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A MODEL OF SEARCH FOR A TARGET MOVING AMONG THREE BOXES: SOME SPECIAL CASES

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Abstract

This paper considers a sequential search model for a target which moves among three boxes according to a moving structure characterized by a Markov transition matrix known to the searcher. We want to derive the search policy which minimizes the expected number of looks needed to detect the target. We shall deal with the model for the *perfect detection* case only. We derive some general properties of this model and obtain optimal policies for models of some paticularly simple moving structure.

1. Introduction

Finding the optimal search policy for a target moving according to the probability law known to the searcher is one of the problems which are highly wanted to be solved. Dobbie [1] mentioned the difficulties of this problem and, at the same time, listed various models of this sort.

M. Klein [2] formulated two search models in terms of the Markovian decision processes. In both models the target is assumed to be not blind so that he sees where the searcher was. In the first one of these models Klein assumes that the target movement is independent of the target location and that the searcher can't know the target location. In the second model he assumes that the target movement depends on the last location of the target and that the searcher can know the last location of the target.

S. M. Pollock [3] has analyzed almost completely the search model for a target between two boxes according to the moving structure known to the searcher. He gave the explicit solution to the perfect detection case and analyzed approximately the partial detection case. In the end of [3] he mentioned the complexities of the posterior probabilities in the case of more than two boxes and said that it would, of course, be interesting to find special cases that yield attractive or useful solutions.

The purpose of this paper is to derive the optimal search policies for some particularly simple models in the three-box case.

2. Description of the Model

We consider a search model for a target which moves among three boxes. At the start of the search, the target is located in box i with *a priori* probability p_i. Put $p=(p_1, p_2, p_3)$ where $p_1+p_2+p_3=1$, $0\leqq p_i\leqq 1$ for $i=1, 2, 3$. The moving structure of the target is indicated by a Markov transition matrix $P=(p_{ij})$ where

$$p_{ij}\equiv\text{Prob.}\{\text{target is in box } j \text{ at time } n \mid \text{target is box } i \text{ at time } n-1\} \quad i, j=1, 2, 3;\ n=1, 2, \cdots.$$

Hereafter we shall use the following notations: $p_{12}=a_1$, $p_{13}=a_2$, $p_{21}=b_1$, $p_{23}=b_2$, $p_{32}=c_1$, $p_{31}=c_2$. The transition flow diagram is as shown in Fig. 1. This moving structure is assumed to be known to the searcher. The search process advances as follows: the searcher looks into some box to detect the target located with the prior probability distribution p. If he detects it, the process stops and if he can't detect it, the target moves with transition matrix P and afterward the searcher looks into some box to detect the target. After this a move and a look occur alternately until detection. It is assumed that the searcher is sure to detect the target if he looks into the box which contains the target. In short we deal with only the *perfect detection* case. The problem is to find an optimal policy, that is, a sequence of boxes indicating the order of looks which minimizes the expected number of looks needed to detect the target.

Let $T_i p$ be the posterior probability distribution with which the target is located in one of the three boxes after the searcher looks into box i and fails to detect the target located with the prior probability distribution p and afterward a new movement of the target occurs. By a straightforward application of Bayes' rule,

$$T_1 p=\left(\frac{b_1p_2+c_2p_3}{1-p_1},\ \frac{(1-b_1-b_2)p_2+c_1p_3}{1-p_1},\ \frac{b_2p_2+(1-c_1-c_2)p_3}{1-p_1}\right)$$

$$T_2 p=\left(\frac{(1-a_1-a_2)p_1+c_2p_3}{1-p_2},\ \frac{a_1p_1+c_1p_3}{1-p_2},\ \frac{a_2p_1+(1-c_1-c_2)p_3}{1-p_2}\right)$$

$$T_3 p=\left(\frac{(1-a_1-a_2)p_1+b_1p_2}{1-p_3},\ \frac{a_1p_1+(1-b_1-b_2)p_2}{1-p_3},\ \frac{a_2p_1+b_2p_2}{1-p_3}\right)$$

Let $V(p)$ be the minimum expected number of looks needed to detect the target when the prior probability distribution is p. Then by the principle of optimality the functional equation for this model is given by

$$V(p)=\min_{i=1,2,3}[D_i:\ 1+(1-p_i)V(T_i p)] \tag{1}$$

where D_i is a symbol that denotes the decision indicating to look into box i. We want to derive the function $V(p)$ satisfying equation (1) and the number of i which attains the minimum of (1).

