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OPTIMAL SELECTION OF SERVICE RATES IN QUEUEING WITH DIFFERENT COST[1]

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Abstract

A single server queueing system with Poisson arrival is considered. The service time is a random variable exponentially distributed with the rate that can be selected from a set of K different rates. The queue has only a limited capacity N and those customers finding system with N persons will not stop. For different cost structure a service rate switching policy is found which minimizes the expected long run average cost.

1. Introduction

Optimization problems in queueing have received a large amount of attention during the past few years. The decision variables in these problems can be related to the arrival stream, service facility or both. In this paper we are concerned with the second type, namely the decision is on service facility.

We are considering a single server queueing system with Poisson

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arrival at rate λ and a finite queue capacity N. Those customers who, upon arrival, find N persons in the system do not stop. The service can be performed with one of the K existing service rates $\mu_1<\mu_2<\cdots<\mu_K$. When service with rate μ_k is used the service time is an exponentially distributed random variable with mean $1/\mu_k$. The server can switch a service rate to another at an arrival time or at a departure time. We wish to find a switching policy which would optimize the long run expected average cost for the following cost structure:

(a) A service cost rate $c(\mu_k)$, the cost incurred in unit time if service with rate μ_k is in effect. It is assumed that $c(\mu_k)$ is non-decreasing in μ_k.

(b) The server will receive a reward A for each customer he serves.

With this cost structure the optimal policy is found to be switch-over or reverse switchover, depending upon whether or not the system is profitable. If the state is defined to be the number of customers in the system, then a switchover policy is a policy which uses higher rates in the higher states and a reverse switchover policy uses lower rates in the higher states.

In the following section we give some definitions and necessary tools. In section 3 the existance of optimal policy for the queueing problem is proved, and in sections 4 and 5, the form of optimal policy is found for two cases: $K=2$, and $K>2$.

2. Definitions and Criteria

As will be seen, the queueing problem will be formulated as a semi-Markov decision process. The notations and definitions used can be found in [4] and [5].

A *semi-Markov decision process* is a process which is observed in each review point and is found to be in one of a possible number of states. The set of all possible states is called the *state space* S. After observing the state, an action a, must be made from a set of possible actions. The set of all possible actions is called the *action space* A. Whenever the state is i, and action a is chosen, then

(i) A transition to state j occurs with the probability $P_{ij}(a)$.

(ii) If the next state of the system is j, then the time until the transition from i to j is a random variable with distribution $F_{ij}(a)$.

(iii) The action a, taken in state i, will specify the cost incurred in the next stage of the process.

A *policy* is a rule for choosing actions when the current state and the past history of the process is given.

A *stationary policy* is a non-randomized policy where the action chosen at a time only depends on the state of the process at that time.

Average Cost Results

Let $Z(t)$ be the total cost incurred by time t, and Z_n be the cost incurred during the n^{th} transition interval, and τ_n be the length of this interval. For each policy π and state i, we define

$$\phi_\pi^1(i)=\lim_{t\to\infty} E_\pi\left[\frac{Z(t)}{t}/X_1=i\right] \tag{2.1}$$

and

$$\phi_\pi^2(i)=\lim_{n\to\infty}\frac{E_\pi\left[\sum_{j=1}^{n} Z_j/X_1=i\right]}{E_\pi\left[\sum_{j=1}^{n} \tau_j/X_1=i\right]} \tag{2.2}$$

where, X_1 is the initial state of the system. Although, ϕ^1 represents the long-run average expected cost, it turns out that ϕ^2 is a "nicer" criteria, in analytical sense, to work with. It also turns out that under some reasonable conditions these two criteria are equal.

Let T be the time of the first return to state i, and f any stationary policy, and suppose that

$$E_f[T/X_1=i]<\infty. \tag{2.3}$$

Then it can be shown (see [4]) that

$$\phi_f^1(i)=\phi_f^2(i).$$

A method to find an optimal stationary policy using ϕ^2 is given in [4]. Thus, if (2.3) is satisfied, then this same policy is optimal for ϕ^1.

Now, let $\bar{c}(i, a)$ denote the expected cost incurred in a transition interval that begins with action a being taken in state i. Also, let $\bar{\tau}(i, a)$ denote the expected length of such a transition interval. It should be noted that, when the relevant criteria is given by equation (2.2), then without loss of generality we may assume that the cost incurred in such an interval, and the length of such an interval, is with probability one $\bar{c}(i, a)$ and $\bar{\tau}(i, a)$. This will be assumed here on.

Note that this implies that $V_\alpha(i)$, the minimal total expected α-discounted cost starting from state i, satisfies

$$(2.4)\qquad V_\alpha(i)=\min_a \{\bar{c}(i, a)+e^{-\alpha\bar{\tau}(i, a)}\sum_j P_{ij}(a)V_\alpha(j)\}, \quad i\geq 0.$$

The following theorem may be found in [5].

