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ON THE EFFECTS OF SMALL DEVIATIONS IN THE TRANSITION MATRIX OF A FINITE MARKOV CHAIN

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1. Introduction

Let $P=\{p_{ij};\ i, j=1, 2, \cdots, r\}$ be the transition matrix of a stationary, regular Markov chain C with r states and $\alpha=\{a_i;\ i=1, 2, \cdots, r\}$ be its limiting vector which represents the stationary distribution of the chain. Suppose that there is another regular Markov chain C' with the transition matrix $P'=\{p'_{ij}\}$ and the limiting vector $\alpha'=\{a'_i\}$. If P' is close to P, we can expect that α' is also close to α. Then how close are they? If P and P' are exactly known, then we can answer the question by calculating both α and α'. However the question is difficult if P' is not exactly known and the only thing being known is that P' is close to P in some measure. Such a situation arises whenever we infer the transition matrix of a Markov chain. In such a case, we can only get an approximate value of the transition matrix, and we are concerned with bounds within which the real limiting vector exists.

This problem is more difficult than it may first appear. Because, each entry a_i of the limiting vector α is written as a quotient of two determinants of matrices, and generally it is not easy to determine bounds of variation of a determinant caused by small changes of its entries.

P. J. Schweitzer [3] gave an answer of this problem by showing a perturbation series expansion of α' in powers of a matrix $U=(P'-P)(I-P+A)^{-1}$ representing the difference between P and P', where A is the matrix with α in each row. However, it is not very easy to guess bounds of the value of U if we only know that P' is close to P. J. L. Smith [4] showed that if $p_{ij}-\Delta p_{ij}^{-}\leqq p'_{ij}\leqq p_{ij}+\Delta p_{ij}^{+}$ for all i and j, then α' lies in a convex cone in the rth order Eucledian space bounded by at most $3r$ hyperplanes. His method gives a precise information about bounds of α', but in order to get it, r linear programming problems must be solved.

In this paper, we obtain simple bounds of α' using special properties of the matrix $(I-P)$. Similar ideas can be applied to other characteristic quantities of a finite Markov chain, *e.g.*, the mean values of first passage times, the variances of first passage times, taboo probabilities, and so on.

The bounds of α' are obtained in Section 2, and bounds of other basic quantities in Section 3. In Sections 4 and 5, the case where p'_{ij} are random variables is treated and simple bounds of the variances of basic quantities are obtained.

2. Bounds of the Limiting Vector α'

2.1 Bounds of a'_k

Let us consider two regular Markov chains C and C' with a common finite state space $S=\{s_1, s_2, \cdots, s_r\}$. We denote their transition matrices by $P=\{p_{ij}\}$ and $P'=\{p'_{ij}\}$ and their limiting vectors by $\alpha=\{a_i\}$ and $\alpha'=\{a'_i\}$. We shall obtain bounds of a'_k under the condition that

$$(1+\epsilon)^{-1}p_{ij}\leqq p'_{ij}\leqq(1+\epsilon)p_{ij} \qquad (i, j=1, 2, \cdots, r;\ i\neq j) \tag{2.1}$$

where ϵ is a positive constant. In our derivation of the bounds in Theorem 1 below, we need not assume that p'_{ii} $(i=1, 2, \cdots, r)$ satisfy inequalities in (2.1). In Section 2.4, it will be shown that the bounds of a'_k in Theorem 1, or in Corollary 1, are very good ones. In this section we state the results only, and the proofs of them are postponed to the next section.

Theorem 1. If (2.1) holds, then

(2.2) $$\frac{a_k}{a_k+(1+\epsilon)^{2r-2}(1-a_k)} \leqq a'_k \leqq \frac{a_k}{a_k+(1+\epsilon)^{-2r+2}(1-a_k)}$$

for every k ($k=1, 2, \cdots, r$).

If ϵ is sufficiently small, the bounds given by (2.2) can be written as in the following corollary. We denote by $O(x)$ a term such that $O(x)/x$ is bounded in a neighbourhood of the origin.

Corollary 1. If (2.1) holds for sufficiently small ϵ, then we have

(2.3) $$|a'_k-a_k| \leqq 2(r-1)(1-a_k)a_k\epsilon+O(\epsilon^2)$$

for every k ($k=1, 2, \cdots, r$).

We can also generalize (2.2) and (2.3) for the case where the range of possible value of p'_{ij} differs among different rows, i.e., the value of ϵ in (2.1) depends on i. Here we show results for the simplest case where P' differs from P only in one row, say, the hth row. We assume that

(2.4) $$(1+\epsilon)^{-1}p_{hj} \leqq p'_{hj} \leqq (1+\epsilon)p_{hj} \qquad (j=1, 2, \cdots, h-1, h+1, \cdots, r)$$

and

(2.5) $$p'_{ij}=p_{ij} \qquad (i,\ j=1,\ 2,\ \cdots,\ r;\ i\neq h).$$

Theorem 2. If (2.4) and (2.5) hold, then

(2.6) $$\frac{a_h}{a_h+(1+\epsilon)(1-a_h)} \leqq a'_h \leqq \frac{a_h}{a_h+(1+\epsilon)^{-1}(1-a_h)},$$

and

(2.7) $$\frac{a_k}{a_k+(1+\epsilon)a_h+(1+\epsilon)^2(1-a_k-a_h)} \leqq a'_k$$
$$\leqq \frac{a_k}{a_k+(1+\epsilon)^{-1}a_h+(1+\epsilon)^{-2}(1-a_k-a_h)}$$

for each k ($k=1, 2, \cdots, h-1, h+1, \cdots, r$).

Corollary 2. If (2.4) and (2.5) hold, and if ϵ is sufficiently small, then

(2.8) $$|a'_h-a_h| \leqq (1-a_h)a_h\epsilon+O(\epsilon^2),$$

and

(2.9) $$|a'_k-a_k|\leqq(2-2a_k-a_h)a_k\epsilon+O(\epsilon^2)$$

for each k $(\neq h)$.

2.2 Proofs of Theorems 1 and 2

Our basic idea of the proofs of Theorem 1 and 2 is as follows. The element a_k of the limiting vector α can be written as a rational function of p_{ij}'s. If a condition of the form as (2.1) holds, then we can obtain bounds of a'_k using the following basic inequalities: If $0<(1+\delta)^{-1}x\leqq x'\leqq(1+\delta)x$ and $0<(1+\delta)^{-1}y\leqq y'\leqq(1+\delta)y$, then

(2.10a) $$(1+\delta)^{-1}(x+y)\leqq x'+y'\leqq(1+\delta)(x+y)$$

(2.10b) $$(1+\delta)^{-2}xy\leqq x'y'\leqq(1+\delta)^2xy$$

(2.10c) $$(1+\delta)^{-2}\frac{y}{x}\leqq\frac{y'}{x'}\leqq(1+\delta)^2\frac{y}{x}$$

(2.10d) $$(1+\delta)^{-1}x-(1+\delta)y\leqq x'-y'\leqq(1+\delta)x-(1+\delta)^{-1}y$$

The inequalities for $x'+y'$, $x'y'$, y'/x' are simple but those for the difference $x'-y'$ are not simple. So it is desirable to make a representation of a_k with no subtractive operation. Fortunately we can make such a representation as in Lemma 2 below.

Now we prove Theorems 1 and 2. By the assumption of regularity, the limiting vector α of the Markov chain C is the unique positive stochastic row vector satisfying $\alpha P=\alpha$. Let $\bar{P}_i$ $(i=1, 2, \cdots, r)$ be the cofactor of the (i, i)th entry of the matrix $(I-P)$, *i.e.*, the determinant of the matrix formed by deleting the ith row and the ith column from the matrix $(I-P)$, where I is the $r\times r$ identity matrix. Let $U=\sum_{i=1}^{r}\bar{P}_i$. We will use the same notations with primes for corresponding quantities of the Markov chain C'.

Lemma 1.

(2.11) $$a_k=\bar{P}_k/U \qquad (k=1, 2, \cdots, r).$$

Proof. This relation is easily derived by Cramer's rule from equations $\alpha P=\alpha$ and $\sum_{i=1}^{r}a_i=1$.