3. The Posterior Probabilities

In this section we shall investigate the state of the posterior probability distribution. In dealing with our model of three-box case, it is of much help to visualize the prior and posterior probabilities by means of a *triangular chart*, that is, by regarding (p_1, p_2, p_3) as the barycentric coordinate of the point p, p may be plotted in the equilateral triangle of height unity where the distance between p and the side opposite to the vertex A_i is p_i.

Substituting $p_3=1-p_1-p_2$ into each component of T_1p we get $T_1p=(X, Y, Z)$ where

$$X=c_2+\frac{(b_1-c_2)p_2}{1-p_1},$$

$$Y=c_1+\frac{(1-b_1-b_2-c_1)p_2}{1-p_1},$$

$$Z=1-c_1-c_2+\frac{(b_2-1+c_1+c_2)p_2}{1-p_1}.$$

Therefore we have

$$\frac{X-c_2}{b_1-c_2}=\frac{Y-c_1}{(1-b_1-b_2)-c_1}=\frac{Z-(1-c_1-c_2)}{b_2-(1-c_1-c_2)}$$

showing that T_1p lies on the segment connecting two points $p'=(b_1, 1-b_1-b_2, b_2)$ and $p''=(c_2, c_1, 1-c_1-c_2)$ in the triangular chart mentioned above. Moreover it can be shown that T_1p is the point which partitions the segment $\overline{p'p''}$ by the ratio p_3: p_2. We can obtain the same results in connection with T_2p, T_3p. We define a *posterior triangle* as a triangle connecting the three points p', p'' and $p'''=(1-a_1-a_2, a_1, a_2)$ in the triangular chart. Then we may summarize the avove results in the following Lemma 1:

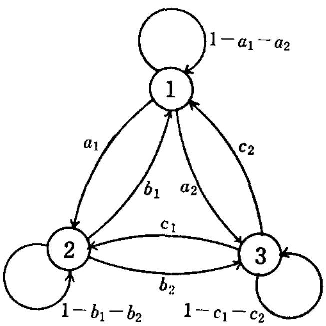


Fig. 1. Moving structure of the target.

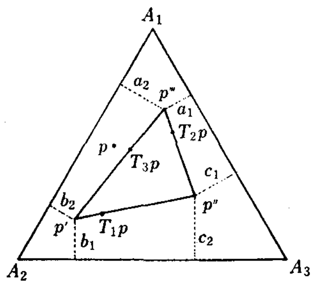


Fig. 2. Posterior triangle.

Lemma 1. For any prior probability distribution p, the posterior probability distribution $T_i p$ lies on the circuit of the posterior triangle which depends on the transition probabilities only. Moreover, $T_1 p$, $T_2 p$, and $T_3 p$ partition respectively three sides of the posterior triangle $\overline{p'p''}$, $\overline{p''p'''}$ and $\overline{p'''p'}$ by the ratio $p_3 : p_2$, $p_1 : p_3$ and $p_2 : p_1$ (see Fig. 2).

4. Some Properties of the Function $V(p)$

We define the function $V_n(p)$ as follows:

$$(2)\qquad \begin{cases} V_0(p)=1 \\ V_n(p)=\min_{i=1,2,3}[1+(1-p_i)V_{n-1}(T_i p)] \end{cases}$$
$$(n=1, 2, 3, \cdots).$$

Then it can be shown that $V_n(p)$ monotonously increases and approaches to $V(p)$ as $n\to\infty$. The verification of this will be omitted since the same discussion is found here and there, for example in [3]. The approximate solution for $V(p)$ may be computed by exact iteration of equation (2). But it is not our purpose to go deeply into the problem of this sort. From the continuity of $V_n(p)$, it may be shown easily that the function $V(p)$ is continuous in p. Moreover we may assert that the function $V(p)$ is concave in p by the same method as one in Pollock [3]. The following Lemma 2 gives an interesting and important property of the function $V(p)$. Let $V(p)=\min_{i=1,2,3} V(p;\ i)$, where

$$V(p;\ i)=1+(1-p_i)V(T_i p)\ .$$

Lemma 2. The function $V(p;\ i)$ is linear on any line segment in the equilateral triangle, which has the vertex A_i as an end point.

Proof. We verify the case $i=1$.

