

OPTIMAL REPLACEMENT POLICIES FOR A REPAIRABLE SYSTEM WITH MARKOVIAN TRANSITION OF STATES

AKIHIKO TAHARA and TOSHIO NISHIDA

Osaka University

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Abstract

For a complex system, it may be too expensive to replace a system by a new one at any failure occasions. Naturally, we have to repair and use it. We shall consider two preventive replacement models in which a system is repairable but cannot recover completely after each repair. More precisely, the state which determines the life time of a system does change by repair of failures one after another and its transition is Markovian. Furthermore, the mean life of a repaired system may decrease and the repair cost may increase with the number of repairs. Optimal replacement policies of a few types for such a system are discussed. Here, the criterion for optimal policies is to minimize the expected total maintenance cost per unit time.

1. Introduction

Optimal replacement and maintenance policies have been studied by many authors. The policies discussed most often are replacement on failure policy, age replacement policy and block replacement policy. In these policies, it is assumed commonly that a maintenance action regenerates the system, that is, the system is as good as new im-

mediately after the completion of maintenance action. This assumption enables us to treat the problem easily by renewal theory.

In many practical situations, however, the maintenance action is not necessarily the replacement of the whole system, but is often the repair or replacement of a part of the system. The recent large-scale and complicated systems enjoy such situations. It is not always valid, therefore, that the maintenance action renews a system completely and the life distribution of a system after maintenance action remains unchanged. Intuitively, we may expect the mean life of a repaired system to be less than that of a new system, that is, a system may deteriorate in terms of mean life.

R.E. Barlow and L. Hunter [1] proposed a maintenance model called minimal repair model which reflects to some degree the situations stated above, but in this model it is assumed that the system failure rate is not disturbed by any repair of failures between the successive replacements. More precisely, the life distribution of a system after minimal repair is given by $[F(\xi+x)-F(\xi)]/[1-F(\xi)]$, where $F(x)$ is that of a new system and ξ is the total operating time until each minimal repair. Notice that if $F(x)$ has a decreasing mean residual life, the mean life of a system after minimal repair is non-increasing with the number of minimal repairs. For this model several preventive replacement policies have been studied by R.E. Barlow and L. Hunter [1], H. Makabe and H. Morimura [3], and H. Morimura [4]. They also proved the existence and uniqueness of the optimal policies in type II and III under the strictly IFR assumption.

The authors [5] proposed a new maintenance model and discussed the optimality of a few types of replacement policies. In this model, we assume the repair can not always renew the failed system, that is, the mean life of the system after repair is smaller than that before it. Moreover, the cost of repair may increase with the number of repairs (failures) of the system. Of course, the system regenerates

completely after a replacement. These three assumptions characterize our maintenance model which will be explained more precisely in the following.

Let X_n be the time to failure of a system which has been repaired $(n-1)$ times before. X_1 is the life time of a new system. We assume that X_n 's are nonnegative and mutually independent random variables, and that $E(X_n)$, the expectation of X_n , is nonincreasing in $n \geq 1$. Let C_n be the expected cost of repair for the n -th failure of the system and it is assumed to be nondecreasing in $n \geq 1$. Finally, let R_n be the expected cost of replacement of the system which has been repaired $(n-1)$ times before. If a system is replaced at the n -th failure, then the replacement cost is R_n .

In the above model, we can consider that a new system is in state 1 and that a system, which has been repaired $(n-1)$ times before, is in state n , then X_n , C_n and R_n mean the life time, the expected repair cost and the expected replacement cost of a system being in state n , respectively. But, in this case the state of a system is merely equal to the number of failures plus one and the transition of states is of course deterministic. Then, this model may describe such a situation that a system deteriorates with the number of failures. In minimal repair model, on the other hand, if we consider that $[F(\xi+x)-F(\xi)]/[1-F(\xi)]$ is the life distribution of a system being in state ξ , then the state of a new system is 0 and the state of a repaired system, whose total operating time until the last failure is ξ , is ξ . In this case, the degree of deterioration of a repaired system does not depend upon the number of failures but upon the total operating time only. For both models, the state number could be used instead of specifying the life distribution of a repaired system, and the larger it is, the larger is the degree of deterioration (in terms of mean life) of the system.

Although the number of failures and the total operating time may

be the most important factors with respect to the degree of deterioration of a repaired system, but in general they could not offer the complete information about it. Then the stochastic models would be required. In the next section, we shall generalize our preceding model on system's states and its transitions, that is, we consider a Markov chain with continuously infinite number of states. Furthermore, we shall propose another preventive maintenance model which is a generalization of minimal repair model. The former will be called Model I and the latter Model II. The situations stated above will be, to a certain extent, reflected in these models.

2. Formulation of Models

In this section, we shall propose two new preventive maintenance models and introduce some replacement policies. In the system under consideration, it is assumed that even though it fails we can completely recover its performance by repair, but the life distribution of a repaired system does change one after another in such a way that its mean life steadily decreases. These situations may be expressed as follows: Let ξ be the 'state' of a system. The distribution of life time, the length of interval between the beginning of operation and the next failure, of a new or repaired system is determined by the state in which the system is found at the beginning of operation. Moreover, the state of a repaired system changes one after another and the law of its transition is Markovian.

We shall now define precisely our two models. Let E_n and X_n be the state and the life time of a system which has been repaired $(n-1)$ times before, respectively. Of course, E_1 and X_1 are those of a new one. In the two-dimensional stochastic process $\{(E_n, X_n); n=1, 2, \dots\}$ defined on a probability space, we assume, in Model I,

$$P[E_1 \leq \xi] = \Phi(\xi)$$

and

$$P[X_n \leq x, E_{n+1} \leq \xi | X_i = x_i, E_j = \xi_j; i=1, 2, \dots, n-1, j=1, 2, \dots, n] = F(x; \xi_n) \cdot \Psi(\xi_n, \xi).$$

On the other hand, in Model II we assume

$$P[E_1 \leq \xi] = \Phi(\xi),$$

$$P[X_n \leq x | E_i = \xi_i; i=1, 2, \dots, n] = F(x; \xi_n)$$

and

$$\alpha E_{n+1} = \alpha E_1 + X_1 + X_2 + \dots + X_n, \quad 0 < \alpha < \infty.$$

In the last expression, it will be natural to define $E_{n+1} = E_1$ or ∞ for all $n \geq 1$ according as $\alpha = \infty$ or 0. The above both processes are the special cases of semi-Markov processes with continuously infinite number of states. Note that in Model I, if the state of a system repaired $(n-1)$ times before, E_n is ξ_n , then the distribution function of X_n , the life time of this system, and its next state, E_{n+1} , are respectively given by $F(x; \xi_n)$ and $\Psi(\xi_n, \xi)$, which are independent of each other, that is, X_n and E_{n+1} are conditionally independent for a given E_n . But in Model II, these are closely related for $0 < \alpha < \infty$, that is, if E_n is ξ_n , then the distribution function of X_n is $F(x; \xi_n)$ and E_{n+1} is determined by ξ_n and X_n such that $\alpha E_{n+1} = \alpha \xi_n + X_n$. In both models, the state number specifying the life time could be interpreted to represent the degree of deterioration of a repaired system. Hence, the deterioration law is Markovian. Furthermore, we could imagine, roughly speaking, such a situation that the degree of deterioration depends mainly upon the number of failures for Model I and the total operating time for Model II. To simplify the discussion, suppose, for both models, the transition of states does not occur at the time of failure but the completion of repair, and the state of a system remains unchanged between successive repairs. First, we shall explain Model I and then Model II.