Lemma 2. For every k ($k=1, 2, \cdots, r$), $\bar{P}_k$ can be written as

(2.12) $$\bar{P}_k=\sum_{J_k} p_{1j_1}p_{2j_2}\cdots p_{k-1j_{k-1}}p_{k+1j_{k+1}}\cdots p_{rj_r}$$

where the summation is taken over a set J_k of ordered $(r-1)$-tuples $(j_1, j_2, \cdots, j_{k-1}, j_{k+1}, \cdots, j_r)$ such that $j_i \neq i$.

This lemma is an immediate corollary of Lemma 9 in Appendix, and here we omit the proof. Our derivation of bounds of a'_k stands on this lemma. An important point of this lemma is that $\bar{P}_k$ can be represented as a sum of terms with plus signs. For example, if $r=3$, then by the definition $\bar{P}_1=\begin{vmatrix}1-p_{22} & -p_{23}\\ -p_{32} & 1-p_{33}\end{vmatrix}=(1-p_{22})(1-p_{33})-p_{23}p_{32}$. However this representation is not convenient to obtain bounds of $\bar{P}_1$, for it contains a term with a minus sign. We can represent $\bar{P}_1$ in a more suitable form as $\bar{P}_1=p_{21}p_{31}+p_{21}p_{32}+p_{23}p_{31}$ using the fact that row sums of P are equal to 1.

Lemma 3. If (2.1) holds, then for every k ($k=1, 2, \cdots, r$) we have

(2.13) $$(1+\epsilon)^{-r+1}\bar{P}_k \leqq \bar{P}'_k \leqq (1+\epsilon)^{r-1}\bar{P}_k$$

and

(2.14) $$(1+\epsilon)^{-r+1}(U-\bar{P}_k) \leqq (U'-\bar{P}'_k) \leqq (1+\epsilon)^{r-1}(U-\bar{P}_k)\,.$$

Proof. This lemma can be easily proved by (2.1), (2.10) and Lemma 2.

Proof of Theorem 1. By Lemmas 1 and 3,

(2.15) $$\begin{aligned} a'_k &= \frac{\bar{P}'_k}{U'}=\frac{\bar{P}'_k}{\bar{P}'_k+(U'-\bar{P}'_k)} \\ &\leqq \frac{(1+\epsilon)^{r-1}\bar{P}_k}{(1+\epsilon)^{r-1}\bar{P}_k+(1+\epsilon)^{-r+1}(U-\bar{P}_k)} \\ &= \frac{a_k}{a_k+(1+\epsilon)^{-2r+2}(1-a_k)}\,. \end{aligned}$$

This proves a half of (2.2) and the other half can be proved in a similar manner.

Theorem 2 can be proved in almost the same way as Theorem 1 using Lemma 4 below, instead of Lemma 3. So we omit the proofs of Lemma 4 and Theorem 2.

Lemma 4. If (2.4) and (2.5) hold, then

(2.16) $\bar{P}'_h=\bar{P}_h$ and $(1+\epsilon)^{-1}(U-\bar{P}_h)\leqq(U'-\bar{P}'_h)\leqq(1+\epsilon)(U-\bar{P}_h)$,

and

(2.17) $(1+\epsilon)^{-1}\bar{P}_k\leqq\bar{P}'_k\leqq(1+\epsilon)\bar{P}_k$

and

(2.18) $(1+\epsilon)^{-1}(U-\bar{P}_k-\bar{P}_h)\leqq(U'_k-\bar{P}'_k-\bar{P}'_h)\leqq(1+\epsilon)(U-\bar{P}_k-\bar{P}_h)$

for each k $(\neq h)$.

2.3 Taylor's Expansion of a'_k

In order to obtain the difference between a'_k and a_k, we shall expand a'_k about a_k in powers of $(p'_{ij}-p_{ij})$. We first introduce some notations.

We can write $\bar{P}_k$ and U as sums of products of transition probabilities with plus signs by Lemma 2. We denote by $\bar{P}^{ij}_k$ the sum of terms in the right hand side of (2.12) containing p_{ij}, and define that $U^{ij}=\sum_{k=1}^{r}\bar{P}^{ij}_k$. Namely,

(2.19) $$\bar{P}^{ij}_k=p_{ij}\frac{\partial}{\partial p_{ij}}\bar{P}_k \qquad (i,\ j,\ k=1,\ 2,\ \cdots,\ r;\ i\neq j)$$

and

(2.20) $$U^{ij}=p_{ij}\frac{\partial}{\partial p_{ij}}U \qquad (i,\ j=1,\ 2,\ \cdots,\ r;\ i\neq j)\ ,$$

where we differentiate $\bar{P}_k$ and U regarding them as functions of p_{ij} $(i,\ j=1,\ 2,\ \cdots,\ r;\ i\neq j)$ and we do not consider that p_{ii} $(i=1,\ 2,\ \cdots,\ r)$ are variables for them.

We define ϵ_{ij} by

(2.21) $$p'_{ij}=p_{ij}(1+\epsilon_{ij}) \qquad (i,\ j=1,\ 2,\ \cdots,\ r;\ i\neq j)\ ,$$

and assume that $|\epsilon_{ij}|\leqq\epsilon$ for sufficiently small ϵ. Then by Taylor's formula we obtain

Theorem 3.

(2.22) $$a'_k=a_k+a_k\sum_{i=1}^{r}\sum_{\substack{j=1\\ j\neq i}}^{r}\left(\frac{\bar{P}^{ij}_k}{\bar{P}_k}-\frac{U^{ij}}{U}\right)\epsilon_{ij}+O(\epsilon^2)$$

for each k ($k=1, 2, \cdots, r$).

Proof. Applying Lemma 1 to the Markov chain C', we have

$$a'_k=\frac{\bar{P}'_k}{U'}\,. \tag{2.23}$$

By Taylor's formula, we can expand it as

$$a'_k=\frac{\bar{P}'_k}{U'}\bigg|_{p'_{ij}=p_{ij}}+\sum_{i=1}^{r}\sum_{\substack{j=1\\ j\neq i}}^{r}\left\{\frac{\partial}{\partial p'_{ij}}\frac{\bar{P}'_k}{U'}\bigg|_{p'_{ij}=p_{ij}}\right\}(p'_{ij}-p_{ij})+R\,, \tag{2.24}$$

where R represents the residual term and it may be replaced by $O(\epsilon^2)$. By (2.19), (2.20) and (2.21),

$$\begin{aligned}
&\left\{\frac{\partial}{\partial p'_{ij}}\frac{\bar{P}'_k}{U'}\bigg|_{p'_{ij}=p_{ij}}\right\}(p'_{ij}-p_{ij}) \\
&=\left\{\frac{1}{U'}\frac{\partial\bar{P}'_k}{\partial p'_{ij}}\bigg|_{p'_{ij}=p_{ij}}\right\}p_{ii}\epsilon_{ij}-\left\{\frac{\bar{P}'_k}{U'^2}\frac{\partial U'}{\partial p'_{ij}}\bigg|_{p'_{ij}=p_{ij}}\right\}p_{ij}\epsilon_{ij} \\
&=\frac{1}{U}\bar{P}^{ij}_k\epsilon_{ij}-\frac{\bar{P}_k}{U^2}U^{ij}\epsilon_{ij} \\
&=\frac{\bar{P}_k}{U}\left(\frac{\bar{P}^{ij}_k}{\bar{P}_k}-\frac{U^{ij}}{U}\right)\epsilon_{ij}\,.
\end{aligned} \tag{2.25}$$

Hence (2.24) becomes

$$a'_k=\frac{\bar{P}_k}{U}+\frac{\bar{P}_k}{U}\sum_{i=1}^{r}\sum_{\substack{j=1\\ j\neq i}}^{r}\left(\frac{\bar{P}^{ij}_k}{\bar{P}_k}-\frac{U^{ij}}{U}\right)\epsilon_{ij}+O(\epsilon^2)\,, \tag{2.26}$$

and using Lemma 1 again, we obtain (2.22).