Let p be any point on the line segment

$$(3)\qquad \frac{p_2}{p_3}=\frac{k}{1-k}\qquad (0\leqq k<1)$$

which has the vertex A_1 as an end point. Then since

$$p=(p_1,\ k(1-p_1),\ (1-k)(1-p_1))\ ,$$
$$T_1p=(k(b_1-c_2)+c_2,\ k(1-b_1-b_2)+c_1(1-k)\ ,$$
$$kb_2+(1-k)(1-c_1-c_2))$$

independently of p_1. Now let $d(p,\ q)$ be Euclidian metric between two points p, q, then

$$\frac{V(p;\ 1)-V(A_1;\ 1)}{d(p,\ A_1)}=\frac{V(T_1p)}{\sqrt{2(1-k+k^2)}}$$

which depends only on k. Therefore $V(p;\ i)$ is linear on the line segment (3). Since $V(p;\ i)$ is continuous in p, the assertion is valid even in the case $k=1$. Q.E.D.

Definition: A set S is *star-convex with respect to* the point $p^0\in S$, if and only if $p\in S$ implies that $\lambda p+(1-\lambda)p^0\in S$ for all $0\leqq\lambda\leqq1$.

Let D_i^* $(i=1,\ 2,\ 3)$ be the optimal decision regions *i.e.*, $D_i^*=\{p;\ V(p)=V(p;\ i)\}$.

Lemma 3. The optimal decision region D_i^* is star-convex with respect to the vertex A_i of the equilateral triangle.

Proof. By means of Lemma 2 and concavity of $V(p)$, for any $p\in D_i^*$

$$\begin{aligned}V[\lambda p+(1-\lambda)A_i]&\leqq V[\lambda p+(1-\lambda)A_i;\ i]\\&=\lambda V(p;\ i)+(1-\lambda)V(A_i;\ i)\\&=\lambda V(p)+(1-\lambda)V(A_i)\\&\leqq V[\lambda p+(1-\lambda)A_i]\ .\end{aligned}$$

Hence $V[\lambda p+(1-\lambda)A_i]=V[\lambda p+(1-\lambda)A_i;\ i]$, implying that $\lambda p+(1-\lambda)A_i\in D_i^*$ Q.E.D.

5. Derivations of Optimal Policies in Some Special Cases

We shall derive in this section the optimal policies in some special cases. We restrict our attention to the cases where $a_1=a_2=a$, $b_1=b_2=b$ and $c_1=c_2=c$.

Case A: $a=b=c$ with $0\leqq a\leqq1/2$.

In this case since the situation is symmetric for each of three boxes in every respect, the optimal decision region $D_i{}^*$ should be given by

$$D_i{}^* = \{p \mid p_i = \max_{j=1,2,3} p_j\} \qquad \text{(see Fig. 3).}$$

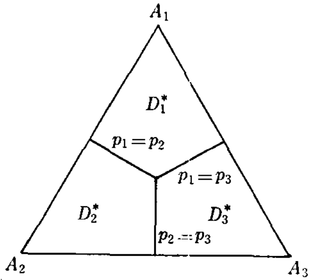
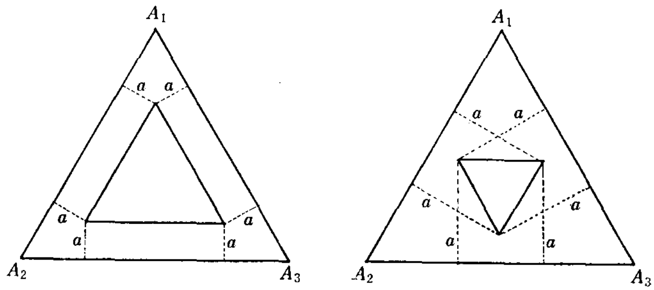


Fig. 3. Optimal decision regions in Case A.



Case A_1: $0 \leqq a \leqq 1/3$ Case A_2: $1/3 \leqq a \leqq 1/2$

Fig. 4. The posterior triangles in Case A.

On the other hand the posterior triangle is as shown in Fig. 4. Therefore from Figs. 3 and 4 and by repetitive applications of Lemma 1 we can derive the optimal policy as follows:

Case A_1: $0 \leqq a \leqq 1/3$. If the prior probability distribution p is contained in Region A of Fig. 5, the optimal search policy is $(1, 2, 3)^\infty$, that is, it is optimal to look into box 1, box 2, box 3 and repeat the search in this order periodically until detection. For other regions of Fig. 5, the optimal policy is $(1, 3, 2)^\infty$ in Region B, $(3, 1, 2)^\infty$

in Region C, $(3, 2, 1)^\infty$ in Region D, $(2, 3, 1)^\infty$ in Region E and $(2, 1, 3)^\infty$ in Region F.

Case A_2: $1/3 \leqq a \leqq 1/2$. For each region of Fig. 6, the optimal policy is 1^∞ in Region A, 2^∞ in Region B and 3^∞ in Region C. In this case the value of $V(p)$ is $1+(1-p_i)/a$ if $p \in D_i^*$ since $V(p)=V(p; i^\infty)$ can be computed actually.