In Model I, the following immediate consequences of the above definitions will be useful in later discussions.

$$P[X_i \leq x_i; i=1, 2, \dots, n | \mathcal{E}_i = \xi_i; i=1, 2, \dots, n] = \prod_{i=1}^n F(x_i; \xi_i),$$

that is, X_1, X_2, \dots, X_n are mutually conditionally independent, given $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n$. For convenience, if we let $X(\xi)$ be the life time of a system being in state ξ , that is, the conditional random variable of X_n given $\mathcal{E}_n = \xi$, then the distribution function of $X(\xi)$ is $F(x; \xi)$. Throughout this paper the words ‘decreasing’ and ‘increasing’ are read to mean nonincreasing and nondecreasing, respectively. We suppose $E[X(\xi)] = \int_0^\infty x dF(x; \xi)$ is decreasing in ξ .

On the other hand, we have

$$P[\mathcal{E}_1 \leq \xi] = \Phi(\xi)$$

and

$$P[\mathcal{E}_{n+1} \leq \xi | \mathcal{E}_i = \xi_i; i=1, 2, \dots, n] = \Psi(\xi_n, \xi),$$

so that $\{\mathcal{E}_n; n=1, 2, \dots\}$ is a homogeneous imbedded-Markov chain in which state space is the set of real numbers. The initial probability distribution function $\Phi(\xi)$ and the transition probability distribution function $\Psi(\xi, \eta)$ of this chain are assumed to satisfy, in addition to the usual conditions, the further conditions $\Phi(\xi) = 0$ for $\xi < 0$ and $\Psi(\xi, \eta) = 0$ for $\eta < \xi$. Hence \mathcal{E}_n is nonnegative and increasing in $n \geq 1$. The functions

$$(1) \quad \Phi_n(\xi) = \int_0^\infty d_\gamma \Phi_{n-1}(\gamma) \Psi(\gamma, \xi), \quad \Phi_1(\xi) = \Phi(\xi)$$

and

$$(2) \quad \Phi(\xi_1, \xi_2, \dots, \xi_n) = \int_0^{\xi_1} d\Phi(\eta_1) \int_{\eta_1}^{\xi_2} d_{\eta_2} \Psi(\eta_1, \eta_2) \dots \\ \times \int_{\eta_{n-2}}^{\xi_{n-1}} d_{\eta_{n-1}} \Psi(\eta_{n-2}, \eta_{n-1}) \Psi(\eta_{n-1}, \xi_n)$$

are the absolute (unconditional) probability distribution function of \mathcal{E}_n and the joint distribution function of $(\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n)$, and are equal to zero unless $\xi \geq 0$ and $\xi_n \geq \xi_{n-1} \geq \dots \geq \xi_1 \geq 0$, respectively.

Combining the assumptions about $E[X(\xi)]$ and $\Psi(\xi, \eta)$, it will be seen that the mean life of a system repaired $(n-1)$ times previously, *i.e.*,

$$(3) \quad E(X_n) = \int_0^{\infty} E[X(\xi)] d\Phi_n(\xi)$$

is decreasing in $n \geq 1$, so that Model I may describe the situation that a system deteriorates in terms of mean life with the number of repairs (failures) of it. To avoid the trivial case, we assume $E(X_1) > 0$.

Furthermore, let $C(\xi)$, being assumed to be increasing in $\xi \geq 0$, be the expected cost of repair for a failed system in state ξ . Let $R(\xi)$ be the expected cost of replacement for a system (not necessarily failed) being in state ξ . Assume $R(\xi)$ is increasing and convex on $\xi \geq 0$. When $C(\xi)$ and $R(\xi)$ are constant independent of ξ , we denote them as C and R , respectively. If we put for $n \geq 1$

$$(4) \quad C_n = \int_0^{\infty} C(\xi) d\Phi_n(\xi) \quad \text{and} \quad R_n = \int_0^{\infty} R(\xi) d\Phi_n(\xi),$$

then C_n and R_n represent the expected repair cost at the time of n -th failure and the expected replacement cost for a system repaired $(n-1)$ times before, respectively.

Since Model I is completely specified by the five elements defined above, that is, the conditional life time $X(\xi)$, the repair cost $C(\xi)$, the replacement cost $R(\xi)$, the initial distribution $\Phi(\xi)$ and the transition distribution $\Psi(\xi, \eta)$, a system in Model I, from now on, will be denoted as $S_1[X(\xi), C(\xi), R(\xi), \Phi(\xi), \Psi(\xi, \eta)]$ for convenience. If we set $\Phi(\xi) = U_1(\xi)$ and $\Psi(\xi, \eta) = U_1(\eta - \xi)$, Model I reduces to our previous maintenance model stated in the introduction, where $U_y(x)$ is the unit step function; $U_y(x) = 1$ or 0 according as $x \geq$ or $< y$.

Again, for convenience we shall summarize the main assumptions about our system $S_1[X(\xi), C(\xi), R(\xi), \Phi(\xi), \Psi(\xi, \eta)]$ as follows:

- i) $E[X(\xi)]$ is decreasing in $\xi \geq 0$;
- ii) $C(\xi)$ is increasing in $\xi \geq 0$;

- iii) $R(\xi)$ is increasing and convex in $\xi \geq 0$;
- iv) $\Psi(\xi, \eta) = 0$ for $\eta < \xi$.

Unless otherwise stated, these are always assumed in Model I.

In Model II, $\{\mathcal{E}_n; n \geq 1\}$ also becomes a homogeneous Markov chain with continuously infinite number of states and its transition probability distribution function, say $\theta(\xi, \eta)$, is given by

$$\theta(\xi, \eta) = P[\mathcal{E}_{n+1} \leq \eta | \mathcal{E}_n = \xi] = F(\alpha(\eta - \xi); \xi).$$

Note $\theta(\xi, \eta) = 0$ for $\eta < \xi$ because X_n is a nonnegative random variable. If we assume $\Phi(\xi) = 0$ for $\xi < 0$ and define for $n \geq 2$

$$(5) \quad \Phi_n(\xi) = \int_0^\infty d_\eta \Phi_{n-1}(\eta) \theta(\eta, \xi) \quad \text{where} \quad \Phi_1(\xi) = \Phi(\xi),$$

then $\Phi_n(\xi)$ denotes the absolute probability distribution function of $\mathcal{E}_n \geq 0$. In the following, similar notations to Model I will be used. Let $X(\xi)$ and $C(\xi)$ be those as defined in Model I, respectively. Again, $E[X(\xi)]$ and $C(\xi)$ are assumed to be decreasing and increasing in $\xi \geq 0$, respectively. Finally, the expected cost of replacement is assumed to be constant independent of the failure history of a system and denoted by R .