2.4 Examples

We have obtained the bounds of a'_k in Theorem 1 and Corollary 1. They only use the information of a_k, and Example 1 below shows that the bounds are nearly attained by a pair of Markov chains having negligibly small weights on states other than two states. It is expected that the bounds of a'_k will be greatly improved if we use all of the information of the limiting vector. Unfortunately we have never been able to obtain such bounds in general case. Example 2 treats a special case where p_{ij} ($i, j=1, 2, \cdots, r$; $i\neq j$) are equal to a constant p, and it seems to show the possibility of improvement of the bounds.

Example 1. Now we shall show that the bounds (2.9) are nearly attained by a pair of Markov chains. Let

$$P=\begin{pmatrix} p_{11} & p_{12} & & & \\ p_{21} & p_{22} & p_{23} & \text{\Large 0} & \\ p_{31} & & p_{33} & p_{34} & \\ p_{41} & \text{\Large 0} & & p_{44} & p_{45} \\ p_{51} & & & & p_{55} \end{pmatrix}. \tag{2.27}$$

Then $\bar{P}_k$'s are given by

$$\begin{aligned} \bar{P}_1&=(p_{21}+p_{23})(p_{31}+p_{34})(p_{41}+p_{45})p_{51} \\ \bar{P}_2&=\quad p_{12}\times(p_{31}+p_{34})(p_{41}+p_{45})p_{51} \\ \bar{P}_3&=\quad p_{12}\times p_{23}\times(p_{41}+p_{45})p_{51} \\ \bar{P}_4&=\quad p_{12}\times p_{23}\times p_{34}\times p_{51} \\ \bar{P}_5&=\quad p_{12}\times p_{23}\times p_{34}\times p_{45}\,. \end{aligned} \tag{2.28}$$

Let us consider the difference between a_5' and a_5. By Theorem 3 we have

$$\begin{aligned} a_5'-a_5=a_5\{ \qquad\qquad & +\epsilon_{12}(1-U^{12}/U) \\ -\epsilon_{21}U^{21}/U & +\epsilon_{23}(1-U^{23}/U) \\ -\epsilon_{31}U^{31}/U & +\epsilon_{34}(1-U^{34}/U) \\ -\epsilon_{41}U^{41}/U & +\epsilon_{45}(1-U^{45}/U) \\ -\epsilon_{51}U^{51}/U & +O(\epsilon^2)\}\,. \end{aligned} \tag{2.29}$$

Now we assume that $p_{21}=p_{31}=p_{41}=\delta$, $p_{12}=p_{23}=p_{34}=p_{34}=\delta^2$ and $p_{51}=\delta^5$ for sufficiently small δ. Then

$$\begin{aligned} &\bar{P}_1=\delta^8(1+\delta)^3\,,\quad \bar{P}_2=\delta^9(1+\delta)^2\,,\quad \bar{P}_3=\delta^{10}(1+\delta)\,, \\ &\bar{P}_4=\delta^{11}\,,\quad \bar{P}_5=\delta^8\,, \end{aligned} \tag{2.30}$$

and we have approximately

$$\begin{aligned} &U^{21}\approx\bar{P}_1\,,\quad U^{31}\approx\bar{P}_1\,,\quad U^{41}\approx\bar{P}_1\,,\quad U^{51}\approx\bar{P}_1\,, \\ &U^{12}\approx\bar{P}_5\,,\quad U^{23}\approx\bar{P}_5\,,\quad U^{34}\approx\bar{P}_5\,,\quad U^{45}\approx\bar{P}_5 \end{aligned} \tag{2.31}$$

and $\qquad U\approx\bar{P}_1+\bar{P}_5\,.$

Hence

$$a_5'-a_5\approx a_5\{(\epsilon_{12}+\epsilon_{23}+\epsilon_{34}+\epsilon_{45})-(\epsilon_{21}+\epsilon_{31}+\epsilon_{41}+\epsilon_{51})\}\bar{P}_1/U\,, \tag{2.32}$$

and if $\epsilon_{12}=\epsilon_{23}=\epsilon_{34}=\epsilon_{45}=\epsilon$ and $\epsilon_{21}=\epsilon_{31}=\epsilon_{41}=\epsilon_{51}=-\epsilon$, then it becomes

(2.33) $$a'_5-a_5\approx 8(1-a_5)a_5\epsilon\ .$$

This coincides with the bound in (2.3) for $r=5$.

Example 2. We consider the extreme case where all p_{ij} $(i, j=1, 2, \cdots, r;\ i\neq j)$ are equal to p (some constant). Then the following lemma can be proved by direct calculations.

Lemma 5. If $p_{ij}=p$ for all i, j such that $i\neq j$, then we have

(2.34) $$\bar{P}_k=r^{r-2}p^{r-1}\ ,$$

(2.35) $$\bar{P}_k^{ij}=r^{r-3}p^{r-1} \quad \text{if } i\neq k \text{ and } j\neq k,\ j\neq i$$
$$=2r^{r-3}p^{r-1} \quad \text{if } i\neq k \text{ and } j=k,$$

(2.36) $$U=r^{r-1}p^{r-1}\ ,$$

(2.37) $$U^{ij}=r^{r-2}p^{r-1} \quad \text{if } j\neq i\ .$$

By Theorem 3, we have

(2.38) $$a'_k-a_k=a_k\frac{1}{r}\Big(\sum_{i\neq k}\epsilon_{ik}-\sum_{j\neq k}\epsilon_{kj}\Big)+O(\epsilon^2)\ .$$

Hence for the particular Markov chain C,

(2.39) $$|a'_k-a_k|\leqq 2(1-a_k)a_k\epsilon+O(\epsilon^2)\ .$$

In this case the multiplier $(r-1)$ in (2.3) vanishes.

3. Bounds of Some Basic Quantities of Finite Markov Chains

In Section 2, we saw that each element a_k of the limiting vector of a regular Markov chain C can be written as a quotient of sums of products of transition probabilities with plus signs. This fact enables us to obtain simple bounds of a'_k of another Markov chain C' in Theorems 1 and 2. The same idea can be applied to other quantities related with finite Markov chains. In this section, we shall show that many quantities related with finite Markov chains can be represented as quotients of sums of products of transition probabilities with plus signs or as sums of such quotients, and that simple bounds can be obtained using such representations.

We will use the following notations.

$s_1, s_2, \cdots, s_r$	states of a Markov chain C
P	transition matrix of the chain
$\boldsymbol{M}_i[\boldsymbol{f}]$	the mean value of a random variable $\boldsymbol{f}$ when the chain is started from state s_i
$\boldsymbol{Var}_i[\boldsymbol{f}]$	the variance of $\boldsymbol{f}$ when the chain is started from state s_i
$G=\{g_{ij}\}$	matrix with entries g_{ij}
$\delta=\{g_i\}$	column vector with entries g_i
ξ	column vector with all entries 1
I	identity matrix
G_{sq}	matrix whose entries are the squares g_{ij}^2 of the entries of G
G_{dg}	diagonal matrix whose ith diagonal entry is g_{ii} of G

3.1 Some basic quantities of absorbing Markov chains

We consider an absorbing Markov chain C with states $s_1, s_2, \cdots, s_r$. We let $\boldsymbol{T}=\{s_1, s_2, \cdots, s_s\}$ be the set of transient states, and $\tilde{\boldsymbol{T}}=\{s_{s+1}, s_{s+2}, \cdots, s_r\}$ be the set of absorbing states. Then the transition matrix $P=\{p_{ij}\}$ of the Markov chain has the form

$$(3.1)\qquad P=\begin{array}{c} \begin{array}{cc} s & r-s \end{array} \\ \left(\begin{array}{c|c} Q & R \\ \hline O & I \end{array}\right) \end{array}\begin{array}{l} \\ s \\ r-s\,. \end{array}$$

We shall deal with the following quantities for the chain.