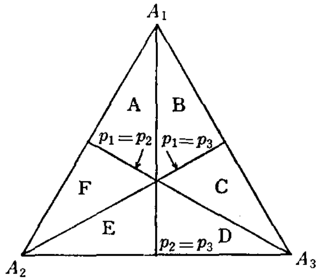


Fig. 5. Optimal policy regions in Case A_1.

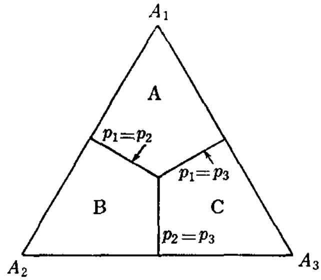


Fig. 6. Optimal policy regions in Case A_2.

The results in Case A coincide with our common sense, that is, if the transition probability is small (Case A_1), we usually look into boxes in the order of the magnitude of the probabilities of the location, choosing the largest as the first to look. If the transition probability is large (Case A_2), we look into only one box having the largest initial probability of the location in the hope that the target comes into the initially searched box.

Case B: $b=c=0$ with $0 \leqq a \leqq 1/2$. The moving structure and the posterior triangle in this case are as shown in Figs. 7 and 8 respectively. Because of the symmetry between boxes 2 and 3, the border between D_2^* and D_3^* should be $p_2=p_3$ and then $T_2 p$, $T_3 p$ can't be contained in D_2^* and D_3^* respectively.

Case B_1: $1/3 \leqq a \leqq 1/2$. Evidently, if $a=1/2$, then $T_2 p \in D_3^*$ and $T_3 p \in D_2^*$ for any p. Considering continuity, it easily follows that

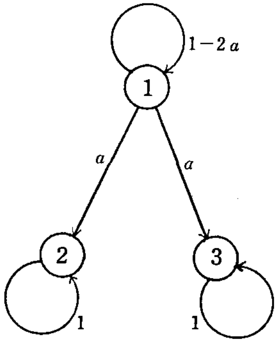


Fig. 7. Moving structure in Case *B*.

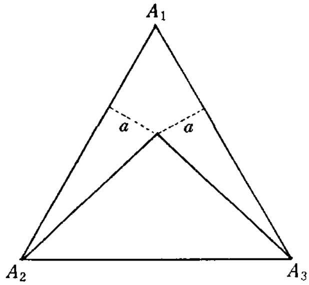


Fig. 8. Posterior triangle in Case *B*.

this is still valid for a slightly smaller than 1/2. For such a, by repetitive applications of Lemma 1, we have

$$V(p;\ 1)=\begin{cases} V[p;\ (1,2,3)]=3-2p_1-p_2 & \text{if}\ \ p_2 \geqq p_3 \\ V[p;\ (1,3,2)]=3-2p_1-p_3 & \text{if}\ \ p_2 \leqq p_3 \end{cases}$$

and $V(p;\ 2)=V[p;\ (2,3)^\infty]=2+\{(1-a)/2a\}p_1-p_2$ for any p. The condition that $V(p;\ 1)=V(p;\ 2)$ gives $p_1=2a/(1+3a)$, as the bounding line between D_1^* and D_2^*, D_3^*. When the point $(1-2a,\ a,\ a)$ is below the line $p_1=2a/(1+3a)$, the above case occurs and then

$$1-2a \leqq \frac{2a}{1+3a} \quad \text{or} \quad \frac{1}{3} \leqq a\ .$$

In this case the optimal policy regions are as shown in Fig. 9 and the optimal policies and the value of $V(p)$ for each region in Fig. 9

Table 1. Solutions in Case B_1.

Region in Fig. 9	Optimal policy	The value of $V(p)$
A	(1, 2, 3)	$3-2p_1-p_2$
B	$(2,\ 3)^\infty$	$2+\frac{1-a}{2a}p_1-p_2$
C	(1, 3, 2)	$3-2p_1-p_3$
D	$(3,\ 2)^\infty$	$2+\frac{1-a}{2a}p_1-p_3$

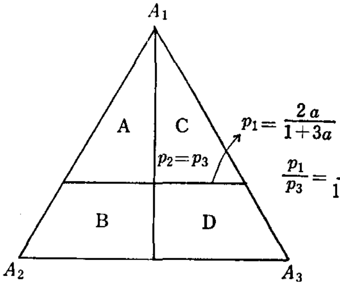


Fig. 9. Optimal policy regions in Case B_1.