Theorem:

If $\bar{c}(i, a)$ is bounded, and if there exists an $M<\infty$ such that

$$(2.5)\qquad |V_\alpha(i)-V_\alpha(0)|<M \text{ for all } \alpha, \text{ all } i$$

then, there exists a bounded function $h(i)$ and a constant g, such that

$$(2.6)\qquad h(i)=\min_a \{\bar{c}(i, a)+\sum_j P_{ij}(a)h(j)-g\bar{\tau}(i, a)\}, \quad i\geq 0$$

and if π^* is a policy which, for each i, prescribes an action which minimizes the right side of (2.6) then

$$g=\phi_{\pi^*}^2(i)=\min_\pi \phi_\pi^2(i), \quad \text{all} \quad i\geq 0$$

and π^* is stationary.

It will be shown in the following section that for our problem

$$g=\min_\pi \phi_\pi^2(i)=\min_\pi \phi_\pi^1(i)$$

Therefore, g is the optimal expected long run average cost. It can be shown that for some sequence $\alpha_n\to 0$, $h(i)=\lim_{n\to\infty}(V_{\alpha_n}(i)-V_{\alpha_n}(0))$ (see [5]).

3. Existence of Stationary Policy

We must define the state of the system, the available actions, and the law of motion. The state of the system is defined as the number of customers in the system, and the state space $S=[0, 1, \cdots, N]$. The action space A is defined as a set of K non-negative integers $A=[1, 2, \cdots, K]$, where action k corresponds to μ_k.

The *transition probabilities* can be easily seen to be

$$(3.1) \qquad P_{ij}(a)=\begin{cases} \dfrac{\mu_a}{\lambda+\mu_a} & j=i-1 \quad N\geq i\geq 1 \\ \dfrac{\lambda}{\lambda+\mu_a} & j=i+1 \quad N>i\geq 1 \\ 1 & j=1 \quad i=0 \\ \dfrac{\lambda}{\lambda+\mu_a} & j=N \quad i=N \\ 0 & \text{otherwise} \end{cases}$$

If $V_\alpha(i)$ is the optimal value function as defined in (2.4), then

$$V_\alpha(0)=\exp\left(-\frac{\alpha}{\lambda}\right)V_\alpha(1)$$

$$V_\alpha(i)=\min_a\left\{\frac{c(\mu_a)-\mu_a A}{\lambda+\mu_a}+\exp\left(-\frac{\alpha}{\lambda+\mu_a}\right)\frac{\lambda}{\lambda+\mu_a}V_\alpha(i+1)\right.$$
$$\left.+\exp\left(-\frac{\alpha}{\lambda+\mu_a}\right)\frac{\mu_a}{\lambda+\mu_a}V_\alpha(i-1)\right\},\quad i=1,\cdots,N-1$$

$$V_\alpha(N)=\min_a\left\{\frac{c(\mu_a)-\mu_a A}{\lambda+\mu_a}+\exp\left(-\frac{\alpha}{\lambda+\mu_a}\right)\frac{\lambda}{\lambda+\mu_a}V_\alpha(N)\right.$$
$$\left.+\exp\left(-\frac{\alpha}{\lambda+\mu_a}\right)\frac{\mu_a}{\lambda+\mu_a}V_\alpha(N-1)\right\}$$

where, $\{c(\mu_a)-\mu_a A\}/(\lambda+\mu_a)$ is the expected cost incurred in a transition interval of length $1/(\lambda+\mu_a)$, the expected transition time when μ_a is in effect, for states $i=1, \cdots, N$.

Lemma 3.1:

(i) $V_\alpha(i)\geq 0$ for all i, and $V_\alpha(i)$ is non-decreasing in i if and only if $c(\mu_a)/\mu_a \geq A$ for all a.

(ii) $V_\alpha(i)<0$ for all i, and $V_\alpha(i)$ is non-increasing in i if and only if $c(\mu_a)/\mu_a < A$ for some a.

Proof:

Define $V_\alpha(i, n)$ as the optimal total expected α-discounted cost starting from state i when there is only n transition remaining in the process. The proof is by induction on n over these functions. These functions converge to $V_\alpha(i)$ as n goes to infinity. The detail is omitted here, for method see [5].

Lemma 3.2:

For the queueing problem

(i) $\bar{c}(i, a)$, the expected cost during a transition interval, is bounded.

(ii) $|V_\alpha(i)-V_\alpha(0)|<M$ for all i and α and for some $M<\infty$.

(iii) $E_f[T/X_1=i]<\infty$, where T, X_1 and f are as defined in section 2.

Proof:

The proof of (i) is immediate. For (ii) let $(r_1, \cdots, r_N)$ be the optimal service rates in states $(1, \cdots, N)$ then the imbedded Markov chain corresponding to transition points is irreducible and positive recurrent (see (3.1)). Let K_i be the expected number of transitions starting from state i before reaching state 0, then $K_i<\infty$ for all i. Let $K=\max_i K_i$, for the case $V_\alpha(i)\leq 0$

$$V_\alpha(i)\geq K \min_a \left\{\frac{c(\mu_a)}{\mu_a}-A+V_\alpha(0)\right\}$$

$$V_\alpha(i)-V_\alpha(0)\geq K \min_a \left\{\frac{c(\mu_a)}{\mu_a}-A\right\}$$

The left side of the above inequality is also bounded by 0.

Hence

$$|V_\alpha(i)-V_\alpha(0)|\leq K\left|\min_a \left\{\frac{c(\mu_a)}{\mu_a}-A\right\}\right|$$

for the case $V_\alpha(i)\geq 0$

$$|V_\alpha(i)-V_\alpha(0)|\leq K\left|\max_a \left\{\frac{c(\mu_a)}{\mu_a}-A\right\}\right|$$

and the results easily follows.