Consequently, a system in Model II is completely specified by the conditional life time $X(\xi)$, the expected repair cost $C(\xi)$, the expected replacement cost R , the initial state distribution $\Phi(\xi)$ and α , so as in Model I, it will be convenient to denote it as $S_{II}[X(\xi), C(\xi), R, \Phi(\xi), \alpha]$. As in the previous model, a system S_{II} may deteriorate in the sense of mean life with the number of repairs because $E[X(\xi)]$ is decreasing in ξ and \mathcal{E}_n is increasing in n . But in this case, the dominating cause of the degree of deterioration is the total operating time rather than the number of failures, since $\alpha \mathcal{E}_n = \alpha \mathcal{E}_1 + X_1 + X_2 + \dots + X_{n-1}$, $n \geq 2$.

In both models, it is assumed that the state of a system—whether operating or failed—is always known with certainty so that a maintenance action is instantly taken whenever a failure is detected, and that the maintenance time is not taken into account. But the last

assumption may be clearly no restriction by regarding maintenance cost as imputed cost including maintenance time. As usual, we shall assume a system is as good as new immediately after a replacement, that is, the probabilistic properties of the first and the successively replaced systems are identical and independent.

The motivation for preventive replacements stems from the fact that continuing to repair a system whenever it fails is often costly compared with replacing it according to an appropriate rule for an infinite time span. If states are not observable, the available information about them at the present time is only the past failure history of the system. Now, for Model I we shall consider the following preventive replacement policies which were introduced for minimal repair model by H. Makabe and H. Morimura [3];

Policy I: Replace a system at time t or k -th failure, whichever occurs first, but for the intervening failures repair it on these occasions. ($k=1, 2, \dots; 0 < t \leq \infty$)

Policy I': Replace a system at the first failure after t hours operating or k -th failure, whichever occurs first, but for the intervening failures repair it on these occasions. ($k=1, 2, \dots; 0 \leq t \leq \infty$)

Policy II: In Policy I, put $k = \infty$.

Policy II': In Policy I', put $k = \infty$.

Policy III: In Policy I or I', put $t = \infty$.

In each policy we shall reschedule preventive replacement under the policy immediately after a replacement.

Our objective is to seek an optimal replacement policy that minimizes the expected total maintenance cost per unit time over an infinite time span. In this paper, this objective will be called cost function. Let us denote the cost functions of Model I under Policy I, I', II, II' and III by $A_I(k, t)$, $A_{I'}(k, t)$, $A_{II}(t)$, $A_{II'}(t)$ and $A_{III}(k)$, respectively. By the definitions, $A_I(\infty, t) = A_{II}(t)$, $A_{I'}(\infty, t) = A_{II'}(t)$ and $A_I(k, \infty) = A_{I'}(k, \infty) = A_{III}(k)$. Moreover, Policy I'(II') improves Policy

I(II), *i.e.*, $A_T(k, t) \leq A_I(k, t)$, because when the operating time reaches t , the system is still operating generally, so we can use it until next failure.

For Model II, we shall consider Policy III only and denote its cost function by $B(k)$.

In the next section, we shall compare the policies of Model I and show that under certain restrictions there is an optimal policy of type III in the category of all policies defined above, and then we shall concentrate upon the optimal schedule of Policy III. In Section 4, we shall discuss Model II. Consequently, for minimal repair model which is a special case of Model II, the existence and uniqueness of the optimal policy in type III will be seen under the decreasing mean residual life assumption.

3. Model I

In this section we shall consider a system $S_I[X(\xi), C(\xi), R(\xi), \Phi(\xi), \Psi(\xi, \eta)]$. First, we shall begin to derive the cost function under each policy. For all $n \geq 0$, put $Z_n = X_0 + X_1 + X_2 + \dots + X_n$, $X_0 = 0$ and define

$$H(x; \xi_0, \xi_1, \xi_2, \dots, \xi_n) = U_0(x) * F(x; \xi_1) * F(x; \xi_2) * \dots * F(x; \xi_n)$$

where $*$ denotes the convolution operation. Z_n means the n -th failure time of a system before replacement and $H(x; \xi_0, \xi_1, \xi_2, \dots, \xi_n)$ is the distribution function of Z_n given $\xi_i = \xi_i; i = 1, 2, \dots, n$. Recall X_1, X_2, \dots, X_n are mutually conditionally independent, given $\xi_1, \xi_2, \dots, \xi_n$. Furthermore, we shall define the following random variables.

$N(x)$: the number of failures in $(0, x)$ when a system is new at time zero and then being repaired on.

$Q(x)$: the residual life of a system at time x , *i.e.*, the interval between x and the time of next failure.

For $n \geq 1$, clearly

$$P[N(x) = n - 1 | \xi_i = \xi_i; i = 1, 2, \dots, n] = H(x; \xi_0, \xi_1, \dots, \xi_{n-1}) - H(x; \xi_0, \xi_1, \dots, \xi_n)$$

and

$$P[N(x)=n-1]=\int\cdots\int P[N(x)=n-1|\mathcal{E}_i=\xi_i; i=1, 2, \dots, n] \\ d\Phi(\xi_1, \xi_2, \dots, \xi_n).$$

Under each policy, the replacement of a system is a regeneration point for the investment process, so that the intervals between successive replacements are independent and identically distributed random variables. Such intervals are known as cycles. The length of cycle under Policy I', say $L(k, t)$, is

$$L(k, t)=\text{Min}[Z_k, t+Q(t)].$$

Because

$$\begin{aligned} & E[L(k, t)|\mathcal{E}_n=\xi_n; n=1, 2, \dots, k] \\ &= E\{\text{Min}[X(\xi_1)+X(\xi_2)+\cdots+X(\xi_k), t+Q(t)]|\mathcal{E}_n \\ &= \xi_n; n=1, \dots, k\} \\ &= E\left[\sum_{n=1}^k Y_n \cdot X(\xi_n)\right] \\ &= \sum_{n=1}^k E[Y_n] \cdot E[X(\xi_n)] \\ &= \sum_{n=1}^k H(t; \xi_0, \xi_1, \dots, \xi_{n-1}) \cdot E[X(\xi_n)], \end{aligned}$$

the expected cycle length is given by

$$E[L(k, t)]=\sum_{n=1}^k \int \cdots \int H(t; \xi_0, \xi_1, \dots, \xi_{n-1}) \cdot E[X(\xi_n)] \\ d\Phi(\xi_1, \xi_2, \dots, \xi_n),$$