$\boldsymbol{n}_i$	number of times in state s_j before absorbing
$\boldsymbol{t}$	number of steps taken before absorption
$\boldsymbol{m}$	total number of transient states entered before absorption
b_{ij}	probability starting in state s_i that the chain is absorbed in state s_j
h_{ij}	probability starting in state s_i that the chain is ever in state s_j

$N=\{n_{ij}\}=\{\boldsymbol{M}_i[\boldsymbol{n}_j]\}$	$(s_i, s_j \in \boldsymbol{T})$
$N_2=\{\boldsymbol{Var}_i[\boldsymbol{n}_j]\}$	$(s_i, s_j \in \boldsymbol{T})$
$\tau=\{\boldsymbol{M}_i[\boldsymbol{t}]\}$	$(s_i \in \boldsymbol{T})$
$\tau_2=\{\boldsymbol{Var}_i[\boldsymbol{t}]\}$	$(s_i \in \boldsymbol{T})$
$\mu=\{\boldsymbol{M}_i[\boldsymbol{m}]\}$	$(s_i \in \boldsymbol{T})$

$B=\{b_{ij}\}$ $(s_i \in T,\ s_j \in \tilde{T})$

$H=\{h_{ij}\}$ $(s_i,\ s_j \in T)$

We can prove that each entry of above matrices or vectors, except τ_2, can be represented as a quotient of two sums of products of transition probabilities with plus signs, or as a sum of such quotients. We denote the matrix $(I-Q)$ by $\bar{Q}$ and the cofactor of the (i, j)th entry of $\bar{Q}$ by $\bar{Q}(i, j)$. By Lemmas 9, 10, 11 and 12 in Appendix, we can easily prove that $|\bar{Q}|$, $\bar{Q}(i, j)$, $\bar{Q}(i, i)-\bar{Q}(i, j)$ and $\bar{Q}(i, i)-|\bar{Q}|$ are written as sums of products of transition probabilities with plus signs. So we shall show that quantities defined above can be represented in terms of p_{ij}, $|\bar{Q}|$, and $\bar{Q}(i, j)$. We use the relations proved in [2].

Since $N=(I-Q)^{-1}=\bar{Q}^{-1}$,

$$M_i[n_j]=n_{ij}=\bar{Q}(j, i)/|\bar{Q}| \,. \tag{3.2}$$

Since $N_2=N(2N_{dg}-I)-N_{sq}$,

$$\begin{aligned} Var_i[n_j]&=2n_{ij}n_{jj}-n_{ij}-n_{ij}^2 \\ &=\frac{\bar{Q}(j, i)}{|\bar{Q}|^2}\{(\bar{Q}(j, j)-|\bar{Q}|)+(\bar{Q}(j, j)-\bar{Q}(j, i))\} \,. \end{aligned} \tag{3.3}$$

Since $\tau=N\xi$,

$$M_i[t]=\sum_{k=1}^{s} n_{ik}=\frac{1}{|\bar{Q}|}\sum_{k=1}^{s}\bar{Q}(k, i) \,. \tag{3.4}$$

Since $\tau_2=(2N-I)\tau-\tau_{sq}$,

$$Var_i[t]=2\sum_{k=1}^{s}\sum_{j=1}^{s} n_{ij}n_{ik}-\sum_{k=1}^{s} n_{ik}-\{\sum_{k=1}^{s} n_{ik}\}^2 \,. \tag{3.5}$$

This quantity cannot be written as a quotient of two sums of products of transition probabilities with plus signs (see Example 3 below).

Since $\mu=(NN_{dg}^{-1})$.

$$M_i[m]=\sum_{k=1}^{s} n_{ik}/n_{kk}=\sum_{k=1}^{s}\bar{Q}(k, i)/\bar{Q}(k, k) \,. \tag{3.6}$$

Since $B=NR$,

$$b_{ij}=\sum_{k=1}^{s} n_{ik}p_{kj}=\frac{1}{|\bar{Q}|}\sum_{k=1}^{s} p_{kj}\bar{Q}(k, i) \,. \tag{3.7}$$

Since $H=(N-I)N_{dg}^{-1}$,

$$(3.8) \qquad h_{ii}=(n_{ii}-1)/n_{ii}=(\bar{Q}(i, i)-|\bar{Q}|)/\bar{Q}(i, i),$$

and for $j\neq i$

$$(3.9) \qquad h_{ij}=n_{ij}/n_{jj}=\bar{Q}(j, i)/\bar{Q}(j, j).$$

Thus we have shown that each quantity considered above can be written as a quotient of sums of products with plus signs except for τ_2. However the following example shows that τ_2 cannot be represented in such a form.

Example 3. We consider an absorbing Markov chain with the transition matrix

$$(3.10) \qquad P=\begin{pmatrix} p_{11} & p_{12} & p_{13} & 0 \\ 0 & p_{22} & 0 & p_{24} \\ 0 & 0 & p_{33} & p_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A tedious calculation shows that in this case

$$(3.11) \qquad \boldsymbol{Var}_1[\boldsymbol{t}]=\{p_{11}p_{24}^2p_{34}^2+(p_{12}+p_{13})(p_{12}p_{22}p_{34}^2+p_{13}p_{24}^2p_{33}) + p_{12}p_{13}(p_{24}-p_{34})^2\}/(p_{12}+p_{13})^2p_{24}^2p_{34}^2,$$

and the numerator of the right hand side of (3.11) cannot be written as a sum of products of p_{ij}'s with plus signs. Hence we cannot adopt our method because it contains a subtractive operation. Only the second order moment $\boldsymbol{M}_i[\boldsymbol{t}^2]$ can be written in a desired form. Since $\boldsymbol{M}_i[\boldsymbol{t}^2]=(2N-I)$,

$$(3.12) \qquad \boldsymbol{M}_i[\boldsymbol{t}^2]=2\sum_{\substack{k=1\\k\neq i}}^{s}\sum_{j=i}^{s}n_{ik}n_{kj}+\sum_{j=1}^{s}(2n_{ii}-1)n_{ij}$$
$$=\frac{1}{|\bar{Q}|^2}\{2\sum_{\substack{k=1\\k\neq i}}^{s}\sum_{j=1}^{s}\bar{Q}(k, i)\bar{Q}(j, k) + \sum_{j=1}^{s}(2\bar{Q}(i, i)-|\bar{Q}|)\bar{Q}(j, i)\}.$$

3.2 Bounds of basic quantities of an absorbing Markov chain C'

We consider two absorbing Markov chains C and C'. We use the notations defined in Section 3.1 for quantities of the chain C, and use the same notations with primes for corresponding quantities of the chain C'. We shall obtain simple bounds for the basic quantities of the chain C' under the assumption that

$$(3.13) \qquad (1+\epsilon)^{-1}p_{ij} \leqq p'_{ij} \leqq (1+\epsilon)p_{ij} \qquad (i=1, 2, \cdots, s,\ j=1, 2, \cdots, r;\ i \neq j)$$

for some positive constant ϵ. For (3.20) and (3.24) below, we need a further assumption that

$$(3.14) \qquad (1+\epsilon)^{-1}p_{ii} \leqq p'_{ii} \leqq (1+\epsilon)p_{ii} \qquad (i=1, 2, \cdots, s).$$

Our main tools are given in the following lemma.

Lemma 6. If (3.13) holds, then

$$(3.15) \qquad (1+\epsilon)^{-s}|\bar{Q}| \leqq |\bar{Q}'| \leqq (1+\epsilon)^{s}\bar{Q}$$

$$(3.16) \qquad (1+\epsilon)^{-s+1}\bar{Q}(i, j) \leqq \bar{Q}'(i, j) \leqq (1+\epsilon)^{s-1}\bar{Q}(i, j)$$

$$(3.17) \qquad (1+\epsilon)^{-s+1}(\bar{Q}(i, i)-\bar{Q}(i, j)) \leqq \bar{Q}'(i, i)-\bar{Q}'(i, j) \leqq (1+\epsilon)^{s-1}(\bar{Q}(i, i)-\bar{Q}(i, j)).$$

If (3.13) and (3.14) hold, then

$$(3.18) \qquad (1+\epsilon)^{-s}(\bar{Q}(i, i)-|\bar{Q}|) \leqq \bar{Q}'(i, i)-|\bar{Q}'| \leqq (1+\epsilon)^{s}(\bar{Q}(i, i)-|\bar{Q}|).$$

Proof. This lemma is easily proved by Lemmas 9, 10, 11 and 12 in Appendix and the inequalities in (2.10).

From (3.2)~(3.9), we can obtain the bounds using Lemma 6. Here we show the upper bounds only, because lower bounds can be obtained by replacing $(1+\epsilon)$ in the upper bounds with $(1+\epsilon)^{-1}$.

