

## A QUEUEING SYSTEM WITH MARKOV ARRIVALS\*

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### Introduction and Summary

The theory of queues has been studied principally under the assumptions of mutual independence of the interarrival times and service times. In practice, however, it seems to be natural that some of these assumptions are not satisfied. Several works in which these assumptions are weakened are as follows: systems with state-dependent interarrival or service times (Ancker and Gafarian [1], [2], Conway and Maxwell [7], Haight [9], [10], Harris [11]), systems with semi-Markovian interarrival or service times (Cinlar [5], [6], Neuts [13]).

In this paper we shall consider a system in which the number of the arrival customers is with Markovian character. After preparing notations (§ 1) and lemmas (§ 2), the transient and the equilibrium behaviors of the virtual waiting-times are discussed in § 2 and § 3, respectively. In

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§4 the steady state expectation  $EW$  is sought, and we give numerical illustrations in §5. As it can be seen from this example, there exists remarkable difference in  $EW$  between the case of independent input and the dependent case. This suggests that the type of work need to be studied.

### 1. Assumptions and Notations

The analysis will be restricted to a system with following assumptions:

1. The possible times of the arrival of the customers and the possible service times are integral multiples of a time period  $T$ . Without any loss of generality, we can put  $T=1$ .

2. The number  $N_n$  of the arrival customers at time  $n$  is Markov chain with a transition matrix  $P=[p_{ij}]$ , where

$$p_{ij} \equiv P_r\{N_n = j | N_{n-1} = i\}$$

3. Customers are served in their order of arrival, but the customers who arrived simultaneously are served in arbitrary order.

4. The service times are strictly positive and mutually independent random variables with identical probability distributions.

Furthermore, throughout this paper we assume that

*Condition I.* 1. The stochastic matrix  $P$  is  $(m+1) \times (m+1)$  (i.e.,  $0 \leq N_n \leq m$  a.s.) and irreducible. 2.  $p_{00}$  is strictly positive.

Let  $W_n$  denote the time to complete whole services for all customers joining the queue at just after  $n$ .  $W_n$  is a discrete analogue of the virtual waiting-time [14], so we shall call it the virtual waiting-time at  $n$ . Let  $X_n$  be the sum of the service times of the customers who arrived at  $n$ , i.e.,

$$(1.1) \quad \begin{aligned} X_n &= \sum_{i=1}^{N_n} S_{n_i} & \text{if } N_n > 0, \\ &= 0 & \text{if } N_n = 0, \end{aligned}$$

where  $S_{n_i}$  is the service time of the  $i$ -th customer arriving at  $n$ . Then we have

$$(1.2) \quad W_n = W_{n-1} - h(W_{n-1}) + X_n,$$

where

$$h(x) = 1 \quad \text{if } x > 0, \\ = 0 \quad \text{if } x = 0.$$

Putting  $P_r\{S_{n_i}=s\}=a_s$  ( $s=1, 2, \dots$ ) and

$$g(z) = \sum_{s=1}^{\infty} a_s z^s,$$

then we get

$$(1.3) \quad g(0) = 0, \quad g(1) = 1,$$

and

$$(1.4) \quad g(z_1) < g(z_2) \quad \text{if } 0 \leq z_1 < z_2.$$

Now we shall introduce the following notations;

$$p_{ij}(x) = P_r \{N_n = j, X_n = x | N_{n-1} = i\}, \\ p_{ij}(r, s) = P_r \{N_n = j, W_n = S | N_{n-1} = i, W_{n-1} = r\}, \\ p_{ij}^{(n)}(r, s) = P_r \{N_n = j, W_n = S | N_0 = i, W_0 = r\}, \\ \psi_{ij}(z) = \sum_{x=0}^{\infty} p_{ij}(x) z^x \quad \text{and} \quad \psi(z) = [\psi_{ij}(z)],$$

and

$$A_{ij}(r; z, w) = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} p_{ij}^{(n)}(r, s) z^s w^n \quad \text{and} \\ A(r; z, w) = [A_{ij}(r; z, w)].$$

*Remark 1.* Since  $N_n \leq X_n$  a.s., we have

$$(1.5) \quad p_{ij}(x) = 0 \quad \text{if } j > x.$$

*Remark 2.* If  $z \geq 0$ ,  $A_{ij}(r; z, w)$  is a monotone and analytic function.