where $Y_n=1$ or 0 according as $X(\xi_1)+\cdots+X(\xi_{n-1})\leq$ or $>t$, so that Y_n and $X(\xi_n)$ are independent. On the other hand, denoting the expected maintenance (repair and replacement) cost over a cycle under Policy I' as $M(k, t)$, we have

$$\begin{aligned} & E[M(k, t)|\mathcal{E}_i=\xi_i; i=1, 2, \dots, k] \\ &= \sum_{n=1}^{k-1} P[N(t)=n-1|\mathcal{E}_i=\xi_i; i=1, \dots, n] \cdot \left[\sum_{i=0}^{n-1} C(\xi_i)+R(\xi_n)\right] \end{aligned}$$

$$+ P[N(t) \geq k-1 | \mathcal{E}_i = \xi_i; i=1, \dots, k] \cdot \left[\sum_{i=0}^{k-1} C(\xi_i) + R(\xi_k) \right],$$

therefore,

$$\begin{aligned} & E[M(k, t)] \\ &= \sum_{n=1}^k \int \dots \int H(t; \xi_0, \xi_1, \dots, \xi_{n-1}) \cdot \{ C(\xi_{n-1}) \\ & \quad + [R(\xi_n) - R(\xi_{n-1})] \} d\Phi(\xi_1, \xi_2, \dots, \xi_n), \end{aligned}$$

where $C(\xi_0) = R(\xi_0) = 0$.

As in the case of replacement on failure policy, using the elementary renewal theorem [2], we can prove that the cost function of Policy I' is equal to

$$A_{I'}(k, t) = \frac{E[M(k, t)]}{E[L(k, t)]}.$$

Note that the dependency of the random variables $L(k, t)$ and $M(k, t)$ does not affect the above expression. Next, the cycle length under Policy I is $\text{Min}(Z_k, t)$ and on account of the assumption, its expected maintenance cost over a cycle is identical with that of Policy I', so that

$$A_I(k, t) = \frac{E[M(k, t)]}{E[\text{Min}(Z_k, t)]}.$$

In particular,

$$A_{II}(t) = \frac{E[M(\infty, t)]}{t}, \quad A_{II'}(t) = \frac{E[M(\infty, t)]}{E[t + Q(t)]}$$

and

$$(6) \quad A_{III}(k) = \frac{E[M(k, \infty)]}{E[L(k, \infty)]} = \frac{\sum_{n=1}^{k-1} C_n + R_k}{\sum_{n=1}^k E(X_n)},$$

where we used $H(\infty; \xi_0, \xi_1, \dots, \xi_n) = 1$ for $n \geq 0$.

Now we shall compare the cost functions of Policy I, I', II, II' and III, and show that under certain restrictions the optimal policy can be found in the policy of type III. As indicated in the previous

section, it suffices to see that the optimal policy in type I' reduces to the policy of type III. For convenience, we shall introduce a supplementary policy in the following;

Policy III': In Policy III, use the schedule k with probability p_k , where $p_k \geq 0$ and $\sum_{k=1}^{\infty} p_k = 1$. This is a random replacement policy of Policy III.

Let us denote the cycle length, the maintenance cost over one cycle and the cost function under Policy III' by $L(p_k)$, $M(p_k)$ and $A_{III'}(p_k)$, respectively. The following lemma may be rather obvious but will play an important role for the comparison of policies.

Lemma 1. The optimum replacement policy in type III' can be nonrandom, that is, it reduces to Policy III.

Proof. By the definition

$$L(p_k) = \sum_{k=1}^{\infty} p_k \cdot L(k, \infty) \quad \text{and} \quad M(p_k) = \sum_{k=1}^{\infty} p_k \cdot M(k, \infty),$$

then using the elementary renewal theorem [2], we have

$$A_{III'}(p_k) = \frac{\sum_{k=1}^{\infty} p_k \cdot E[M(k, \infty)]}{\sum_{k=1}^{\infty} p_k \cdot E[L(k, \infty)]} = \sum_{k=1}^{\infty} q_k \cdot A_{III}(k)$$

where

$$q_k = \frac{p_k \cdot E[L(k, \infty)]}{\sum_{k=1}^{\infty} p_k \cdot E[L(k, \infty)]} \geq 0 \quad \text{and} \quad \sum_{k=1}^{\infty} q_k = 1,$$

which completes the proof.

If the transition of states of a system is deterministic, *i.e.*, $P[\mathcal{E}_n = \xi_n; n=1, 2, \dots] = 1$, we can easily see

$$A_{I'}(k, t) = A_{III'}(p_n)$$

where

$$p_n = \begin{cases} P[N(t) = n-1] & \text{for } n=1, 2, \dots, k-1 \\ P[N(t) \geq k-1] & \text{for } n=k \\ 0 & \text{for } n \geq k+1. \end{cases}$$

Notice that this does not mean the identity of both policies of type I' and III'. For example, set $k = \infty$, then replacement before t can occur under Policy III', but never under Policy I'(II'). Irrespectively of this fact, Lemma 1 yields.

Theorem 1. In Model I, if the transition of states is deterministic, Policy III is optimal in the class of our policies.

This theorem may assert that Policy II', in comparison with Policy III, is disadvantageous because the number of failures in its successive cycles fluctuates one after another, that is, it will be too fast to replace in some cases and too slow in the other. Then, Policy II is disadvantageous because, in addition to the above fact, it wastes the residual life of a system. Whereas, the replacement under Policy II is usually taken before failure, as compared with Policy III after failure. Further, under Policy II a system is replaced periodically at times nt ($n=1, 2, \dots$) independent of the failure history, so we do not require the keeping of records on system use. But these practical advantages of Policy II, which will appear as the difference of replacement costs for both policies, are neglected in our models, then the foregoing statements are rather a matter of course. In the discussions made thus far, the assumptions i), ii), iii) and iv) in Section 2 are not necessary. In the following of this section, of course, these are always assumed.

Now, our problem is to expand the discussion to the case of stochastic transition of states. However, avoiding the difficulty of general cases, throughout the comparison of policies, we shall assume

$$(7) \quad F(x; \xi) \leq F(x; \eta) \quad \text{for all } x \geq 0 \text{ if } \xi \leq \eta$$

and

$$(8) \quad \Psi(\xi, \xi + \zeta) \geq \Psi(\eta, \eta + \zeta) \quad \text{for all } \zeta \geq 0 \text{ if } \xi \leq \eta,$$

that is, $X(\xi)$ is stochastically decreasing in ξ , and $(E_{n+1} - E_n)$ is stochastically increasing in E_n . Note the equality in (8) means that $\{E_n; n \geq 1\}$ is a spatial homogeneous chain.