(iii) also easily follows from the properties of the defined chain, since the number of transitions before the first return to i is bounded for all i, and so is the expected time of each transition.

With this lemma, the optimal stationary policy can be found from the optimality conditions (2.6), and this policy will minimize $\phi^1(i)$, the expected long run average cost starting from state i.

4. Derivation of Optimal Policy

In this section we first give some general properties which are true for all K and then the form of optimal policy will be found for case $K=2$. The case $K>2$ will be treated in the next section.

The optimal stationary policy is one which prescribes the minimizing action in the following equations.

$$h(i)=\min_a \{\bar{c}(i, a)+\sum_j P_{ij}(a)h(j)-g\bar{\tau}(i, a)\}$$

$P_{ij}(a)$ were defined in (3.1) and

$$\bar{c}(0, a)=0$$

$$\bar{c}(i, a)=\frac{\mu_a}{\lambda+\mu_a}\left\{\frac{c(\mu_a)}{\mu_a}-A\right\}, \quad i=1, \cdots, N$$

$$\bar{\tau}(0, a)=\frac{1}{\lambda}$$

$$\bar{\tau}(i, a)=\frac{1}{\lambda+\mu_a}, \quad i=1, \cdots, N.$$

Hence,

$$\begin{cases} h(0)=h(1)-\dfrac{g}{\lambda} \\ h(i)=\min\limits_a\left\{\dfrac{c(\mu_a)-\mu_a A}{\lambda+\mu_a}+\dfrac{\lambda}{\lambda+\mu_a}h(i+1)+\dfrac{\mu_a}{\lambda+\mu_a}h(i-1)\right. \\ \qquad\qquad \left.-\dfrac{g}{\lambda+\mu_a}\right\}, \quad i=1, \cdots, N-1 \end{cases}$$

$$\left|\begin{array}{l} h(N)=\min_a \left\{\dfrac{c(\mu_a)-\mu_a A}{\lambda+\mu_a}+\dfrac{\mu_a}{\lambda+\mu_a}h(N-1)+\dfrac{\lambda}{\lambda+\mu_a}h(N)\right. \\ \qquad\qquad \left. -\dfrac{g}{\lambda+\mu_a}\right\} \end{array}\right.$$

or since $1/(\lambda+\mu_a)>0$ for all a,

$$\begin{cases} g=\lambda[h(1)-h(0)] \\ g=\min\limits_a \{c(\mu_a)-\mu_a A+\lambda[h(i+1)-h(i)]-\mu_a[h(i)-h(i-1)]\} \\ \qquad i=1, \cdots, N-1 \\ g=\min\limits_a \{c(\mu_a)-\mu_a A-\mu_a[h(N)-h(N-1)]\}. \end{cases}$$

The following lemma is proved using the stationary probabilities for the $M/M/1$ queueing system without use of the semi-Markov decision process.

Lemma 4.1:

(a) If $A\geq c(\mu_K)/\mu_K$ and $c(\mu_K)/\mu_K=\min\limits_a\{c(\mu_a)/\mu_a\}$, then the optimal policy is to use μ_K in all states.

(b) If $A\leq c(\mu_1)/\mu_1$ and $c(\mu_1)/\mu_1=\min\limits_a\{c(\mu_a)/\mu_a\}$, then the optimal policy is to use μ_1 in all states, where $\mu_1<\mu_2<\cdots<\mu_K$.

Proof:

Let $(r_1, r_2, \cdots, r_N)$ be the set of service rates used in states $(1, 2, \cdots, N)$ where $r_i\epsilon[\mu_1, \cdots, \mu_K]$, and $\{P_i\}_{i=0}^{\infty}$ be the stationary probabilities of the queueing system, where P_i is the proportion of the time that the system spends in state i.

Then

$$P_n=\frac{\lambda^n}{r_1 r_2\cdots r_n}P_0, \quad n=1, 2, \cdots, N$$

$$P_n=0, \quad n>N$$

$$P_0=\frac{1}{1+\dfrac{\lambda}{r_1}+\dfrac{\lambda^2}{r_1 r_2}+\cdots+\dfrac{\lambda^N}{r_1 r_2\cdots r_N}}$$

Treating r_i's as continuous variables, it is easy to see

$$\frac{\partial P_N}{\partial r_i}<0, \quad i=1, 2, \cdots, N$$

Then P_N is maximized if $r_i=\mu_1$ for all i, and minimized if $r_i=\mu_K$ for all i.