(3.19)	$\boldsymbol{M}'_i[\boldsymbol{n}'_j] \leqq (1+\epsilon)^{2s-1}\boldsymbol{M}_i[\boldsymbol{n}_j]$	under (3.13)
(3.20)	$\boldsymbol{Var}'_i[\boldsymbol{n}'_j] \leqq (1+\epsilon)^{4s-1}\boldsymbol{Var}_i[\boldsymbol{n}_j]$	under (3.13) and (3.14)
(3.21)	$\boldsymbol{M}'_i[\boldsymbol{t}'] \leqq (1+\epsilon)^{2s-1}\boldsymbol{M}_i[\boldsymbol{t}]$	under (3.13)
(3.22)	$\boldsymbol{M}'_i[\boldsymbol{m}')] \leqq (1+\epsilon)^{2s-2}\boldsymbol{M}_i[\boldsymbol{m}]$	under (3.13)
(3.23)	$b'_{ij} \leqq (1+\epsilon)^{2s}b_{ij}$	under (3.13)
(3.24)	$h'_{ii} \leqq (1+\epsilon)^{2s-1}h_{ii}$	under (3.13) and (3.14)
(3.25)	$h'_{ij} \leqq (1+\epsilon)^{2s-2}h_{ij} \qquad (i \neq j)$	under (3.13)

Some of these bounds can be improved using the relations

$$\bar{Q}(i, i)=\bar{Q}(i, j)+(\bar{Q}(i, i)-\bar{Q}(i, j)) \tag{3.26}$$

or

$$|\bar{Q}|=\sum_{\substack{k=1\\k\neq i}}^{s} p_{ik}(\bar{Q}(i, i)-\bar{Q}(i, k))+(\sum_{k=s+1}^{r} p_{ik})\bar{Q}(i, i)\,. \tag{3.27}$$

For example, (3.25) can be improved as

$$h'_{ij}\leqq h_{ij}/\{h_{ij}+(1+\epsilon)^{-2s+2}(1-h_{ij})\}\,, \tag{3.28}$$

and (4.19) for $i=j$ can be improved as

$$M'_i[n'_i]\leqq(1+\epsilon)n_{ii}/\{(\sum_{k=s+1}^{r} p_{ik})n_{ii}+(1+\epsilon)^{-2s+2}\sum_{\substack{k=1\\k=i}}^{s} p_{ik}(n_{ii}-n_{ki})\}\,. \tag{3.29}$$

3.3 Some basic quantities of regular Markov chains

We let $s_1, s_2, \cdots, s_r$ be the states of a regular Markov chain C and $P=\{p_{ij}\}$ be its transition matrix. We shall consider the following quantities.

$\alpha=\{a_i\}$ limiting vector (stationary distribntion of the chain)

A matrix with each row α

$Z=\{z_{ij}\}=(I-P+A)^{-1}$ (fundamental matrix)

$M=\{m_{ij}\}$ matrix of mean number of steps required to reach s_j for the first time, starting in s_i

$W=\{w_{ij}\}$ matrix of variances for the number of steps required to reach s_j, starting in s_i

These quantities except for off-diagonal entries of Z and W can be written as quotients of sums of products of transition probabilities with plus signs or as sums of such quotients. We can easily guess that off-diagonal entries of Z and W cannot be represented in such forms. Off-diagonal entries of Z have both possibilities of taking positive values and nagative values. So they cannot be represented as quotients of positive terms. Off-diagonal entries of W are essentially the same as entries τ_2 in Section 3.1.

Now we shall show that above quantities can be written in desired forms. We will use the notations defined in Section 2.1 again. Besides we denote by $\bar{P}_k(i, j)$ $(i, j, k=1, 2, \cdots, r;\ i\neq k,\ j\neq k)$ the cofactor of the entry $-p_{ij}$ (or $1-p_{ii}$, if $i=j$) of the matrix formed by deleting the kth row and the kth column from $(I-P)$. As proved in Lemma 10 in Appendix, $\bar{P}_k(i, j)$ can also be represented as a sum of products of transition probabilities with plus signs.

By Lemma 1,

$$a_k=\frac{\bar{P}_k}{U}\,. \tag{3.30}$$

By Lemma 13 in Appendix,

$$z_{kk}=\frac{\bar{P}_k}{U}+\frac{1}{U^2}\sum_{\substack{j=1\\ j\neq k}}^{r}\sum_{\substack{i=1\\ i\neq k}}^{r}\bar{P}_j\bar{P}_k(i, j)\,. \tag{3.31}$$

Since $m_{kk}=1/a_k$,

$$m_{kk}=\frac{U}{\bar{P}_k}\,. \tag{3.32}$$

By changing the notations in (3.4),

$$m_{jk}=\frac{1}{\bar{P}_k}\sum_{\substack{i=1\\ i\neq k}}^{r}\bar{P}_k(i, j) \qquad (j\neq k)\,. \tag{3.33}$$

Since $w_{kk}=(2z_{kk}-a_k)/a_k^2$,

$$w_{kk}=\frac{U}{\bar{P}_k}+\frac{2}{\bar{P}_k^2}\sum_{\substack{j=1\\ j\neq k}}^{r}\sum_{\substack{i=1\\ i\neq k}}^{r}\bar{P}_j\bar{P}_k(i, j)\,. \tag{3.34}$$

3.4 Bounds of basic quantities of a regular Markov chain C'

We consider two regular Markov chains C and C'. We use the notations defined in Section 3.3 for quantities of the chain C, and the same notations with primes for corresponding quantities of the chain C'. We shall obtain simple bounds of basic quantities of the chain C' under the assumption (2.1).

Our main tools are Lemma 3 and the following.

Lemma 7. If (2.1) holds, we have

$$(1+\epsilon)^{-r+2}\bar{P}_k(i, j)\leqq\bar{P}_k'(i, j)\leqq(1+\epsilon)^{r-2}\bar{P}_k(i, j)\,. \tag{3.35}$$

Proof. This is an immediate corollary of Lemma 10.

From (3.30)~(3.34), we can obtain the bounds of the quantities using Lemmas 3 and 7 as follows. Here we show only the upper bounds under the condition (2.1), because the lower bounds can be obtained by replacing $(1+\epsilon)$ in the upper bounds by $(1+\epsilon)^{-1}$.

(3.36) $a'_k \leqq (1+\epsilon)^{2r-2} a_k$

(3.37) $z'_{kk} \leqq (1+\epsilon)^{4r-5} z_{kk}$

(3.38) $m'_{kk} \leqq (1+\epsilon)^{2r-2} m_{kk}$

(3.39) $m'_{jk} \leqq (1+\epsilon)^{2r-3} m_{jk} \quad (j \neq k)$

(3.40) $w'_{kk} \leqq (1+\epsilon)^{4r-5} w_{kk}$

Some of these bounds can be improved, *e.g.*, (3.36) can be improved as Theorem 1 in Section 2.1. However we omit discussions in detail here.

4. An Upper Bound of the Variance of a'_k

In Section 2, we got simple bounds of a'_k. In the case where p'_{ij}'s are random variables, we can also obtain a simple bound of the variance of a'_k. Here we consider the simplest case where p'_{ij} $(i, j = 1, 2, \cdots, r;\ j \neq i)$ are mutually independent random variables distributed about p_{ij}. (We can easily modify the results for the case where row vectors of P' are mutually independent random vectors. See the note at the end of this section.) We will use italic letter M and Var to represent means and variances with respect to the random variables p'_{ij}. We assume that p'_{ij} $(i \neq j)$ satisfy the following three conditions.