Lemma 2. (see page 52 of [2]) Let $G_1(x)$ and $G_2(x)$ be distribution functions. We assume $G_1(0)=G_2(0)$ and $G_1(x)\geq G_2(x)$ for all $x\geq 0$. If $a(x)$ is decreasing in $x\geq 0$, then

$$\int_0^{\infty} a(x)dG_1(x)\geq\int_0^{\infty} a(x)dG_2(x);$$

and if $a(x)$ is increasing in $x\geq 0$, then the reverse inequality holds.

For convenience we say that a n -variate function $a(\xi_1, \xi_2, \dots, \xi_n)$ is increasing (decreasing) in $\xi_1, \xi_2, \dots, \xi_n$ if, and only if,

$$a(\xi_1, \xi_2, \dots, \xi_n)\underset{\leq}{\geq} a(\eta_1, \eta_2, \dots, \eta_n) \text{ for } \xi_i\leq\eta_i; i=1, 2, \dots, n,$$

where words and symbols in parentheses should be read together.

Lemma 3. Under (7),

- (i) $E[X(\xi)]$ is decreasing in $\xi\geq 0$.
- (ii) $H(x; \xi_0, \xi_1, \dots, \xi_n)$ is an increasing function of $\xi_1, \xi_2, \dots, \xi_n$ for all fixed $x\geq 0$.

Proof. (i) This is a direct result of the above lemma.

(ii) Obvious by the definition and hypothesis.

Lemma 4. Under (8),

- (i) if $a(\xi, \eta)$ is increasing (decreasing) in both ξ and $\eta\geq 0$, then

$$\int_{\xi}^{\infty} a(\xi, \eta)d_{\eta}\Psi(\xi, \eta) \text{ is increasing (decreasing) in } \xi\geq 0.$$

- (ii) if $a(\xi)$ is increasing and convex in $\xi\geq 0$, then

$$\int_{\xi}^{\infty} [a(\eta)-a(\xi)]d_{\eta}\Psi(\xi, \eta) \text{ is increasing in } \xi\geq 0.$$

- (iii) if $a(\xi_1, \xi_2, \dots, \xi_n)$ and $b(\xi_1, \xi_2, \dots, \xi_n)$ are both increasing in $\xi_1, \xi_2, \dots, \xi_n$, then

$$\begin{aligned} & \int \cdots \int a(\xi_1, \dots, \xi_n)b(\xi_1, \dots, \xi_n)d\Phi(\xi_1, \dots, \xi_n) \\ & \geq \int \cdots \int a(\xi_1, \dots, \xi_n)d\Phi(\xi_1, \dots, \xi_n) \\ & \quad \cdot \int \cdots \int b(\xi_1, \dots, \xi_n)d\Phi(\xi_1, \dots, \xi_n), \end{aligned}$$

where $\Phi(\xi_1, \dots, \xi_n)$ is defined by (2), and if $a(\xi_1, \xi_2, \dots, \xi_n)$ is increasing and $b(\xi_1, \xi_2, \dots, \xi_n)$ is decreasing in $\xi_1, \xi_2, \dots, \xi_n$, then the reverse inequality holds.

Proof. (i) By the hypothesis and Lemma 2, we have

$$\int_0^\infty a(\xi, \xi + \zeta) d_\zeta \Psi(\xi, \xi + \zeta) \stackrel{(\leq)}{=} \int_0^\infty a(\xi, \xi + \zeta) d_\zeta \Psi(\xi', \xi' + \zeta) \stackrel{(\geq)}{=} \int_0^\infty a(\xi', \xi' + \zeta) d_\zeta \Psi(\xi', \xi' + \zeta)$$

for $\xi \leq \xi'$ which concludes (i).

(ii) Evident due to (i), because a two-variate function $[a(\xi + \zeta) - a(\xi)]$ is increasing in both ξ and $\zeta \geq 0$.

(iii) We shall prove only the first assertion. The second part may be shown similarly. The proof proceeds by induction. If we put

$$\hat{a}(\xi) = a(\xi) - \int a(\xi) d\Phi(\xi) \quad \text{and} \quad \hat{b}(\xi) = b(\xi) - \int b(\xi) d\Phi(\xi),$$

then $\hat{a}(\xi)$ and $\hat{b}(\xi)$ are increasing in $\xi \geq 0$ and $\int \hat{a}(\xi) d\Phi(\xi) = \int \hat{b}(\xi) d\Phi(\xi) = 0$. Noting $\hat{a}(\xi) \cdot [\hat{b}(\xi) - \hat{b}(\xi^*)] \geq 0$ for all $\xi \geq 0$, where ξ^* is such that $\hat{a}(\xi^* - 0) \leq 0 \leq \hat{a}(\xi^* + 0)$, we have

$$\int \hat{a}(\xi) \hat{b}(\xi) d\Phi(\xi) = \int \hat{a}(\xi) \cdot [\hat{b}(\xi) - \hat{b}(\xi^*)] d\Phi(\xi) \geq 0,$$

which confirms (iii) for $n=1$. From this, we get

$$\begin{aligned} & \int a(\xi_1, \dots, \xi_{n-1}, \xi_n) b(\xi_1, \dots, \xi_{n-1}, \xi_n) d_{\xi_n} \Psi(\xi_{n-1}, \xi_n), \\ & \geq \int a(\xi_1, \dots, \xi_{n-1}, \xi_n) d_{\xi_n} \Psi(\xi_{n-1}, \xi_n) \\ & \quad \cdot \int b(\xi_1, \dots, \xi_{n-1}, \xi_n) d_{\xi_n} \Psi(\xi_{n-1}, \xi_n), \end{aligned}$$

where the both integrals in the right-hand side are increasing in $\xi_1, \xi_2, \dots, \xi_{n-1}$ due to (i) of this lemma. Thus, the asserted inequality follows inductively.

With the preparation made thus far, the following theorem can

be easily proven by the similar arguments to the deterministic case.

Theorem 2. If the assumptions (7) and (8) are hold, then in Model I, Policy III is optimal in the class of all our policies.