Now let λ_e be the effective departure rate or long run average departure rate, then $\lambda_e=\lambda(1-P_N)$. Since $c(r_i)$ is the cost rate when r_i is used, then $\sum_{i=1}^{N} P_i c(r_i)$ is the average service cost rate. The average return rate will be:

$$G=\lambda(1-P_N)A-\sum_{i=1}^{N} P_i c(r_i)$$

Let $g=\max G \max_{r_i \in [\mu_1, \cdots, \mu_K]} \{\lambda(1-P_N)A-\sum^{N} P_i c(r_i)\}$

(a) $$g=\max_{r_i}\left\{\lambda(1-P_N)A-\sum_{i=1}^{N} P_i r_i \frac{c(r_i)}{r_i}\right\}$$

$$\leq \max_{r_i}\left\{\lambda(1-P_N)A-\frac{c(\mu_K)}{\mu_K}\sum_{i=1}^{N} P_i r_i\right\}$$

Now since $\sum_{i=1}^{N} P_i r_i$ is the long run average rate or the effective service rate, then

$$\sum_{i=1}^{N} P_i r_i=\lambda(1-P_N)$$

Hence

$$g\leq \max_{r_i \in [\mu_1, \cdots, \mu_K]} \lambda(1-P_N)\left[A-\frac{c(\mu_K)}{\mu_K}\right] \tag{4.1}$$

But by assumption $A-c(\mu_K)/\mu_K\geq 0$ then, the right side of (4.1) is maximized if P_N is minimized. P_N is minimized if $r_i=\mu_K$ for all i, and for the same choice of rates we will have equality in (4.1).

Q.E.D.

(b) The proof is similar to part (a) replacing

$$\frac{c(\mu_K)}{\mu_K} \text{ by } \frac{c(\mu_1)}{\mu_1} \text{ and using } A-\frac{c(\mu_1)}{\mu_1}<0.$$

Form of Optimal Policy When $K=2$

We are now ready to find the form of optimal policy for the case $K=2$.

Lemma 4.2:

Let $R=\{c(\mu_2)-c(\mu_1)\}/(\mu_2-\mu_1)-A$. Then, the optimal policy is to use μ_1 in state i if and only if

(4.2) $$h(i)-h(i-1)\leq R.$$

Proof:

Let $f^*(i)$ be the optimal action when in state i, then $f^*(i)=1$ if and only if

(4.3) $$c(\mu_1)-\mu_1 A+\lambda[h(i+1)-h(i)]-\mu_1[h(i)-h(i-1)]$$
$$\leq c(\mu_2)-\mu_2 A+\lambda[h(i+1)-h(i)]-\mu_2[h(i)-h(i-1)]$$

or

$$h(i)-h(i-1)\leq\frac{c(\mu_2)-c(\mu_1)}{\mu_2-\mu_1}-A.$$

Q.E.D.

Theorem 1:

If $A\geq c(\mu_a)/\mu_a$ for at least one of the rates, then there exists a state J, $0\leq J\leq N$, such that

$$f^*(i)=1,\quad i=1, 2, \cdots, J$$
$$f^*(i)=2,\quad i=J+1, \cdots, N.$$

That is, the optimal policy is switchover, and $J+1$ is the switching state.

Proof:

Let k be the smallest state such that $f^*(k)=2$ and $f^*(k-1)=1$. We consider two cases:

Case 1:

$k=1$, then for state 0

(4.4) $$h(0)=h(1)-\frac{g}{\lambda}$$

(4.5) $$g=\lambda[h(1)-h(0)]>\lambda R$$

by Lemma 4.1. For state 1, since $f^*(1)=2$

$$(4.6)\qquad \begin{aligned} &g=c(\mu_2)-\mu_2 A+\lambda[h(2)-h(1)]-\mu_2[h(1)-h(0)]\\ &g<c(\mu_2)-\mu_2 A+\lambda[h(2)-h(1)]-\mu_2 R. \end{aligned}$$

From (4.5) and (4.6)

$$\lambda R<c(\mu_2)-\mu_2 A+\lambda[h(2)-h(1)]-\mu_2 R$$

or

$$(4.7)\qquad \lambda R+\mu_2 R-c(\mu_2)+\mu_2 A<\lambda[h(2)-h(1)].$$

Now for $c(\mu_1)/\mu_1\leq c(\mu_2)/\mu_2$ we have $\mu_2 R-c(\mu_2)+\mu_2 A\geq 0$ and from (4.7)

$$h(2)-h(1)>R$$

or by Lemma 3, $f^*(2)=2$. Continuing in this manner we can show $f^*(i)=2$ for $i=3, 4, \cdots, N$. But for $c(\mu_2)/\mu_2<c(\mu_1)/\mu_1$ by Lemma (4.1) we know μ_2 is optimal in all states. This completes the proof for case 1.

Case 2:

$k\geq 2$. By Lemma 4.2

$$(4.8)\qquad \begin{cases} h(k-1)-h(k-2)\leq R, & f^*(k-1)=1\\ h(k)-h(k-1)>R, & f^*(k)=2 \end{cases}$$

and for state k and $k-1$,

$$\begin{cases} g=c(\mu_1)-\mu_1 A+\lambda[h(k)-h(k-1)]-\mu_1[h(k-1)-h(k-2)]\\ g=c(\mu_2)-\mu_2 A+\lambda[h(k+1)-h(k)]-\mu_2[h(k)-h(k-1)] \end{cases}$$

or by (4.8)

$$\begin{cases} g>c(\mu_1)-\mu_1 A+\lambda R-\mu_1 R\\ g<c(\mu_2)-\mu_2 A-\mu_2 R+\lambda[h(k+1)-h(k)]. \end{cases}$$

Then

$$\lambda[h(k+1)-h(k)]>c(\mu_1)-c(\mu_2)-(\mu_1-\mu_2)A+(\mu_2-\mu_1)R+\lambda R.$$

Hence,

$$h(k+1)-h(k)>R$$

or $f^*(k+1)=2$. Continuing in the same manner we can show $f^*(i)=2$ for $i=k+2, \cdots, N$ which completes the proof of the theorem.

