.24)}\qquad \min\left[\sum_{q=u}^{v}p_{i,q},\ \sum_{q=u+1}^{v+1}p_{j,q}\right]\leqq\min\left[\sum_{q=u}^{v}p_{j,q},\ \sum_{q=u+1}^{v+1}p_{i,q}\right]$$
$$(1\leqq u<v\leqq m-1)$$

hold, then the same conclusions as that in I hold.

Proof. By theorem 6 it is sufficient to show that (5.9) hold for case I.

Case I. (1) In case $p_{i,k}\leqq p_{j,k+1}$ for each k ($k=1\sim m-1$), it becomes $p_{i,k}\leqq p_{i,k+1}$ from (5.23) and so $\sum_{q=k-r-1}^{k-1}p_{i,q}$ is a smallest term in (5.9) for all r and k. Hence (5.9) hold.

(2) In case $p_{i,k}\leqq p_{j,k+1}$ ($k=1\sim k_1-1$), $p_{j,k_1+1}<p_{i,k_1}$, it becomes $p_{j,k+1}\leqq p_{i,k},\ p_{j,k}$ for each k ($k_1<k\leqq m-1$) by assumption and then $\sum_{q=k-r-1}^{k-1}p_{i,q}$ or $\sum_{q=k-r-1}^{k_1-1}p_{i,q}+\sum_{q=k_1+1}^{k}p_{j,q}$, $\sum_{q=k-r}^{k}p_{j,q}$ is a smallest term in (5.9).

(3) In case $p_{j,2}<p_{i,1}$, it holds always $p_{j,k+1}\leqq p_{i,k},\ p_{j,k}$ for each k ($2\leqq k\leqq m-1$) by assumption and $\sum_{q=k-r}^{k}p_{j,q}$ is a smallest term in (5.9).

Q.E.D.

Relation (A) is schematically shown in Fig. 2. Theorem 7 can be used as a criterion in algorithm for determining the optimal sequence

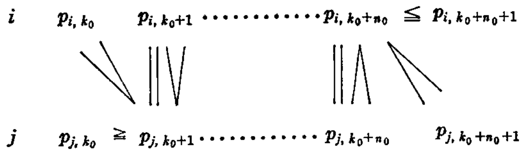

Fig. 2. Relation (*A*).

for m machines ($m \geqq 3$) in flow shop under consideration.

5.3 Generalization of Theorem 4 under Johnson Working Rule

It will be shown that generalization of theorem 4 ($m=3$) which is the case where *Johnson working rule* for two machines case is used, is a special case of general theorem 7.

Generalization of theorem 4 to m machines case ($m \geqq 2$) is stated as below:

Corollary. If job i directly precedes job j on each adjacent two machines M_k, M_{k+1} under *Johnson working rule* by using

(5.23) $\min[p_{i,k}, p_{j,k+1}] \leqq \min[p_{j,k}, p_{i,k+1}]$

for each k ($k=1 \sim m-1$), then job i directly precedes job j on m machines case, that is, if an optimal sequence on each adjacent two machines M_k, M_{k+1} ($k=1 \sim m-1$) coincides with each other, then this sequence becomes an optimal sequence on m machines, when *Johnson working rule* is employed in flow shop where no passing is allowed.

Proof. It is sufficient to show that case II in theorem 7 does not occur. If $(m-1)$ inequalities (5.23) hold in case II and Johnson working rule is used for two jobs i, j where i is smaller than j (j is smaller than i), then, as known from Fig. 2 and (5.23), job j precedes job i on two machines M_k, M_{k+1}; $k=k_0$, or $k_0+1, \cdots, k_0+n_0-1$ ($k=k_0+n_0$, or $k_0+n_0-1, \cdots, k_0+1$) which contradicts to the assumption.

Q.E.D.

6. Approximate Algorithms

The problem is to decide the approximate sequence which has sufficiently small makespan on m machines in flow shop where no passing is allowed.

The methods of approximation are classified into following three types, that is,

1) Determination of the position of n_k jobs at stage k to constract the approximate sequence of n jobs. Usually the case where $n_k=1$ is considered.

2) Systematic reformation of the order of some adjacent jobs in the suitably determined initial sequence of n jobs.

3) Separation of the set of n jobs into r subsets followed by suitable connection of the optimal or approximate subsequence of each subset.

Next, some approximate algorithms related to these types will be presented by using the inequalities in theorem 6.

6.1 Approximate Algorithms 1

Let n jobs be named along $1, 2, \cdots, n$ as usual.

Step 1. Each of the optimal sequences obtained by each inequality of (5.1) and (5.2) is determined by general working rule as the same way for two machines case, under transitive property of the order of jobs determined by each inequality.

Step 2. For each pair (i, j) among n jobs, frequency of the order ij and frequency of the order ji in all optimal sequences determined at step 1 are counted respectively. Then the order among two orders ij, ji which has larger number of frequency is determined. If the number of frequency is equal to each other, then both orders ij, ji are taken.