By (1.2) and (1.5) we get

$$(1.6) \quad \begin{aligned} p_{ij}(r, s) &= p_{ij}(s-r+1) && \text{if } i=0, 1 \leq r \leq s+1, j \leq s, \\ & && \text{or if } 0 < i \leq r \leq s+1, j \leq s, \\ &= p_{0j}(s) && \text{if } i=r=0, j \leq s, \\ &= 0 && \text{otherwise,} \end{aligned}$$

$$(1.7) \quad p_{ij}^{(*)}(r, s) = 0 \quad \text{if } i > r \text{ or } j > s,$$

and by (1.1) and the independence of the service times,

$$(1.8) \quad \psi_{ij}(z) = p_{ij} g^j(z).$$

We introduce the three inequality notations between two matrices

$$\mathbf{A} = [a_{ij}] \text{ and } \mathbf{B} = [b_{ij}]:$$

$$\mathbf{A} \leq \mathbf{B} \quad \text{if } a_{ij} \leq b_{ij} \text{ for all } i, j,$$

$$\mathbf{A} \leq \mathbf{B} \quad \text{if } \mathbf{A} \leq \mathbf{B} \text{ and } \mathbf{A} \neq \mathbf{B}$$

$$\mathbf{A} < \mathbf{B} \quad \text{if } a_{ij} < b_{ij} \text{ for all } i, j.$$

These inequalities are also used for vectors.

## 2. Transient Behavior of the Virtual Waiting-Time

In order to determine  $A_{ij}(r; z, w)$  explicitly, the properties of a maximal eigenvalue of the matrix  $\psi(z)$  play main role, so at first we shall list them.

*Lemma 1.* (Debreu and Herstein; Theorem I in [8].)

Let  $\mathbf{A} \geq 0$  be an irreducible matrix. Then  $\mathbf{A}$  has an eigenvalue  $\lambda_0 > 0$  such that

- (i) with  $\lambda_0$  can be associated an eigenvector  $\mathbf{x}_0 > 0$ ;
- (ii) if  $\alpha$  is any other eigen value of  $\mathbf{A}$ , then  $|\alpha| < \lambda_0$ ;
- (iii)  $\lambda_0$  increases when any component of  $\mathbf{A}$  increases;
- (iv)  $\lambda_0$  is a simple root of  $|\lambda \mathbf{I} - \mathbf{A}| = 0$ .

*Lemma 2.* Let  $\lambda_0(z)$  be the maximal eigenvalue of  $\psi(z)$ . Under Condition I, for  $z \geq 0$

- (i)  $\lambda_0(z)$  is determined uniquely;

(ii)  $\lambda_0(z)$  is a continuous, strictly positive and strictly increasing function;

(iii)  $\lambda_0(z)$  is a simple root of  $|\lambda I - \psi(z)| = 0$ ;

(iv) all components of the eigenvector  $\mathbf{x}_0(z)$  associated with  $\lambda_0(z)$  have the same sign, more precisely,  $\mathbf{x}_0(0) \geq 0$  (or  $\leq 0$ ) and for  $z > 0$   $\mathbf{x}_0(z) > 0$  (or  $< 0$ ).

*Proof.* Obviously  $\lambda_0(z)$  is continuous (cf. Bellman [3], pp. 60–61).

Suppose  $z > 0$ . Then from Condition I and (1.8)  $\psi(z) \geq 0$  is irreducible, and from (1.4), (1.8)

$$\psi(z_1) \leq \psi(z_2) \quad \text{if } z_1 < z_2.$$

By Lemma 1 the lemma holds for  $z > 0$ .

Next suppose  $z = 0$ . We get  $\psi(0) = [p_{ij} \delta_{j0}]$  by (1.3) and (1.8), where  $\delta_{ij}$  is Kronecker's delta. From Condition I, 2,

$$(2.1) \quad \lambda_0(0) = p_{00} > 0,$$

and the other eigenvalues are equal to 0. Therefore  $\lambda_0(0)$  is a simple root of  $|\lambda I - \psi(0)| = 0$ . The proof of the lemma is complete.

*Lemma 3.* (Debreu and Herstein; Lemma\* in [8])

Let  $A \geq 0$  be a square matrix and let  $\lambda_0$  be its maximal eigenvalue. If for an  $\mathbf{x} > 0$   $A\mathbf{x} \geq s\mathbf{x}$ ,  $\lambda_0 \geq S$ .