Theorem 2:

If $A<c(\mu_a)/\mu_a$ for $a=1, 2$. Then, there exists a state J, $0\leq J\leq$

N, such that

$$f^*(i)=2 \quad \text{for } i=1, \cdots, J$$
$$f^*(i)=1 \quad \text{for } i=J+1, \cdots, N.$$

That is, the optimal policy is a reverse switchover, and $J+1$ is the switching state.

Proof:

The proof is similar to the proof of Theorem 1. Let k be the smallest state such that $f^*(k)=1$ and $f^*(k-1)=2$, consider two cases:

Case 1:

$k=1$, $f^*(1)=1$ and

$$h(1)-h(0)\leq R.$$

For state 0,

$$g=\lambda[h(1)-h(0)]\leq \lambda R. \tag{4.9}$$

For state 1,

$$g=c(\mu_1)-\mu_1 A+\lambda[h(2)-h(1)]-\mu_1[h(1)-h(0)] \tag{4.10}$$
$$g\geq c(\mu_1)-\mu_1 A+\lambda[h(2)-h(1)]-\mu_1 R.$$

From (4.9) and (4.10)

$$\lambda[h(2)-h(1)]\leq \lambda R+\mu_1 R-c(\mu_1)+\mu_1 A. \tag{4.11}$$

Now, if $c(\mu_1)/\mu_1\geq c(\mu_2)/\mu_2$, then

$$\mu_1 R-c(\mu_1)+\mu_1 A=\mu_1\frac{c(\mu_2)-c(\mu_1)}{\mu_2-\mu_1}-\mu_1 A-c(\mu_1)+\mu_1 A$$
$$=\frac{1}{\mu_2-\mu_1}[\mu_1 c(\mu_2)-\mu_2 c(\mu_1)]\leq 0$$

and (4.11) reduces to

$$h(2)-h(1)\leq R$$

or $f^*(2)=1$. Continuing in the same manner we can show $f^*(i)=1$ for $i=3, 4, \cdots, N$. If $c(\mu_1)/\mu_1<c(\mu_2)/\mu_2$ then by Lemma 4.1 the optimal policy is to use μ_1 in all states, which completes the proof of Case 1.

Case 2:

The proof is similar to the proof of Case 2 in Theorem 1 and details are omitted here.

A Computational Approach

Knowing the from of optimal policy we can easily find the switching state. Let Q be the proportion of time the system spends using rate μ_1 in the optimal switchover policy, and P_i's be the set of stationary probabilities. Since the cost rate is $c(\mu_1)-\mu_1 A$ and $c(\mu_2)-\mu_2 A$ for rates μ_1 and μ_2 respectively, then

(4.12) $$g=Q[c(\mu_1)-\mu_1 A]+(1-Q-P_0)[c(\mu_2)-\mu_2 A]$$

Now let Q^j be the proportion of time that the system spends using μ_1 where j is the switching state, and P_i^j's the corresponding stationary probabilities, then

$$Q^j=\sum_{i=1}^{j-1} P_i^j,\quad j=1, 2, \cdots, N+1$$

$Q^1=0$ corresponds to using μ_1 in all states and Q^{N+1} corresponds to using μ_2 in all states.

We now compute $g^1, g^2, \cdots, g^{N+1}$ from (4.12) using $Q^1, Q^2, \cdots, Q^{N+1}$ respectively in the right hand side of the relation, then

$$g=g^{J+1}=\min_j \{g^j\}$$

and $J+1$ is the optimal switching state.

The same method can be used in the nonprofitable system, except the switching is from rate μ_2 to rate μ_1.

5. *K* Service Rates, $K>2$

In this section, we will generalize the results of the last section for the case $K>2$. As we showed, the stationary policy π^* is optimal if it prescribes the minimizing action in the following relations:

(5.1) $$\begin{cases} g=\lambda[h(1)-h(0)] \\ g=\min_a \{c(\mu_a)-\mu_a A+\lambda[h(i+1)-h(i)]-\mu_a[h(i)-h(i-1)]\} \\ \qquad i=1, 2, \cdots, N-1 \\ g=\min_a \{c(\mu_a)-\mu_a A-\mu_a[h(N)-h(N-1)]\} \end{cases}$$

We now show that certain rates are not used in the optimal policy and can be eliminated from further consideration.

Let C be the set of all points $(\mu_k, c(\mu_k))$, and consider the piecewise linear convex function H joining $(\mu_1, c(\mu_1))$ and $(\mu_K, c(\mu_K))$ which bounds the set C from below and changes slope only at the points of the set (see Figure 1). We prove the following:

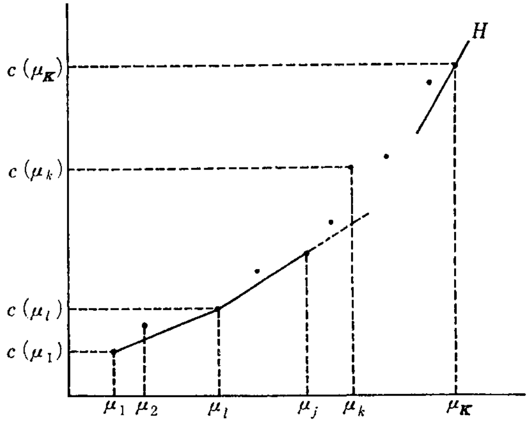

Fig. 1. Set C of points $(\mu_j, c(\mu_j))$ and function H.