(4.1) $M[\epsilon_{ij}]=0\,,$ $\quad$ *i.e.*, $M[p'_{ij}]=p_{ij}$

(4.2) $M[\epsilon_{ij}^2]=\sigma_{ij}^2 \leqq \sigma^2\,,$ $\quad$ *i.e.*, $Var[p'_{ij}]=p_{ij}^2\sigma_{ij}^2 \leqq p_{ij}^2\sigma^2$

(4.3) $M[\epsilon_{ij}^3]=O(\sigma^3)\,,$ $\quad$ *i.e.*, $M[|p'_{ij}-p_{ij}|^3]=O(\sigma^3)$

Theorem 4. If p'_{ij} $(i, j=1, 2, \cdots, r;\ i \neq j)$ are mutually independent random variables satisfying the three conditions (4.1), (4.2) and (4.3), then

$$M[a_k'] = a_k + O(\sigma^2) \tag{4.4}$$

and

$$Var[a_k'] = a_k^2 \sum_{i=1}^{r} \sum_{\substack{j=1 \\ j\neq i}}^{r} \left(\frac{\bar{P}_k^{ij}}{\bar{P}_k} - \frac{U^{ij}}{U} \right)^2 \sigma_{ij}^2 + O(\sigma^3) . \tag{4.5}$$

Proof. (4.4) is an obvious consequence of Theorem 3 with a slight change in the residual term. (4.5) can be easily proved by taking the expectation of the square of (2.22).

Theorem 5. If p'_{ij} $(i, j=1, 2, \cdots, r;\ i\neq j)$ are mutually independent random variables and if the three conditions (4.1), (4.2) and (4.3) are satisfied, then

$$Var[a_k'] \leqq 2(r-1) a_k^2 (1-a_k)^2 \sigma^2 + O(\sigma^3) . \tag{4.6}$$

Proof. By Theorem 4, we have

$$Var[a_k'] \leqq \sigma^2 a_k^2 \sum_{i=1}^{r} \left\{ \sum_{\substack{j=1 \\ j\neq i}}^{r} \left(\frac{\bar{P}_k^{ij}}{\bar{P}_k} - \frac{U^{ij}}{U} \right)^2 \right\} + O(\sigma^3) . \tag{4.7}$$

The sums in the braces in (4.7) are dominated by quadratic functions of a_k and a_i as follows. When $i\neq k$,

$$\begin{aligned} \sum_{j\neq i} \left(\frac{\bar{P}_k^{ij}}{\bar{P}_k} - \frac{U^{ij}}{U} \right)^2 &= \frac{1}{(\bar{P}_k U)^2} \sum_{j\neq i} \{ \bar{P}_k^{ij}(U-\bar{P}_k) - \bar{P}_k(U^{ij} - \bar{P}_k^{ij}) \}^2 \\ &\leqq \frac{1}{(\bar{P}_k U)^2} [\sum_{j\neq i} \{ \bar{P}_k^{ij}(U-\bar{P}_k) \}^2 + \sum_{j\neq i} \{ \bar{P}_k(U^{ij} - \bar{P}_k^{ij}) \}^2] \\ &\leqq \frac{1}{(\bar{P}_k U)^2} [\{ \sum_{j\neq i} \bar{P}_k^{ij}(U-\bar{P}_k) \}^2 + \{ \sum_{j\neq i} \bar{P}_k(U_k^{ij} - \bar{P}_k^{ij}) \}^2] \end{aligned} \tag{4.8}$$

Since $\sum_{j\neq i} \bar{P}_k^{ij} = \bar{P}_k$ and $\sum_{j\neq k} U^{kj} = U - \bar{P}_k$, the right hand side of the above inequality is equal to

$$\begin{aligned} &\frac{1}{(\bar{P}_k U)^2} [\{ \bar{P}_k(U-\bar{P}_k) \}^2 + \{ \bar{P}_k(U-\bar{P}_i-\bar{P}_k) \}^2] \\ &= (1-a_k)^2 + (1-a_i-a_k)^2 \\ &= 2(1-a_k)^2 - 2(1-a_k)a_i + a_i^2 \end{aligned} \tag{4.9}$$

When $i=k$, since $\bar{P}_k^{kj}=0$,

$$\sum_{j=k} \left(\frac{\bar{P}_k^{kj}}{\bar{P}_k} - \frac{U^{kj}}{U} \right)^2 \leqq \frac{1}{U^2} (\sum_{k=j} U^{kj})^2 = (1-a_k)^2 . \tag{4.10}$$

Hence (4.7) becomes

$$Var\{a'_k\} \leqq \sigma^2 a_k^2 [\sum_{\substack{i=1\\ i\neq k}}^{r} \{2(1-a_k)^2 - 2(1-a_k)a_i + a_i^2\} + (1-a_k)^2] + O(\sigma^3) \tag{4.11}$$
$$\leqq \sigma^2 a_k^2 [(2r-1)(1-a_k)^2 - 2(1-a_k)\sum_{i\neq k} a_i + (\sum_{i\neq k} a_i)^2] + O(\sigma^3)$$
$$= 2(r-1)\sigma^2 a_k^2 (1-a_k)^2 + O(\sigma^3),$$

and this completes the proof.

For the case where P' differs from P in one row, we have the following theorem. Its proof is essentially contained in the preceding proof.

Theorem 6. If p'_{hj} $(j=1, 2, \cdots, h-1, h+1, \cdots, r)$ are mutually independent random variables satisfying the three conditions (4.1), (4.2) and (4.3), and $p'_{ij}=p_{ij}$ $(i, j=1, 2, \cdots, r;\ i\neq h)$, then when $k\neq h$,

$$Var\{a'_k\} \leqq a_k^2\{(1-a_k)^2 + (1-a_h-a_k)^2\}\sigma^2 + O(\sigma^3), \tag{4.12}$$

and when $k=h$

$$Var\{a'_h\} \leqq a_h^2(1-a_h)^2\sigma^2 + O(\sigma^3). \tag{4.13}$$

Example 4. The bound in Theorem 5 is the best one of those which use the information of a_k only. In order to show this fact, we shall consider the Markov chain in Example 1 again. By the approximation (2.32) we have

$$Var[a'_5] = M[(a'_5-a_5)^2] \tag{4.14}$$
$$\approx a_5^2(1-a_5)^2\{\sigma_{12}^2+\sigma_{23}^2+\sigma_{34}^2+\sigma_{45}^2+\sigma_{21}^2+\sigma_{31}^2+\sigma_{41}^2+\sigma_{51}^2\}.$$

If $\sigma_{12}^2=\sigma_{23}^2=\cdots=\sigma_{51}^2=\sigma^2$, then it follows that

$$Var[a'_5] \approx 8a_5^2(1-a_5)^2\sigma^2, \tag{4.15}$$

and this coincides with the bound in Theorem 5 for $r=5$ except for a term of order σ^3.

Example 5. We shall again consider the extreme case where all p_{ij} $(i, j=1, 2, \cdots, r;\ i\neq j)$ are equal to p (some constant). In this case, by Theorems 4 and 5, we have

$$Var[a'_k] = a_k^2 \frac{1}{r^2}\{\sum_{i\neq k}\sigma_{ik}^2 + \sum_{j\neq k}\sigma_{kj}^2\} + O(\sigma^3). \tag{4.16}$$

If $\sigma_{ik}^2=\sigma_{kj}^2=\sigma^2$ for all i $(\neq k)$ and j $(\neq k)$, then it follows that

$$(4.17) \qquad Var[a'_k]=2(r-1)\frac{1}{r^4}\sigma^2+O(\sigma^3)\,.$$

Note: Up to now, we assume that p'_{ij}'s are mutually independent random variables. We can also obtain a simple bound of variance of a'_k under a weaker condition.

Theorem 7. If row vectors $\{p'_{ij};\ j=1, 2, \cdots, r\}$ $(i=1, 2, \cdots, r)$ are mutually independent random vectors and if the three conditions (4.1), (4.2) and (4.3) are satisfied for all p'_{ij}, then

$$(4.13) \qquad Var[a'_k]\leqq 2(2r-3)a_k^2(1-a_k)^2\sigma^2+O(\sigma^3)\,.$$

The proof of this theorem is similar to that of Theorem 5, so here we omit the proof.

5. Upper Bounds of the Variances of Basic Quantities

The method for obtaining a simple bound of the variance of a'_k can be applied to some basic quantities of the Markov chain C' defined in Sections 3.1 and 3.3. Again we assume that p'_{ij} $(i, j=1, 2, \cdots, r;\ i\neq j)$ are mutually independent random variables satisfying the three conditions (4.1), (4.2) and (4.3). (We can easily modify the results for the case where row vectors of P' are mutually independent random vectors. See the notes at the ends of Sections 5.1 and 5.2.) The means of the quantities of the chain C' coincide with the corresponding quantities of the chain C except for terms of order σ^2. Here we show simple bounds of the variances of the quantities. Since the procedures for obtaining the bounds are essentially the same as in Section 4, we omit the proofs.