On the other hand we need a definition. Assume that  $H_{ij}(x)$  is a distribution function with a moment generating function

$$f_{ij}(v) = \int_{-\infty}^{\infty} e^{vx} dH_{ij}(x),$$

which exists. Let  $C = [c_{ij}]$  be a finite ergodic transition matrix and  $C(v) = [c_{ij} f_{ij}(v)]$ .

*Definition 1.*  $C(v)$  is degenerate if it is of the form

$$C(v) = e^{\beta v} D(v) C D^{-1}(v)$$

where  $D(v) = \text{diag} [\exp(\beta_i V)]$  ( $\beta, \beta_i$ ; real). The following two lemmas are due to Miller [12].

*Lemma 4.* (Theorem 1 (b) in [12])

The maximal eigenvalue  $\alpha(v)$  of  $C(v)$  is of the form  $e^{\beta v}$  ( $\beta$ ; real) if and only if  $C(v)$  is degenerate.

*Lemma 5.* (Theorem 2 in [12])

If  $C(v)$  is not degenerate, then  $\alpha(v)$  is a strictly convex function of  $v$ , where  $v$  is a real number.

Using Lemmas 2 to 5, we have the following lemmas.

*Lemma 6.* Under Condition I the maximal eigenvalue  $e^{-v}\lambda_0(e^v)$ , of the matrix  $e^{-v}\psi(e^v)$  is a strictly convex function of  $v$  ( $v$ ; real).

*Proof.* Let  $H_0(x)$  and  $H_1(x)$  be the distributions such as:

$$\begin{aligned} dH_0(x) &= a_j & \text{if } x = j-1, j = 1, 2, \dots, \\ &= 0 & \text{otherwise,} \end{aligned}$$

and

$$\begin{aligned} dH_1(x) &= a_j & \text{if } x = j, j = 1, 2, \dots, \\ &= 0 & \text{otherwise,} \end{aligned}$$

i.e.,

$$\int_{-\infty}^{\infty} e^{vx} dH_0(x) = e^{-v} g(e^v),$$

$$\int_{-\infty}^{\infty} e^{vx} dH_1(x) = g(e^v).$$

Denoting the convolution operator by a symbol  $*$ , we define the distribution function

$$\begin{aligned} H_{i0}(x) &= 1 & \text{if } x \geq -1 \\ &= 0 & \text{if } x < -1, \end{aligned}$$

and further inductively the distribution functions

$$H_{ij}(x) = H_0 * H_1^{(j-1)*}(x) \quad \text{for } j = 1, 2, \dots.$$

Hence, the  $(i, j)$  component of  $e^{-v}\psi(e^v)$  is rewritten as

$$p_{ij} e^{-v} g^j(e^v) = p_{ij} \int_{-\infty}^{\infty} e^{vx} dH_{ij}(x).$$

On the other hand, since  $\lambda_0(0) = p_{00} \neq 0$  and  $\lambda_0(z)$  is a strictly increasing function of  $z$  by Lemma 2,  $\lambda_0(z)$  is not of the form  $z^{\beta+1}$  ( $\beta$ ; real). Hence,  $e^{-v}\lambda_0(e^v)$  is not of the form  $e^{\beta v}$ . Since  $P$  is obviously ergodic by Condition I, it follows by Lemma 4 that  $e^{-v}\psi(e^v)$  is not degenerate. Thus, the maximal eigenvalue of  $e^{-v}\psi(e^v)$  is a strictly convex function by Lemma 5.

Lemma 7. Under Condition I, the equation

$$(2.2) \quad |zI - w\psi(z)| = 0$$

has a unique root  $\xi_0(w)$  in  $0 < z < 1$ , for any  $0 < w < 1$ .

$\xi_0(w)$  is an increasing function of  $w$  and a unique root of  $z - w\lambda_0(z) = 0$  in  $0 < z < 1$ .

Proof. The equation (2.2) is equivalent to the condition that there exists an eigenvalue  $\lambda(z)$  of  $\psi(z)$  which satisfies

$$(2.3) \quad z - w\lambda(z) = 0.$$

(i) First, under the condition (2.3) we will show that components  $x_i(z)$  ( $i=0, 1, \dots, m$ ) of the eigenvector  $\mathbf{x}(z)$  associated with  $\lambda(z)$  are all real and have the same sign.