Example ($m=3$, $n=6$). For three machines case, inequalities become (4.8), (4.12) and (4.18) in theorem 3, a special case of theorem 6.

i	1	2	3	4	5	6
$p_{i,1}$	6	12	4	3	6	2
$p_{i,2}$	7	2	6	11	8	14
$p_{i,3}$	3	3	8	7	10	12

Processing Time

Then optimal sequences by (4.8), (4.12) and (4.18) become as below respectively (Step 1):

(1) $\begin{cases} 643152 \\ 643512 \end{cases}$ (2) 235641 (3) $\begin{cases} 345612 \\ 354612 \end{cases}$

And the order of two jobs of each pair (i, j) is determined as below. (Step 2):

(1, 2)	12	(2, 4)	42	(3, 5)	35	(3, 6)	36
(1, 3)	31	(3, 4)	34	(4, 5)	45, 54	(4, 6)	64
(2, 2)	32	(1, 5)	51	(1, 6)	61	(5, 6)	56
(1, 4)	41	(2, 5)	52	(2, 6)	62		

Step 3. Each job i which is increasing from $i=2$ to $i=n$ is classified into one of two subsets N_1, N_2 defined as follows. Subset N_1 is composed of the job i which has the unique order for all job j $(<i)$. Subset N_2 is composed of the job i which has two orders ij, ji for certain job j $(<i)$. Job 1 belongs to N_1 only when unique order among the pair (1, 2) is determined.

Under the above preparations, the procedures in next step 4 construct the approximate sequence successively.

Step 4. First the order of the jobs in N_1 along the increasing number of job is determined by following each unique order of two jobs (i, j).

Next the position of each job in N_2 is determined along the increasing number of job by following each unique or twofold order of two jobs (i, j) such that already determined order of jobs is not changed. Lastly sequence of n jobs can be obtained.

If any sequence of n jobs cannot be determined by the above ways, then the possible sequences that satisfy almost all unique or twofold orders are taken. Then the sequence having minimum makespan among them is the approximate sequence.

Example. (*Continued*)

$$N_1=\{1, 2, 3, 4, 6\}, \quad N_2=\{5\}. \quad \text{(Step 3).}$$

Construction of the approximate sequence is made as below (Step 4):

Stage.	1	2	3	4
N_1:	12	312	3412	36412

N_2: (Stage 5). Order 45 contradicts to the unique orders 56, 64 and then the order 54 determines one approximate sequence 356412 with makespan 57 which is just an optimal sequence in this example.

6.2 Remarks

1) Product-latine method [16] of the matrix for Hamiltonian path can be used after step 2 in the above algorithm 1 by taking both order ij, ji as the element corresponding to twofold order of (i, j).

2) Mean approximate ratio for 15 examples of three machines and six jobs was 98.45% where

$$\text{(Approximate ratio)}=\frac{\text{(Optimal time)}}{\text{(Approximate time)}}\times 100\%.$$

3) Additional steps to make the makespan of the sequence of n jobs obtained in step 4 more smaller by exchanging suitable jobs of this sequence may be considered.

6.3 Other Related Algorithms

1) In the above algorithm 1, only some of the inequalities (5.1), (5.2) can be taken instead of all inequalities (5.1), (5.2) and then approximate sequence will be obtained by following the same steps.

For example, Johnson approximation for three machines [14] is obtained by using only the inequality (4.18) in the above example and general formula of this type for m machines case is next inequality where $u=1$ and $v=m-1$ in (5.2):

$$\min\left[\sum_{k=1}^{m-1} p_{i,k},\ \sum_{k=2}^{m} p_{j,k}\right] \leqq \min\left[\sum_{k=r}^{m-1} p_{j,k},\ \sum_{k=2}^{m} p_{i,k}\right]. \tag{6.1}$$

But two additional inequalities:

$$\min\left[\sum_{k=1}^{m-2} p_{i,k},\ \sum_{k=2}^{m-1} p_{j,k}\right] \leqq \min\left[\sum_{k=1}^{m-2} p_{j,k},\ \sum_{k=2}^{m-1} p_{i,k}\right], \tag{6.2}$$

$$\min\left[\sum_{k=2}^{m-1} p_{i,k},\ \sum_{k=3}^{m} p_{j,k}\right] \leqq \min\left[\sum_{k=2}^{m-1} p_{j,k},\ \sum_{k=3}^{m} p_{i,k}\right]. \tag{6.3}$$

for instance may yield better approximate sequence.

2) If, in any above algorithm, we use only step 1 and then decide the approximate sequence as the sequence with least makespan among all optimal sequences for each inequality corresponding to (5.1) and/or (5.2), or similar inequalities, we obtain another algorithm. Algorithm in [13] can be considered as this type.

3) An algorithm to decide one job of the approximate sequence successively from the first job is formulated by using the relations

$$T_k(ij) \leqq T_k(ji) \text{ for all } k\ (k=2\sim m) \tag{6.4}$$

that are the same as (3.3) and (5.6), or one relation

$$T_m(ij) \leqq T_m(ji) \tag{6.5}$$

where each $T_k(ij)$, $T_k(ji)$ can be calculated by recurrence relations (3.1), (3.2) and so on successively.

This algorithm is stated as below:

At each stage k $(k=1\sim n-1)$ there has been already decided the presubsequence $i_1 \cdots i_{k-1}$ with $(k-1)$ jobs, then for each pair (i, j) of two jobs among remained $(n-k+1)$ jobs, the order ij is determined by relations (6.4) where both orders ij, ji are taken when inequalities with reverse direction hold for the pair (i, j), or by relation (6.5).

Next one job i_k which precedes all other jobs j becomes k th job of the approximate sequence. If such job does not exist, the jobs i_k that have a large number of jobs preceded by i_k are considered as the k th jobs of the approximate sequence.

Then the approximate sequence with least makespan among all sequences of n jobs obtained at stage n is determined.

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