5.1 Upper bounds of the variances of basic quantities of an absorbing Markov chain C'

If p'_{ij} $(i\neq j)$ are mutually independent random variables satisfying the three conditions (4.1), (4.2) and (4.3), then

$$(5.1) \qquad Var[M'_i[n'_j]]\leqq(2s-1)\sigma^2\{M_i[n_j]\}^2+O(\sigma^3)$$

(5.2) $$Var[M_i'[t']] \leqq (2s-1)\sigma^2\{M_i[t]\}^2 + O(\sigma^3)$$

(5.3) $$Var[M_i'[m']] \leqq 2(s-1)\sigma^2\{M_i[m]\}^2 + O(\sigma^3)$$

(5.4) $$Var[b_{ij}'] \leqq 2s\sigma^2\{b_{ij}\}^2 + O(\sigma^3)$$

(5.5) $$Var[h_{ij}'] \leqq 2(s-1)\sigma^2\{h_{ij}\}^2 + O(\sigma^3) \qquad (i \neq j)$$

For the quantities $Var_i'[n_j']$ and h_{ii}', we cannot get upper bounds by this method. Because the representations (3.3) and (3.8) contain $(\bar{Q}(j, j)-|\bar{Q}|)$ and $(\bar{Q}(i, i)-|\bar{Q}|)$, and in order to write $(\bar{Q}(i, i)-|\bar{Q}|)$ as a sum of products of transition probabilities with plus signs we must use p_{ii} in addition to p_{ij} $(j \neq i)$. Since p_{ij}' $(j=1, 2, \cdots, r)$ are dependent, we cannot apply our method for these quantities. However we can get bounds of them under a weaker condition (see the following note).

Note: We can modify the above results for the case where row vectors of P' are mutually independent random vectors and all p_{ij}' satisfy the three conditions (4.1), (4.2) and (4.3). In this case the modified bounds are given by multiplying the bounds in (5.1)~(5.5) by two. The factor "two" results from the difference between the following two sequences of inequalities. If a_i, $b_i > 0$ and $\sum_i a_i = \sum_i b_i = c$, then

(5.6) $$\sum_i (a_i - b_i)^2 \leqq \sum_i a_i^2 + \sum_i b_i^2 \leqq (\sum_i a_i)^2 + (\sum_i b_i)^2 = 2c^2$$

and

(5.7) $$\{\sum_i |a_i - b_i|\}^2 \leqq (\sum_i a_i + \sum_i b_i)^2 = 4c^2 .$$

However here we omit discussions in detail.

In this case we can also obtain upper bounds of $Var_i'[n_j']$ and h_{ii}'.

(5.8) $$Var[Var_i'[n_j']] \leqq (4s-1)\sigma^2\{Var_i[n_j]\}^2 + O(\sigma^3)$$

(5.9) $$Var[h_{ii}'] \leqq (2s-1)\sigma^2\{h_{ii}\}^2 + O(\sigma^3)$$

5.2 Upper bounds of the variances of basic quantities of a regular Markov chain C'

If p_{ij}' $(i \neq j)$ are mutually independent random variables satisfying the three conditions (4.1), (4.2) and (4.3), then

(5.10) $$Var[a_k'] \leqq 2(r-1)\sigma^2\{a_k\}^2 + O(\sigma^3)$$

(5.11) $Var[z'_{kk}] \leqq 8(r-1)\sigma^2\{z_{kk}\}^2 + O(\sigma^3)$

(5.12) $Var[m'_{kk}] \leqq 2(r-1)\sigma^2\{m_{kk}\}^2 + O(\sigma^3)$

(5.13) $Var[m'_{jk}] \leqq (2r-3)\sigma^2\{m_{jk}\}^2 + O(\sigma^3) \quad (j \neq k)$

(5.14) $Var[w'_{kk}] \leqq 8(r-1)\sigma^2\{w_{kk}\}^2 + O(\sigma^3)$

Note: We can also modify the above results for the case where row vectors of P' are mutually independent random vectors and all p'_{ij} satisfy the three conditions (4.1), (4.2) and (4.3). In this case the modified bounds are given by multiplying the bounds in (5.10)~(5.14) by two.

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Appendix

Here we shall prove several lemmas related with a matrix defined by (A.1) and (A.2) below. Main results of this paper are based on Lemma 9 below, and essentially the same result as this lemma has been established by Bott and Mayberry [1] in relation to arborescences of a graph in order to calculate the determinants of certain matrices met with in economics. However we might as well prove the lemma here again.

We consider a matrix $X=\{x_{ij};\ i,\ j=1,\ 2,\ \cdots,\ n\}$ with components

(A.1) $$x_{ij}=-y_{ij} \qquad (i,\ j=1,\ 2,\ \cdots,\ n;\ i\neq j)$$

and

(A.2) $$x_{ii}=\sum_{k=1}^{n} y_{ik} \qquad (i=1,\ 2,\ \cdots,\ n),$$

where y_{ij} $(i,\ j=1,\ 2,\ \cdots,\ n)$ are some constants or variables. The following Lemmas 9, 10, 11 and 12 show fundamental properties of such a matrix. We prepare some notations and a lemma.

Let $X(j_1,\ j_2,\ \cdots,\ j_n)$ be the matrix formed from X by substituting $y_{1j_1}=y_{2j_2}=\cdots=y_{nj_n}=1$ and $y_{ij}=0$ for $j\neq j_i$. Also we conveniently denote an analogous $(n-1)\times(n-1)$ matrix by a similar notation with a prime as $X'(j_1,\ j_2,\ \cdots,\ j_{n-1})$.

Lemma 8. $|X(j_1,\ j_2,\ \cdots,\ j_n)|=0$ or $+1$.

Proof. If $j_i\neq i$ for every i, then $|X(j_1,\ j_2,\ \cdots,\ j_n)|=0$ because row sums of $X(j_1,\ j_2,\ \cdots,\ j_n)$ are equal to zero. If $j_i=i$ for specific i, then the i-th row of $X(j_1,\ j_2,\ \cdots,\ j_n)$ is a unit vector, namely, the ith entry of it is 1 and all other entries are 0. So if we expand the determinant of $X(j_1,\ j_2,\ \cdots,\ j_n)$ by the ith row, then it is reduced to the determinant of $X'(j_1,\ \cdots,\ j_{i-1},\ j_{i+1},\ \cdots,\ j_n)$. Hence the lemma is proved by the mathematical induction on n.

Lemma 9. Let X be the matrix defined by (A.1) and (A.2). Then its determinant is written as a sum of products of y_{ij}'s with plus signs:

(A.3) $$|X|=\sum_{J} y_{1j_1}y_{2j_2}\cdots y_{nj_n}$$

where the summation is taken over a set J of ordered n-tuples $(j_1,\ j_2,\ \cdots,\ j_n)$.

Proof. Since

(A.4) $$|X|=\begin{vmatrix} y_{11}+\cdots+y_{1n} & -y_{12} & \cdots & -y_{1n-1} & -y_{1n} \\ -y_{21} & y_{21}+\cdots+y_{2n} & \cdots & -y_{2n-1} & -y_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ -y_{n-11} & -y_{n-11} & \cdots & y_{n-11}+\cdots+y_{n-1n} & -y_{n-1n} \\ -y_{n1} & -y_{n2} & \cdots & -y_{nn-1} & y_{n1}+\cdots+y_{nn} \end{vmatrix},$$

we can write it as a sum of products of y_{ij}'s with coefficients:

$$|X|=\sum_{J^*}\delta(j_1, j_2, \cdots, j_n)y_{1j_1}y_{2j_2}\cdots y_{nj_n}, \tag{A.5}$$

where $\delta(j_1, j_2, \cdots, j_n)$ are integers and the summation is taken over the set J^* of all ordered n-tuples $(j_1, j_2, \cdots, j_n)$ of indices. We shall prove that $\delta(j_1, j_2, \cdots, j_n)=0$ or $+1$. We note that $\delta(j_1, j_2, \cdots, j_n)$ is equal to the determinant of $X(j_1, j_2, \cdots, j_n)$. So this lemma is an immediate corollary of Lemma 8.