Lemma 5.1:

The set of service rates corresponding to the points $(\mu_k, c(\mu_k))$ not on the function H will not be used in the optimal policy.

Proof:

$f^*(i)=k$, *i.e.*, rate μ_k is optimal in state i, if

$$(5.2)\qquad \begin{aligned} &c(\mu_k)-\mu_k A+\lambda[h(i+1)-h(i)]-\mu_k[h(i)-h(i-1)] \\ &\leq c(\mu_j)-\mu_j A+\lambda[h(i+1)-h(i)]-\mu_j[h(i)-h(i-1)] \end{aligned}$$

for all $j\neq k$

or

$$c(\mu_k)-\mu_k A-c(\mu_j)+\mu_j A\leq(\mu_k-\mu_j)[h(i)-h(i-1)] \quad \text{for } j\neq k$$

or

$$(5.3)\qquad \begin{cases} h(i)-h(i-1) > \dfrac{c(\mu_k)-c(\mu_j)}{\mu_k-\mu_j} - A & \text{if } \mu_k > \mu_j \\ h(i)-h(i-1) \leq \dfrac{c(\mu_j)-c(\mu_k)}{\mu_j-\mu_k} - A & \text{if } \mu_j > \mu_k . \end{cases}$$

Now assume that $(\mu_k, c(\mu_k))$ is above H, and let μ_m be the largest rate to the left of μ_k where $(\mu_m, c(\mu_m))$ is on H and μ_n be the smallest rate to the right of μ_k and $(\mu_n, c(\mu_n))$ on H.

Then from definition of H,

$$\frac{c(\mu_k)-c(\mu_m)}{\mu_k-\mu_m} > \frac{c(\mu_n)-c(\mu_k)}{\mu_n-\mu_k}$$

or

$$\frac{c(\mu_k)-c(\mu_m)}{\mu_k-\mu_m} - A > \frac{c(\mu_n)-c(\mu_k)}{\mu_n-\mu_k} - A.$$

But this violates (5.3). Therefore, $f^*(i)=k$ is not possible and μ_k can be eliminated. Now let $\mu_1<\mu_2<\cdots<\mu_L$ be those rates such that the corresponding $(\mu, c(\mu))$ points are on H. (Rearrange the indexes so that this is true). Then,

$$(5.4)\qquad \frac{c(\mu_2)-c(\mu_1)}{\mu_2-\mu_1} - A < \frac{c(\mu_3)-c(\mu_2)}{\mu_3-\mu_2} - A < \cdots < \frac{c(\mu_L)-c(\mu_{L-1})}{\mu_L-\mu_{L-1}} - A .$$

We define

$$(5.5)\qquad R_j = \frac{c(\mu_j)-c(\mu_{j-1})}{\mu_j-\mu_{j-1}} - A .$$

Lemma 5.2:

$$(5.6)\qquad f^*(i)=j \text{ if and only if } R_j < h(i)-h(i-1) \leq R_{j+1} .$$

The proof of this lemma is immediate from (5.3) and (5.4).

Lemma 5.3

Let $(r_1, \cdots, r_N)$ be the optimal rates in states $(1, 2, \cdots, N)$ for a given cost function $c(\mu_1)\leq\cdots\leq c(\mu_L)$, and assume rate μ_t is not used in the optimal policy. Now let us consider a new problem with the same structure but a different cost function $c'(\cdot)$, where

$$\begin{cases} c'(\mu_i)=c(\mu_i), & i\neq t \\ c'(\mu_t)>c(\mu_t). \end{cases}$$

Then μ_t is not used in the optimal policy for the new problem.

Proof:

The lemma is intuitively clear since increasing the cost of a non-optimal rate should not change the optimal policy. Formally, let p_i be the stationary probability, the proportion of time the system spends in state i, then

$$g=\sum_{i=1}^{N} p_i c(r_i)-\lambda(1-p_N)A$$

Now assume that a different set of rates $(r'_1, r'_2, \cdots, r'_N)$ are optimal for the new problem and p_i''s are the corresponding stationary probabilities. Then if μ_t is used in the optimal policy in state j, g' the optimal cost rate for the new problem is

$$g'=\sum_{i=1}^{N} p'_i c(r'_i)-\lambda(1-p'_N)A<g.$$

(g is an upper bound since $(r_1, r_2, \cdots, r_N)$ can be used for the new problem which gives the cost rate equal to g.)

Then

$$g'=\sum_{i\neq j} p'_i c(r'_i)+p'_j c'(\mu_t)-\lambda(1-p'_N)A$$
$$>\sum_{i\neq j} p'_i c(r'_i)+p'_j c(\mu_t)-\lambda(1-p'_N)A. \tag{5.7}$$

But the right side of (5.7) is the cost rate using the original cost function and gives lower cost rate than g, which contradicts the fact that g was the optimal cost rate.

