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ON AN OPTIMAL-LOCATION PROBLEM

HIROSHI YANAI

Keio University

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Abstract

An optimal-location problem concerned with target searching is formulated by the method of dynamic programming. A graphical procedure to solve the associated recurrence relation including non-linear programming problems is presented. Validity of the dynamic programming formulation is also discussed.

1. Problem

A wrecked fisher-boat is to be searched in a rectangular region on the ocean (Fig. 1). It was decided to search the exact location by para-

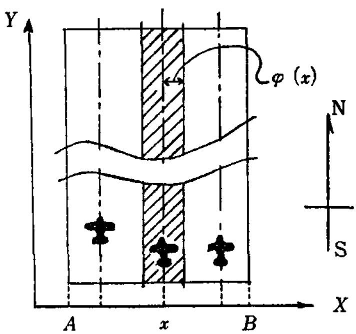

Fig. 1.

llel-sweep with several aeroplanes. The aeroplane flying north along the meridian at longitude x sweeps the meridional strip of longitudinal interval $[x-\varphi(x), x+\varphi(x)]$. The width of the strip $2\varphi(x)$ viz. the effective sweep width is given by a function of the longitude as in Fig. 2.

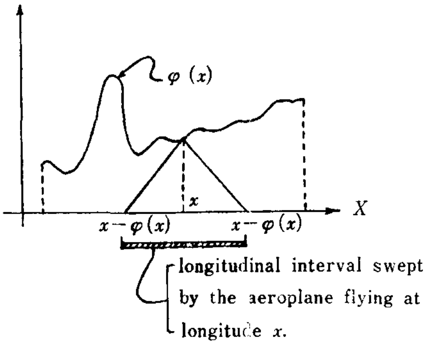

Fig. 2.

The problem is to find the minimum number of aeroplanes necessary to sweep the whole region completely and to determine the longitudes at which the aeroplanes fly.

2. Mathematical Formulation

We formulate this problem by the method of dynamic programming. Denoting by

x_n: the maximum range of the sweep by n aeroplanes i.e. the interval $[A, x_n]$ of the longitude is swept by n aeroplanes if they are optimally located

u_n: the optimal location of the n-th aeroplane,

we obtain the recurrence relation of the form

$$x_n = u_n + \varphi(u_n) = \max_{\substack{A \leq \xi \leq B \\ \xi - \varphi(\xi) \leq x_{n-1}}} [\xi + \varphi(\xi)] \tag{1}$$

$$n = 1, 2, 3, \cdots$$

$$x_0 = A$$

where $\xi+\varphi(\xi)$, the content of the brackets, represents the east-end of the sweep swept by the n-th aeroplane flying along the meridian at longitude ξ. There might exist more than two ξ's at which $\xi+\varphi(\xi)$ attains the maximum. In this case we take, to be specific, the maximum ξ among them. The constraint $\xi-\varphi(\xi)\leq x_{n-1}$ demands the strip swept by the n-th aeroplane not to apart from that of the n-1-st. The other constraint $A\leq\xi\leq B$ is conveniently introduced and can be omitted in some cases e.g. if some of the aeroplanes are permitted to fly outside the original rectangular region and the function $\varphi(\xi)$ is known also for the outside of the interval $[A, B]$.

The sequence of the optimal locations u_n is obtained by solving the recurrence relations (1) from $n=1$ until we obtain the first $x_n\geq B$; we denote by N the first n at which $x_n\geq B$. Obviously, we have the inequality relationship

$$u_1 < u_2 < \cdots < u_n < u_{n+1} < \cdots < u_N$$

for this sequence provided $\varphi(x)>0$ for $x\in[A, B]$.

Remark 1

As a variation of the procedure (1), we may also begin with the right (viz. east) end of the interval $[A, B]$; in this case, we use the recurrence relation

$$(2) \qquad y_m = v_m-\varphi(v_m) = \min_{\substack{A\leq\xi\leq B \\ \xi+\varphi(\xi)\geq y_{m-1}}} [\xi-\varphi(\xi)]$$

$$m = 1, 2, \cdots$$

$$y_0 = B$$

where y_m represents the west end of the sweep $[y_m, B]$ swept by m aeroplanes which are optimally located at $v_m, v_{m-1}, \cdots, v_1$. With this recurrence relation, we determine y_m's and v_m's successively from $y_0=B$ until the first $y_m\leq A$, which we denote by y_M.

The location given by the sequence $v_M, v_{M-1}, \cdots, v_1$ will not in general coincide with those of $u_1, u_2, \cdots, u_N$ determined by (1), but their cardinalities M and N are expected to be identical by the validity of the dynamic programming formulations as in the remark below.