Lemma 10. Let X be the matrix defined by (A.1) and (A.2). Then cofactor $X(i, j)$ of the entry x_{ij} of X is written as a sum of products of y_{kh}'s $(k\neq i)$ with plus signs:

$$X(i, j)=\sum_{J_{ij}} y_{1j_1}y_{2j_2}\cdots y_{i-1j_{i-1}}y_{i+1j_{i+1}}\cdots y_{nj_n} \tag{A.6}$$

where the summation is taken over a set J_{ij} of ordered $(n-1)$-tuples $(j_1, j_2, \cdots, j_{i-1}, j_{i+1}, \cdots, j_n)$.

Proof. $X(i, j)$ is written as

$$\begin{aligned} X(i, j)& \\ &=\sum\delta_{ij}(j_1, \cdots, j_{i-1}, j_{i+1}, \cdots, j_n)y_{1j_1}\cdots y_{i-1j_{i-1}}y_{i+1j_{i+1}}\cdots y_{nj_n}. \end{aligned} \tag{A.7}$$

The coefficient $\delta_{ij}(j_1, \cdots, j_{i-1}, j_{i+1}, \cdots, j_n)$ is equal to the determinant $X(i, j)$ for $y_{1j_1}=\cdots=y_{nj_n}=1$ and $y_{kh}=0\ (h\neq j_k)$. It can be easily proved that if $i\neq j$ and $j_j=j$, then $\delta_{ij}(j_1, \cdots, j_{i-1}, j_{i+1}, \cdots, j_n)=0$. Also it can be proved that if $i=j$ or if $i\neq j$ and $j_j\neq j$, then $\delta_{ij}(j, \cdots, j_{i-1}, j_{i+1}, \cdots, j_n)$ is equal to the determinant of $X'(j'_1, \cdots, j'_{i-1}, j'_{i+1}, \cdots, j'_n)$ where

$$j'_k=\begin{cases} k & \text{if } j_k=i \\ j_k & \text{otherwise} \end{cases} \tag{A.8}$$

So this lemma is proved by Lemma 8.

Lemma 11. Let J_{ij} $(i, j=1, 2, \cdots, n)$ be the set of $(n-1)$-tuples defined in Lemma 10. If $i\neq j$, then

$$J_{ii}\supsetneqq J_{ij}, \tag{A.9}$$

or equivalently, $X(i, i)-X(i, j)$ can be written as a sum of products of y_{kh}'s $(k\neq i)$ with plus signs:

(A.10) $$X(i, i)-X(i, j) = \sum_{J_{ii}-J_{ij}} y_{1j_1}y_{2j_2}\cdots y_{i-1j_{i-1}}y_{i+1j_{i+1}}\cdots y_{nj_n}\,.$$

Proof. $X(i, i)-X(i, j)$ can be considered as the determinant of the matrix formed by replacing the ith row of X by a vector which has $+1$ in the ith entry, -1 in the jth entry and 0's in other entries. This matrix has a special form of X with $y_{ij}=1$ and $y_{ik}=0$ for $k=1, 2, \cdots, j-1, j+1, \cdots, n$. Thus this lemma can be proved immediately from Lemma 9.

Lemma 12. Let X be the matrix defined by (A.1) and (A.2), and define $y_{io}=1-(y_{i1}+y_{i2}+\cdots+y_{in})$ for given i $(i=1, 2, \cdots, n)$. Then $X(i, i)-|X|$ is written as a sum of products of y_{io} and y_{kh}'s $(k, h=1, 2, \cdots, n)$ with plus signs:

(A.11) $$X(i, i)-|X|=\sum_{J^{**}} y_{1j_1}y_{2j_2}\cdots y_{nj_n}$$

where the summation is taken over a set J^{**} of ordered n-tuples $(j_1, j_2, \cdots, j_n)$.

Proof. By expanding $|X|$ in cofactors by the ith row, we have

(A.12) $$|X|=(y_{i1}+y_{i2}+\cdots+y_{in})X(i, i)-\sum_{k\neq i} y_{ik}X(i, k)\,.$$

It follows that

(A.13) $$X(i, i)-|X|=y_{io}X(i, i)+\sum_{k\neq i} y_{ik}X(i, k)\,.$$

Hence by Lemma 10, $X(i, i)-|X|$ is written as (A.11).

Finally we shall prove the equation (3.31).

Lemma 13. For each k,

(A.14) $$z_{kk}=\frac{\bar{P}_k}{U}+\frac{1}{U^2}\sum_{\substack{j=1\\ j\neq k}}^{r}\sum_{\substack{i=1\\ i\neq k}}^{r}\bar{P}_j\bar{P}_k(i, j)\,.$$

Proof. We first show that $|I-P+A|=U$. If we add all the columns from the first to the $(r-1)$st of the matrix $(I-P+A)$ to the last column, then the last column becomes to ξ which is the column vector with all entries 1. Next for each $i=1, 2, \cdots, r-1$, we subtract $a_i\xi$ from the ith column. Then the matrix becomes to one formed from $(I-P)$ by replacing the last column by ξ. Since this matrix has the same determinant as that of $(I-P+A)$, if we expand the determinant of the matrix by the last column, we have

$$\text{(A.15)} \qquad |I-P+A|=\sum_{k=1}^{r}\bar{P}(k, r),$$

where $\bar{P}(k, r)$ is the cofactor of the entry $-p_{kr}$ (or $1-p_{rr}$, if $k=r$) of the matrix $(I-P)$. It can be easily proved that $\bar{P}(k, r)=\bar{P}_k$. Hence by the definition of U in Section 2, (A.15) becomes

$$\text{(A.16)} \qquad |I-P+A|=U.$$

Since $Z=(I-P+A)^{-1}$,

(A.17)

$$z_{kk}=\frac{1}{U}\begin{array}{r} \\ \\ \\ \\ k) \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} \phantom{p_{12}+\cdots+p_{1r}+a_1} & \phantom{p_{21}+\cdots+p_{2r}+a_2} & k & & \phantom{p_{r1}+\cdots+p_{rr-1}+a_r} \end{array} \\ \begin{vmatrix} p_{12}+\cdots+p_{1r}+a_1 & -p_{12}+a_2 & \cdots & \cdots & -p_{1r}+a_r \\ -p_{21}+a_1 & p_{21}+\cdots+p_{2r}+a_2 & \cdots & \cdots & -p_{2r}+a_r \\ \vdots & \vdots & & & \vdots \\ \hline \vdots & \vdots & & & \vdots \\ -p_{r1}+a_1 & -p_{r2}+a_2 & \cdots & \cdots & p_{r1}+\cdots+p_{rr-1}+a_r \end{vmatrix} \end{array},$$

Each column of the determinant in (A.17) is a sum of two column vectors, one of which is of the form $a_i\xi$. Hence the determinant can be written as a sum of 2^{r-1} determinants, each of which is formed from $(I-P)$ by deleting the kth row and the kth column and replacing some columns by vectors of the form $a_i\xi$. Of such determinants those which have two or more columns of the form $a_i\xi$, are equal to zero. Hence the determinant in (A.17) can be written as a sum of r determinants $\bar{P}_k$ and $F_k(j)$ $(j=1, 2, \cdots, k-1, k+1, \cdots, r)$, where $F_k(j)$ is the determinant of a matrix formed from $(I-P)$ by deleting the kth row and the kth column and replacing the jth column by $a_j\xi$. If we expand $F_k(j)$ by the jth column, then

$$\text{(A.18)} \qquad F_k(j)=\sum_{\substack{i=1\\ i\neq k}}^{r} a_j\bar{P}_k(i, j).$$

Hence we have

(A.19) $$z_{kk}=\frac{1}{U}[\bar{P}_k+\sum_{\substack{j=1\\j\neq k}}^{r}\sum_{\substack{i=1\\i\neq k}}^{r}a_j\bar{P}_k(i,\ j)]\,,$$

and using Lemma 1, we can obtain the expression (A.14).