Proof. By the hypotheses and the above lemmas, we have

$$\begin{aligned} & E[L(k, t)] \\ & \leq \sum_{n=1}^k \int \cdots \int H(t; \xi_0, \xi_1, \dots, \xi_{n-1}) d\Phi(\xi_1, \dots, \xi_n) \\ & \quad \cdot \cdots \int E[X(\xi_n)] d\Phi(\xi_1, \dots, \xi_n) \\ & = \sum_{n=1}^{\infty} p_n \cdot \sum_{i=1}^n E(X_i) \end{aligned}$$

and

$$\begin{aligned} & E[M(k, t)] \\ & = \int R(\xi_1) d\Phi(\xi_1) + \sum_{n=2}^k \int \cdots \int H(t; \xi_0, \xi_1, \dots, \xi_{n-1}) \\ & \quad \cdot \left\{ C(\xi_{n-1}) + \int [R(\xi_n) - R(\xi_{n-1})] d_{\xi_n} \Psi(\xi_{n-1}, \xi_n) \right\} d\Phi(\xi_1, \dots, \xi_{n-1}) \\ & \geq \int R(\xi_1) d\Phi(\xi_1) + \sum_{n=2}^k \int \cdots \int H(t; \xi_0, \xi_1, \dots, \xi_{n-1}) d\Phi(\xi_1, \dots, \xi_{n-1}) \\ & \quad \cdot \int \cdots \int \left\{ C(\xi_{n-1}) + \int [R(\xi_n) - R(\xi_{n-1})] d_{\xi_n} \Psi(\xi_{n-1}, \xi_n) \right\} \\ & \quad d\Phi(\xi_1, \dots, \xi_{n-1}) \\ & = \sum_{n=1}^{\infty} p_n \cdot \left[\sum_{i=1}^{n-1} C_i + R_n \right], \end{aligned}$$

where

$$p_n = \begin{cases} P[N(t) = n-1] & \text{for } n=1, 2, \dots, k-1 \\ P[N(t) \geq k-1] & \text{for } n=k \\ 0 & \text{for } n \geq k+1. \end{cases}$$

Hence, $A_{IV}(k, t) \geq A_{III}(p_n)$, so that Lemma 1 applies. The proof is complete.

This theorem insists that comparing with Policy III, Policy II' and II are disadvantageous not only in the deterministic case, but

also in the case of the stochastic transition of states of a system, because of the reason stated under Theorem 1. One of the two conditions (7) and (8) is not sufficient for the above theorem, which will be easily seen in the following

Example 1. Let $X(\xi)=2$ for $0 \leq \xi < 1$, $F(x; \xi)=[U_1(x)+U_3(x)]/2$ for $1 \leq \xi < 2$ and $X(\xi)=2-U_3(\xi)-U_4(\xi)$ for $\xi \geq 2$, and let $C(\xi)=1$, $R(\xi)=4$ and $\Phi(\xi)=U_0(\xi)$ for $\xi \geq 0$, and finally let $\Psi(\xi, \eta)=[U_1(\eta-\xi)+U_2(\eta-\xi)]/2$ or $U_2(\eta-\xi)$ according as $0 \leq \xi < 1$ or $\xi \geq 1$, then $C_n=1$ and $R_n=4$ for $n \geq 1$, and $E(X_1)=E(X_2)=2$, $E(X_3)=1/2$ and $E(X_n)=0$ for $n \geq 4$, so that $A_{III}(1)=2$, $A_{III}(2)=5/4$ and $A_{III}(k)=2(k+3)/9$ for $k \geq 3$, but $A_{IV}(t)=21/17$ if $3 < t < 4$.

Example 2. Let $X(\xi)=2-U_2(\xi)-U_4(\xi)$, $C(\xi)=1$, $R(\xi)=4$ and $\Phi(\xi)=U_0(\xi)$ for $\xi \geq 0$, and let $\Psi(\xi, \eta)=[U_1(\eta-\xi)+U_2(\eta-\xi)]/2$, $U_3(\eta-\xi)$ or $U_1(\eta-\xi)$ according as $0 \leq \xi < 1$, $1 \leq \xi < 2$ or $\xi \geq 2$, then $C_n=1$ and $R_n=4$ for $n \geq 1$, and $E(X_1)=2$, $E(X_2)=3/2$, $E(X_3)=1/2$ and $E(X_n)=0$ for $n \geq 4$, so that $A_{III}(1)=2$, $A_{III}(2)=10/7$ and $A_{III}(k)=(k+3)/4$ for $k \geq 3$, but $A_{IV}(t)=11/8$ if $3 < t < 4$.

Note that the above two examples satisfy all assumptions for Model I but (7) or (8), respectively.

Cor. 1. Denote the optimal schedules of Policy II, II' and III by t^* , t^{**} and k^* , respectively. If the transition of states is deterministic or the assumptions (7) and (8) hold, then $t^* < \infty$ means $t^{**} < \infty$ and $t^{**} < \infty$ means $k^* < \infty$.

Proof. Since $A_{III}(k^*) \leq A_{IV}(t) \leq A_{III}(t)$ for all $t \geq 0$ and $A_{III}(\infty)=A_{IV}(\infty)=A_{III}(\infty)$, if $k^* = \infty$, then $t^{**} = \infty$ and if $t^{**} = \infty$, then $t^* = \infty$ which completes the proof.

We are now in a position to find out the optimal policy of type III so as to minimize the cost function $A_{III}(k)$ given by (6). In the remainder of section, the assumption (7) is not necessary. But the assumption (8) will be used again for the convexity of R_n . The following lemma is elementary, but since it will play an important role in our arguments, we state it here.

Lemma 5. Let $a(n)=b(n)/c(n)$, ($n=1, 2, \dots$), where $b(n) > 0$ and $c(n) > 0$

for all $n \geq 1$. If $b(n)$ is convex and $c(n)$ is concave in $n \geq 1$, then $-a(n)$ is unimodal (in the wide sense) unless it is monotone increasing or decreasing in $n \geq 1$. Furthermore, if the convexity of $b(n)$ or the concavity of $c(n)$ is strict, then $\Delta a(n) = 0$ for at most only one $n \geq 1$, say n^* , that is, $a(n)$ is strictly decreasing on $1 \leq n \leq n^*$ and strictly increasing on $n \geq n^* + 1$.

Proof. The difference $\Delta a(n)$ of $a(n)$ is

$$\Delta a(n) = \frac{\Delta b(n) \cdot c(n) - b(n) \cdot \Delta c(n)}{c(n) \cdot c(n+1)}.$$

By hypothesis, the difference of the numerator of the right-hand side is nonnegative; $\Delta^2 b(n) \cdot c(n+1) - b(n+1) \cdot \Delta^2 c(n) \geq 0$. Noting $c(n) \cdot c(n+1) > 0$, this implies that if $\Delta a(n) \geq 0$, then $\Delta a(m) \geq 0$ for all $m \geq n$, which confirms the first assertion of the lemma and the second is similarly proved.

Lemma 6. (i) If $a(\xi)$ is a monotone increasing (decreasing) function, then

$$E[a(\mathcal{E}_n)] = \int_0^\infty a(\xi) d\Phi_n(\xi)$$

is monotone increasing (decreasing) in $n \geq 1$.

(ii) Under (8), if $b(\xi)$ is an increasing and convex function, then $E[b(\mathcal{E}_n)]$ is convex in $n \geq 1$.

Proof. Since $\mathcal{E}_n \leq \mathcal{E}_{n+1}$, (i) is obvious. Then, using (ii) of Lemma 4, we can easily see

$$\Delta E[b(\mathcal{E}_n)] = \int_0^\infty \int_\xi^\infty [b(\eta) - b(\xi)] d_\eta(\xi, \eta) d\Phi_n(\xi)$$

is increasing in $n \geq 1$, which implies (ii).