Remark 2

In the above, we have formulated the problem by the method of dynamic programming, in which the value to be maximized was the range swept by the aeroplanes, from the point $x_0 = A$ toward the east. But in the original problem, the minimum number of the aeroplanes was also required. So one might suspect that it could be possible to find some other sequence $\{z_n\}^N{}_0$, among those with $z_0 < A$, which could cover the interval with smaller N. The *best* among such sequences is, naturally, the one covering the maximum range from its initial point $z_0 (< A)$. This sequence is given by the recurrence relation

$$z_n = f(z_{n-1}) = \max_{\substack{A \le \xi \le B \\ \xi - \varphi(\xi) \le z_{n-1}}} [\xi + \varphi(\xi)]$$

$$n = 1, 2, 3, \cdots$$

$$z_0 < A .$$

But since the function

$$f(z_{n-1}) = \max_{\substack{A \le \xi \le B \\ \xi - \varphi(\xi) \le z_{n-1}}} [\xi + \varphi(\xi)] = \max_{\substack{\xi - \eta \le z_{n-1} \\ \eta \le \varphi(\xi) \\ A \le \xi \le B}} [\xi + \eta]$$

(cf. (3.1)*) is non-decreasing (cf. Fig. 3), we can prove inductively that

$$z_n \le x_n , \qquad n = 0, 1, 2, \cdots ,$$

from which the above possibility is ruled out.

* Double numbers in parentheses refer to equations in another section.

3. Graphical Solution

The sequence of the optimization problems of the form (2.1) can be solved by a sequence of simple graphical procedures.

The optimization problem in (2.1) is equivalent to the optimization problem of the form

$$\text{(1)}\qquad \begin{aligned} &\text{maximize} && \xi+\eta \\ &\text{subject to} && \xi-\eta \leq x_{n-1} \\ & && \eta \leq \varphi(\xi) \\ & && A \leq \xi \leq B\,. \end{aligned}$$

The graphical aspect of the above optimization problem is shown in Fig. 3, in which the shaded area indicates the feasible region. The procedure to obtain the solution is obvious: Moving up and down a ruler parallel to the straight line $\xi+\eta=$const., the point (ξ, η) giving maximum to $\xi+\eta$ in the feasible region will be easily obtained. This point is the solution which we denoted by $(u_n, \varphi(u_n))$ in (2.1).

Repeating this procedure, from $x_0=A$ until we reach the first $x_n \geq B$,

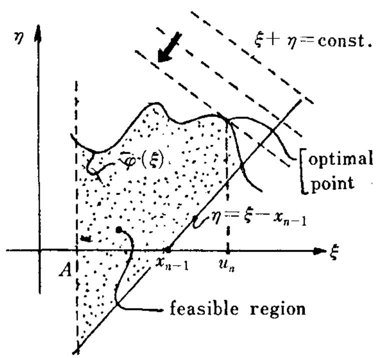

Fig. 3.

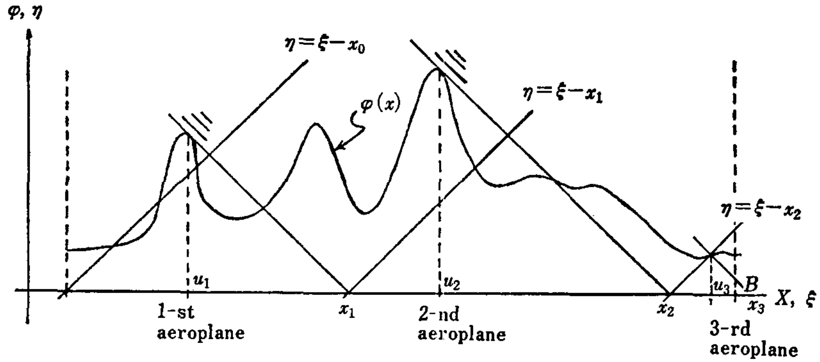

Fig. 4.

we obtain the sequence $\{x_n\}$ and $\{u_n\}$, the maximum ranges and the optimal longitudes. —An example is worked out in Fig. 4.

Remark 3

As it may be obvious from Fig. 3, it is not necessary to solve the non-linear programming problem of the form (1), if

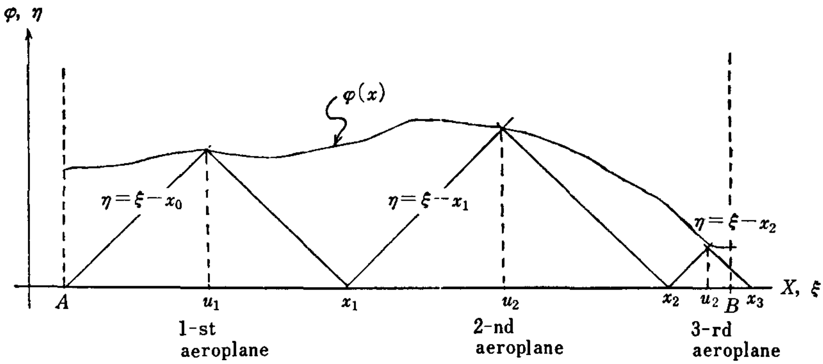

Fig. 5.

$$\left|\frac{d\varphi(x)}{dx}\right| < 1 \qquad \text{for} \quad \forall x \in [A, B],$$

but to solve the equation

$$\varphi(\xi) = \xi - x_{n-1},$$

since the optimal point is always exists at the point of intersection between $\eta = \varphi(\xi)$ and $\eta = \xi - x_{n-1}$. The uniqueness of the solution of the equation is obvious from the condition above. (Fig. 5)

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