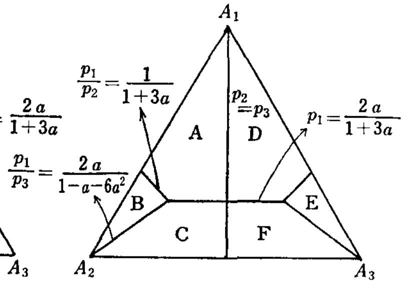


Fig. 10. Optimal policy regions in Case B_2.

are as shown in Table 1.

Case B_2: $a^* \leqq a < 1/3$ where $a^*(\fallingdotseq 0.267)$ is a unique root of $12a^3 - 6a^2 - 3a + 1 = 0$ on [0 1/3).

We consider the case that the point $(1-2a, a, a)$ is slightly above the line $p_1 = 2a/(1+3a)$ but that the recurrent point $T_3 T_2 p$ is below the line $p_1 = 2a/(1+3a)$. Then $(1-2a)^2/(1-a) \leqq 2a/(1+3a)$ or $12a^3 - 6a^2 - 3a + 1 \leqq 0$ or $a \geqq a^* \fallingdotseq 0.267$.

In this case, D_2^* is divided into two regions, that is, Region B where $T_2 p \in D_1^*$ and Region C where $T_2 p \in D_3^*$ and $T_3 T_2 p \in D_2^*$. The optimal policy regions and solutions in this case are as shown in Fig. 10 and Table 2 respectively.

Table 2. Solutions in Case B_2.

Region in Fig. 10	Optimal policy	The value of $V(p)$
A	(1, 2, 3)	$3-2p_1-p_2$
B	(2, 1, 3, 2)	$3-(1-3a)p_1-2p_2$
C	$(2, 3)^\infty$	$2+\frac{1-a}{2a}p_1-p_2$
D	(1, 3, 2)	$3-2p_1-p_3$
E	(3, 1, 2, 3)	$3-(1-3a)p_1-2p_3$
F	$(3, 2)^\infty$	$2+\frac{1-a}{2a}p_1-p_3$

Case B_3: $0 \leqq a < a^*$ where a^* is above mentioned. In this case the recurrent point T_3T_2p is above the line $p_1 = 2a/(1+3a)$ and then $T_3T_2p \in D_1^*$. The solutions in this case are shown in Fig. 11 and Table 3.

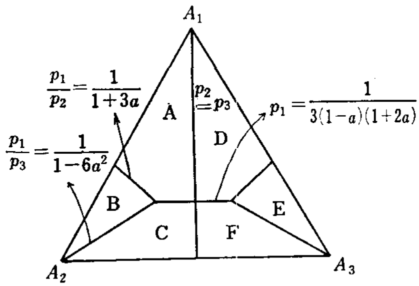


Fig. 11. Optimal policy regions in Case B_3.

Table 3. Solutions in Case B_3.

Region in Fig. 11	Optimal policy	The value of $V(p)$
A	(1, 2, 3)	$3-2p_1-p_2$
B	(2, 1, 3, 2)	$3-(1-3a)p_1-2p_2$
C	(2, 3, 1, 2, 3)	$2+(1+3a-6a^2)p_1-p_2$
D	(1, 3, 2)	$3-2p_1-p_3$
E	(3, 1, 2, 3)	$3-(1-3a)p_1-2p_3$
F	(3, 2, 1, 3, 2)	$2+(1+3a-6a^2)p_1-p_3$

6. Discussion

1. In the models analyzed in this paper, we assumed perfect detection and symmetry between boxes 2 and 3. In the non-symmetric case, we have to investigate some more general properties, since the calculations and results would be more complicated. Moreover, in the partial detection case, it seems that only approximate properties are derived similarly to the two-box case of Pollock [3].

2. At first thought it seems that examples in Section 5 can be reduced to the two-box case because of the symmetry between boxes

2 and 3. For example, Case A in Section 5 seems to be reduced to

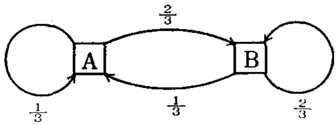


Fig. 12. Moving structure in the modified model.

the two-box case with the moving structure as shown in Fig. 12 in which the box A is identical with the box 1 and box B consists of boxes 2 and 3. But since the look into box B means the look into box 2 or 3, the detection probability of the look into box B can't usually be unity and changes on each stage of the search process. Therefore this example can't be reduced to the two-box case.

3. The concept of the posterior triangle may be extended to the N-box case. But it is complicated and will be difficult to obtain the explicit solution even for any special case.

Acknowledgement

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