Thus, $E(X_n)$ in (3) and C_n in (4) are decreasing and increasing in $n \geq 1$, respectively. On the other hand, R_n in (4) is convex in $n \geq 1$ under (8). Then, Lemma 5 yields the following theorem which will give a convenient criterion to seek an optimal policy of type III. In the following discussions, the cost function $A_{\text{III}}(k)$ will be denoted by

$A(k)$ for brevity.

Theorem 3. If the expected replacement cost $R(\xi)$ is constant or the assumption (8) is satisfied, then for a system $S_i[X(\xi), C(\xi), R(\xi), \Phi(\xi), \Psi(\xi, \eta)]$ the optimal policy of type III is to replace the system at each k^* -th failure such as

$$\text{Min} [k | \Delta A(k) \geq 0] \leq k^* \leq \text{Min} [k | \Delta A(k) > 0]$$

unless the cost function $A(k)$ steadily decreases. In the last case, k^* is of course infinity. Furthermore, if $E(X_n)$ is strictly decreasing or C_n is strictly increasing or R_n is strictly convex, then the uniqueness of the optimal policy fails only when the succeeding two values of k give the same cost function.

Notice that the two extreme values of optimum k^* are 1 or ∞ . For the first case, it is optimal to replace the system at each failure and for the second, which perhaps will never occur, it is optimal to repair it every time of its failure.

Finally, we shall be concerned with the optimal policies of two systems whose parameters are different from each other. In the above theorem, the optimal policy is not determined uniquely for the case in which the successive several values of k give the same cost function. But hereafter, we shall choose the smallest one out of them, that is, $k^* = \text{Min} [k | \Delta A(k) \geq 0]$ or ∞ if $\Delta A(k) < 0$ for all $k \geq 1$. We shall consider the following amount associated with a system $S_i[X(\xi), C(\xi), R(\xi), \Phi(\xi), \Psi(\xi, \eta)]$:

$$(9) \quad N(k) = (C_k + \Delta R_k) \cdot \sum_{n=1}^k E(X_n) - \left(\sum_{n=1}^{k-1} C_n + R_k \right) \cdot E(X_{k+1}).$$

Referring to Lemma 5, we can easily see the following

Lemma 7. Consider two systems $S_i^1[X^i(\xi), C^i(\xi), R^i(\xi), \Phi^i(\xi), \Psi^i(\xi, \eta)]$ ($i=1, 2$), where unless $R^i(\xi)$ are constant, we assume (8) about $\Psi^i(\xi, \eta)$ ($i=1, 2$), respectively. If we let $N_i(k)$ be $N(k)$ of (9) corresponding to system S_i^i ($i=1, 2$), respectively, then $N_i(k)$ are increasing in $k \geq 1$ and $k_i^* = \text{Min} [k | N_i(k) \geq 0]$ or ∞ if $N_i(k) < 0$ for all $k \geq 1$, so that $k_1^* \leq k_2^*$

if, and only if, $N_1(k) \geq 0$ for all k which satisfies $N_2(k) \geq 0$.

In the following theorem, the systems under consideration have common $\Phi(\xi)$ and $\Psi(\xi, \eta)$, so for convenience we shall omit them and further drop the suffix I, then we write $S_i[X^i(\xi), C^i(\xi), R^i(\xi)]$ for $S_i^I[X^i(\xi), C^i(\xi), R^i(\xi), \Phi(\xi), \Psi(\xi, \eta)]$.

Theorem 4. We assume the condition (8) unless the expected replacement cost $R(\xi)$ is constant. Let k_i^* be the smallest optimal policies for the respective systems S_i .

(i) Consider two systems $S_1[X(\xi), C(\xi), R(\xi)]$ and $S_2[Y(\xi), D(\xi), T(\xi)]$. If $\{E[X(\xi)] - E[Y(\xi)]\} \leq 0$ is decreasing, $[C(\xi) - D(\xi)] \geq 0$ is increasing, and $[R(\xi) - T(\xi)] \leq 0$ is increasing in $\xi \geq 0$, then $k_1^* \leq k_2^*$.

(ii) Consider the following systems; $S_1[X(\xi), C(\xi), R(\xi)]$, $S_2[aX(\xi), C(\xi), R(\xi)]$, $S_3[X(\xi), aC(\xi), R(\xi)]$, $S_4[X(\xi), C(\xi), aR(\xi)]$, $S_5[aX(\xi), aC(\xi), R(\xi)]$, $S_6[aX(\xi), C(\xi), aR(\xi)]$, $S_7[X(\xi), aC(\xi), aR(\xi)]$ and $S_8[aX(\xi), aC(\xi), aR(\xi)]$. If $a \geq 1$, then $k_8^* = k_5^* \leq k_1^* = k_2^* = k_7^* = k_3^* \leq k_4^* = k_6^*$.

Proof. Let $N_i(k)$ be $N(k)$ of (9) with respect to S_i ($i=1, 2, \dots$), respectively. We apply the foregoing lemma to each assertion.

(i) For convenience, we shall introduce a supplementary system $S_3[Y(\xi), C(\xi), R(\xi)]$. By the hypothesis and (i) in Lemma 6, $[E(Y_n) - E(X_n)] \geq 0$ is increasing in $n \geq 1$, so $E(Y_n)/E(X_n)$ is increasing in n because $[E(Y_n) - E(X_n)]/E(X_n)$ is increasing in n , then

$$\frac{E(Y_{k+1})}{E(X_{k+1})} \geq \frac{\sum_{n=1}^k E(Y_n)}{\sum_{n=1}^k E(X_n)},$$

which implies if $N_3(k) \geq 0$, then $N_1(k) \geq 0$. On the other hand, since $(C_n - D_n) \geq 0$ and $(R_n - T_n) \leq 0$ are increasing in $n \geq 1$ by the similar argument to the above, we can easily see $N_3(k) - N_2(k) \geq 0$ for all $k \geq 0$. Thus, if $N_2(k) \geq 0$, then $N_1(k) \geq 0$.

(ii) It suffices to prove $k_7^* \leq k_4^*$, because $A_1(k) = aA_2(k) = a^{-1}A_7(k) = A_3(k)$, $A_3(k) = aA_1(k)$ and $A_4(k) = aA_6(k)$ where $A_i(k)$ are the cost

functions of S_i , respectively, and further $k_1^* \geq k_2^*$ is a direct conclusion of (i). If $N_i(k) \geq 0$, then equivalently

$$C_k \cdot \sum_{n=1}^k E(X_n) - \left(\sum_{n=1}^{k-1} C_n \right) \cdot E(X_{k+1}) \\ \geq a [R_k \cdot E(X_{k+1}) - \Delta R_k \cdot \sum_{n=1}^k E(X_n)],$$

which implies $N_1(k) \geq 0$, because the left-hand side in the above inequality is nonnegative.