Lemma 5.4:

(a) Let $c=c(\mu_k)/\mu_k=\min_a\{c(\mu_a)/\mu_a\}$, $a=1, 2, \cdots, L$ and $c\leq A$. Then $g>\lambda(c-A)$.

(b) Let $c=c(\mu_k)/\mu_k=\min_a\{c(\mu_a)/\mu_a\}$, $a=1, 2, \cdots, L$ and $c>A$. Then $g<\lambda(c-A)$.

Proof:

(a) Let $(r_1, r_2, \cdots, r_N)$ be the optimal rates in states $1, 2, \cdots, N$ and p_i's the corresponding stationary probabilities then

$$g=\sum_{i=1}^{N} p_i c(r_i)-\lambda(1-p_N)A=\sum_{i=1}^{N} r_i p_i \frac{c(r_i)}{r_i}-\lambda(1-p_N)A$$
$$\geq c\sum_{i=1}^{N} r_i p_i-\lambda(1-p_N)A=\lambda(1-p_N)(c-A)>\lambda(c-A).$$

The proof of (b) is similar and will be omitted here.

The following Lemma is a generalization of Lemma 4.1.

Lemma 5.5:

Let $c=c(\mu_k)/\mu_k=\min_a \{c(\mu_a)/\mu_a\}$, $a=1, 2, \cdots, L$.

(a) If $c\leq A$, then all rates μ_j such that $\mu_j<\mu_k$ can be eliminated from further consideration.

(b) If $c>A$, then all rates faster than $\mu_k (\mu_j>\mu_k)$ can be eliminated from further consideration.

Proof:

(a) Let us assume for simplicity that $k=2$. We must show μ_1 will not be used in the optimal policy.

Consider the same system but a different cost function as follows:

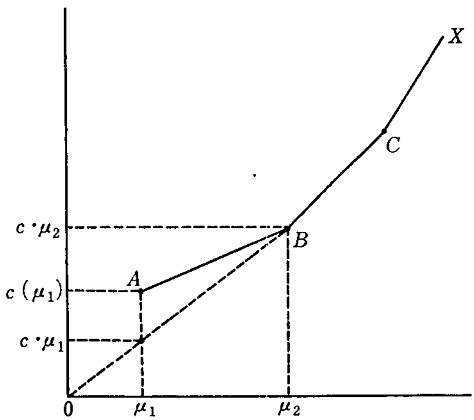

Fig. 2.
ABCX: The cost function $c(\cdot)$.
OBCX: The cost function $c'(\cdot)$.

$$\begin{cases} c'(\mu_1)=c\cdot\mu_1 \text{ where } c=\dfrac{c(\mu_2)}{\mu_2},\ c(\mu_1)>c\cdot\mu_1 \\ c'(\mu_i)=c(\mu_i), \quad i=2,\cdots,L \end{cases}$$

(see Figure 2).

We first show that μ_1 will not be used in the optimal policy for the new problem. The proof of the lemma then follows from Lemma 5.3. In order for μ_1 to be optimal in a state i in the new problem, we must have

$$h(i)-h(i-1)\leq\frac{c(\mu_2)-c\mu_1}{\mu_2-\mu_1}-A=c-A.$$

But by Lemma 5.4

$$g=\lambda[h(1)-h(0)]>\lambda(c-A)$$

then

$$h(1)-h(0)=\frac{g}{\lambda}>c-A.$$

Therefore, $f^*(1)\neq 1$.

Now, let $f^*(1)=t$ where $t\geq 2$. Then

$$h(1)-h(0)>R_t>c-A$$
$$g=\lambda[h(1)-h(0)]>\lambda R_t.$$

For state 1

$$g=c(\mu_t)-\mu_t A+\lambda[h(2)-h(1)]-\mu_t[h(1)-h(0)]$$

and

$$g<c(\mu_t)-\mu_t A+\lambda[h(2)-h(1)]-\mu_t R_t.$$

Then

$$\lambda R_t<c(\mu_t)-\mu_t A+\lambda[h(2)-h(1)]-\mu_t R_t$$

or

$$\lambda(h(2)-h(1))>\lambda R_t+\mu_t R_t-c(\mu_t)+\mu_t A.$$

But by definition of function H and function OBCX (see figure 2)

$$(\mu_t-\mu_{t-1})[\mu_t R_t-c(\mu_t)+\mu_t A]=\mu_{t-1}c(\mu_t)-\mu_t c(\mu_{t-1})\geq 0 \text{ for } t>2$$

Hence,

$$h(2)-h(1)>R_t \text{ or } f^*(2)\geq t.$$

Continuing in the same way, it can be shown $f^*(i) \geq t$ for $i=3, 4, \cdots, N$. This completes the proof of the lemma for the new problem, but by Lemma 5.3 the same will be true for the original problem.

By this part of the lemma, we can now assume that in the case of a profitable system, H passes through origin and

$$\frac{c(\mu_1)}{\mu_1} \leq \frac{c(\mu_2)}{\mu_2} \leq \cdots \leq \frac{c(\mu_L)}{\mu_L}.$$

(b) This case is intuitively clear since all rates faster than μ_k are more expensive and their use will increase the rate of the number of customers served.

The proof closely follows the method to prove part (a). First we consider a new problem with the cost of rates faster than μ_k reduced to $c \cdot \mu_j$, and show for this problem that the optimal policy does not use rates μ_j, $\mu_j > \mu_k$, and then by Lemma 5.3 complete the proof. Details are omitted here.