From this theorem, we can say that the system whose $E[X(\xi)]$ is constant independent of ξ has a largest optimal schedule in the class of all systems which have common $C(\xi)$, $R(\xi)$, $\Phi(\xi)$ and $\Psi(\xi, \eta)$.

4. Model II

Let us consider a system $S_{II}[X(\xi), C(\xi), R, \Phi(\xi), \alpha]$ defined in Section 2. But unfortunately, it seems to be difficult to find out the optimal one among our all policies. Here, we shall discuss only the policy of type III, but this does not mean the optimality of Policy III. As in the case of Model I, we can easily show that the cost function under Policy III is equal to

$$B(k) = \frac{\sum_{n=1}^{k-1} C_n + R}{\sum_{n=1}^k E(X_n)},$$

where

$$(10) \quad E(X_n) = \int_0^\infty E[X(\xi)] d\Phi_n(\xi)$$

and

$$(11) \quad C_n = \int_0^\infty C(\xi) d\Phi_n(\xi).$$

Lemma 8. $S_I[X(\xi), C(\xi), R, \Phi(\xi), F(\alpha(\eta - \xi); \xi)] \sim S_{II}[X(\xi), C(\xi), R, \Phi(\xi), \alpha]$, where \sim represents that both cost functions of the above two systems under Policy III are identical, and set $F(\alpha(\eta - \xi); \xi) = U_\infty(\eta - \xi)$ or $U_0(\eta - \xi)$

according as $\alpha=0$ or ∞ .

Proof. Since both $\Phi_n(\xi)$'s in (1) and (5) are identical, both $E(X_n)$'s in (3) and (10), and both C_n 's in (4) and (11) are identical, respectively, which implies $A_{III}(k)=B(k)$.

Recall Model I and II are quite different, but by this lemma we can say that Model I includes Model II in the sense of the cost function under Policy III. Consequently, using the results of the previous section, we have

Theorem 3'. For a system $S_{III}[X(\xi), C(\xi), R, \Phi(\xi), \alpha]$, the optimal policy in type III is to replace the system at each k^* -th failure such as

$$\text{Min } [k | \Delta B(k) \geq 0] \leq k^* \leq \text{Min } [k | \Delta B(k) > 0]$$

unless the cost function $B(k)$ steadily decreases. In the last case, k^* is infinity. Furthermore, if $E(X_n)$ is strictly decreasing or C_n is strictly increasing, then the uniqueness of the optimal policy fails only when the succeeding two values of k give the same cost function.

Theorem 4'. Consider two systems $S_{II}^1[X(\xi), C(\xi), R, \Phi(\xi), \alpha]$ and $S_{II}^2[X(\xi), D(\xi), T, \Phi(\xi), \alpha]$. If $[C(\xi) - D(\xi)] \geq 0$ is increasing in $\xi \geq 0$ and $(R - T) \leq 0$, then $k_1^* \leq k_2^*$, where k_i^* are the smallest optimal policies for S_{II}^i ($i=1, 2$), respectively.

Next, we shall see the fact that minimal repair model [3, 4] is a special case of our Model II. In minimal repair model it is assumed that the failure rate of the system is not disturbed after performing minimal repair, that is, the failure distribution of the system after minimal repair is given by

$$(12) \quad \frac{F(\xi + x) - F(\xi)}{1 - F(\xi)},$$

where $F(x)$ is the failure distribution function of a new system and ξ is the total operating time without down time until each minimal repair.

On the other hand, consider a system $S_{II}[X(\xi), C(\xi), R, \Phi(\xi), \alpha]$ in

which the distribution function $F(x; \xi)$ of $X(\xi)$ is given by (12)—we shall denote such a system by $S_n[F, C(\xi), R, \Phi(\xi), \alpha]$. But by our assumption on $E[X(\xi)]$, we have to assume $F(x)$ is DMRL (decreasing mean residual life), *i.e.*,

$$E[X(\xi)] = \int_0^\infty x dF(x; \xi) = \int_\xi^\infty \frac{1-F(x)}{1-F(\xi)} dx$$

is decreasing in $\xi \geq 0$. DMRL is a sufficiently wide class for preventive maintenance problems. IFR is of course DMRL, but the reverse is not necessarily true. If $F(x)$ is strictly DMRL and differentiable everywhere, then $E_n < E_{n+1}$ a.s. for $0 < \alpha < \infty$, so referring to Lemma 6, we get $E(X_n)$ is strictly decreasing in $n \geq 1$, and hence the second assertion in Theorem 3' holds.

Further, consider a system $S_n[F, C, R, U_0(\xi), 1]$, which is clearly a system discussed in original minimal repair model. In this case, the initial state E_1 is zero, the state E_{n+1} of a system which has been repaired n times before is equal to its total operating time Z_n until n -th failure, and the conditional random variable $X(\xi)$ represents the residual life of a system which has been used over ξ hours.

In generalized minimal repair model $S_n[F, C(\xi), R, \Phi(\xi), \alpha]$, $\Phi(\xi)$ may imply the possibility that a new system is not completely new, but is used some (a small amount) of time, and $C(\xi)$ may imply that the expected repair cost is increasing according to the total operating time. But in this case it may be more general to define

$$C_n = \int_0^\infty C(\xi) d\Phi_{n+1}(\xi), \quad n \geq 1,$$

instead of (4). Finally, α may represent the force of recovery of (minimal) repair. We shall put $\beta = (\alpha - 1)/(\alpha + 1)$, $-1 \leq \beta \leq 1$, and call it 'coefficient of recovery'. For the case of $E_1 = 0$,

$$E_{n+1} = \begin{cases} 0 & \text{for } \beta = 1 \quad (\text{complete renewal}) \\ Z_n & \text{for } \beta = 0 \quad (\text{minimal repair case}) \\ \infty & \text{for } \beta = -1 \quad (\text{recovery to state } \infty). \end{cases}$$

If $F(x)$ has unbounded failure rate, $\beta = -1$ implies we can not repair the failed system.

5. Conclusions

In this paper we have proposed two preventive maintenance models for a repairable system. The state which determines the life time of a repaired system does change one after another and its transition is Markovian. In Model I, X_n (duration of \mathcal{E}_n) and \mathcal{E}_{n+1} (next state) are conditionally independent given \mathcal{E}_n (present state), but in Model II, these are closely related; $\alpha\mathcal{E}_{n+1} = \alpha\mathcal{E}_n + X_n$.

The main results are as follows:

(i) In Model I, Policy III (replacement by the number of failures) is optimal in the class of all our policies under certain reasonable restrictions, which is mainly due to the conditional independency of X_n and \mathcal{E}_{n+1} given \mathcal{E}_n . But, we can say nothing about the above fact for Model II.

(ii) For both models, a convenient criterion to find out the optimal policy in type III is given.

(iii) Model II is a generalization of minimal repair model.

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