By this part of the lemma, we can now assume that in the case of a nonprofitable system

$$\frac{c(\mu_1)}{\mu_1} \geq \frac{c(\mu_2)}{\mu_2} \geq \cdots \geq \frac{c(\mu_L)}{\mu_L}$$

Theorem 3:

Let $\mu_1 < \mu_2 < \cdots < \mu_L$ be L service rates such that $R_2 < R_3 < \cdots < R_L$, where R_j's are as defined in (5.5), and $c \leq A$ (profitable system). Then $f^*(i)$ is a nondecreasing function of i, *i.e.*,

$$f^*(1) \leq f^*(2) \leq \cdots \leq f^*(N)$$

which is a switchover policy.

Proof:

We show that if $f^*(i)=t$ then $f^*(j) \geq t$ for all $j > i$. We consider two cases:

Case 1:

If $f^*(1)=t$ then by the same method in the proof of Lemma 5.5,

$$f^*(i) \geq t \quad \text{for all } i > 1.$$

Case 2:

Now let i be the smallest state such that $f^*(i)>f^*(1)$, $N\geq i>1$. Assume that $f^*(i)=k$ and $f^*(i-1)=t$ where $t<k$. For state $i-1$ and i we have

$$R_t=\frac{c(\mu_t)-c(\mu_{t-1})}{\mu_t-\mu_{t-1}}-A<h(i-1)-h(i-2)$$
$$\leq\frac{c(\mu_{t+1})-c(\mu_t)}{\mu_{t+1}-\mu_t}-A=R_{t+1}$$
$$R_k=\frac{c(\mu_k)-c(\mu_{k-1})}{\mu_k-\mu_{k-1}}-A<h(i)-h(i-1)$$
$$\leq\frac{c(\mu_{k+1})-c(\mu_k)}{\mu_{k+1}-\mu_k}-A=R_{k+1}$$

$$\begin{cases} g=c(\mu_t)-\mu_t A+\lambda[h(i)-h(i-1)]-\mu_t[h(i-1)-h(i-2)] \\ g=c(\mu_k)-\mu_k A+\lambda[h(i+1)-h(i)]-\mu_k[h(i)-h(i-1)]. \end{cases}$$

But

$$R_k\geq R_{t+1}>R_t\,,\quad k\geq t+1$$

then

$$\begin{cases} g>c(\mu_t)-\mu_t A+\lambda R_k-\mu_t R_k \\ g<c(\mu_k)-\mu_k A+\lambda[h(i+1)-h(i)]-\mu_k R_k \end{cases}$$

or

$$c(\mu_k)-\mu_k A+\lambda[h(i+1)-h(i)]-\mu_k R_k>c(\mu_t)-\mu_t A+\lambda R_k-\mu_t R_k \tag{4.12}$$
$$\lambda[h(i+1)-h(i)]>-c(\mu_k)+c(\mu_t)+\mu_k A-\mu_t A+\lambda R_k +(\mu_k-\mu_t)R_k\,.$$

But

$$(\mu_k-\mu_t)R_k-[c(\mu_k)-c(\mu_t)]+(\mu_k-\mu_t)A$$
$$=(\mu_k-\mu_t)\left\{\frac{c(\mu_k)-c(\mu_{k-1})}{\mu_k-\mu_{k-1}}-\frac{c(\mu_k)-c(\mu_t)}{\mu_k-\mu_t}\right\}\geq 0$$

the last inequality holds by Lemmas 5.1 and 5.5. Then,

$$h(i+1)-h(i)>R_k$$

or

$$f^*(i+1)\geq k\,.$$

Continuing in the same manner, we can show $f^*(j)\geq k$ for $j=i+1, i+2, \cdots, N$, which completes the proof of Theorem 3.

Theorem 4:

Let $\mu_1<\mu_2<\cdots<\mu_L$ be L service rates such that $R_2<R_3<\cdots<R_L$ and $c>A$ (nonprofitable system), then $f^*(i)$ is nonincreasing function of i, *i.e.*,

$$f^*(1)\geq f^*(2)\geq\cdots\geq f^*(N).$$

or the optimal policy is reverse switchover.

Proof:

It will be shown that if $f^*(i)=t$, then $f^*(j)\leq t$ for all $j>i$. Details are omitted here since method of proof is closely related to the proof of Theorem 3.

References

[1] Blackwell, D., "Discrete Dynamic Programming," *Annals of Mathematical Statistics*, **33** (1962), 719-726.

[2] Crabill, T.B., "Optimal Control of a Oueue with Variable Service Rates," Technical Report No. 75 Department of Operations Research, Cornell University, Ithaca, New York, June 1969.

[3] Lippman, S.A. and S.M. Ross, "The Streetwalker's Dilemma: A Job Shop Model," ORC 69-39, Operations Research Center, University of California, Berkeley, December 1969.

[4] Ross, S.M., "Average Cost Semi-Markov Decision Processes," ORC 69-27, Operations Research Center, University of California, Berkeley, September 1969.

[5] Ross, S.M., *Applied Probability Models with Optimization Applications*, Holden Day Publishing Co., San Francisco, California, April 1970.