Since  $z$  and  $w$  are real,  $\lambda(z)$  and  $\psi(z)$  are also real. Thus, taking conjugate of  $\lambda(z)\mathbf{x}(z) = \psi(z)\mathbf{x}(z)$ , we get

$$\lambda(z)\bar{\mathbf{x}}(z) = \psi(z)\bar{\mathbf{x}}(z),$$

which shows that  $\bar{\mathbf{x}}(z)$  and  $\mathbf{x}(z)$  are the same eigenvector, i.e.,  $\bar{\mathbf{x}}(z) = \mathbf{x}(z)$ . Hence,  $x_i(z)$  are all real for  $i=0, 1, \dots, m$ .

Now, we assume that  $x_0(z) \leq 0$  and that there exist  $i$  and  $j$  so that  $x_i(z) < 0$ ,  $x_j(z) > 0$ . Multiplication of a proper permutation matrix  $\Pi(z)$  to  $\mathbf{x}(z)$  implies that

$$\Pi(z)\mathbf{x}(z) = \begin{bmatrix} \eta_1(z) \\ \eta_2(z) \end{bmatrix},$$

$$\eta_1(z) = [x_{j_0}(z), \dots, x_{j_{k-1}}(z)]^T \leq 0,$$

and

$$\eta_2(z) = [x_{j_k}(z), \dots, x_{j_m}(z)]^T > 0,$$

where  $(j_0, j_1, \dots, j_m)$  is a permutation of  $(0, 1, \dots, m)$  and  $2 \leq k \leq m$ . In this case, by the assumption  $x_0(z) \leq 0$  we may have suppose that the  $(0, 0)$  component of  $\Pi(z)$  is equal to 1.

Using (2.3),

$$(2.4) \quad \frac{z}{w} \begin{bmatrix} \eta_1(z) \\ \eta_2(z) \end{bmatrix} = \begin{bmatrix} B_{11}(z), B_{12}(z) \\ B_{21}(z), B_{22}(z) \end{bmatrix} \begin{bmatrix} \eta_1(z) \\ \eta_2(z) \end{bmatrix},$$

where

$$\begin{bmatrix} B_{11}(z), B_{12}(z) \\ B_{21}(z), B_{22}(z) \end{bmatrix} = \Pi(z) \psi(z) \Pi^{-1}(z).$$

Of course  $B_{11}(z), B_{22}(z)$  are square matrices. Thus, since the  $(0, 0)$  component of  $\Pi(z)$  is 1, we have

$$\Pi(z) \psi(z) \Pi^{-1}(z) = \begin{bmatrix} \psi_{00}(z) & \psi_{0j_1}(z) & \dots & \psi_{0j_m}(z) \\ \psi_{i_1 0}(z) & \psi_{i_1 j_1}(z) & \dots & \psi_{i_1 j_m}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{i_m 0}(z) & \psi_{i_m j_1}(z) & \dots & \psi_{i_m j_m}(z) \end{bmatrix},$$

where  $(i_1, i_2, \dots, i_m)$  is a permutation of  $(1, 2, \dots, m)$  and similarly  $(j_1, j_2, \dots, j_m)$  is so. From (1.8) and  $g(z) \leq z$  for  $0 < z < 1$ , we get

$$\sum_{h=k}^m \psi^{i_j h}(z) \leq z \quad \text{if } 0 < z < 1,$$

and hence

$$(2.5) \quad \lambda_{B_{22}(z)} \leq z$$

where  $\lambda_{B_{22}(z)}$  is the maximal eigenvalue of  $B_{22}(z)$ .

On the other hand, we have

$$B_{22}(z) \eta_2(z) \geq \frac{z}{w} \eta_2(z),$$

which may be proved easily by rewriting (2.4) as

$$B_{21}(z) \eta_1(z) = \left[ \frac{z}{w} I - B_{22}(z) \right] \eta_2(z),$$



and using  $B_{21}(z) \geq 0, \eta_1(z) \leq 0$ . For  $0 < z < 1$  and  $0 < w < 1$  it follows that

$$\lambda_{B_{22}}(z) \geq \frac{z}{w} > z,$$

by Lemma 3. This contradicts to (2.5). So, if  $x_0(z) \leq 0$  we get  $x_i \leq 0$  for all  $i=1, 2, \dots, m$ .

If  $x_0(z) > 0$ , then  $x_i(z) \geq 0$  ( $i=1, 2, \dots, m$ ) can be shown by the same argument. Thus,  $x_i(z)$  ( $i=0, 1, \dots, m$ ) have the same sign.

(ii) We will show that  $\lambda(z)$  which is an eigenvalue of  $\psi(z)$  is equal to the maximal eigenvalue  $\lambda_0(z)$ .

For any vector  $y(z)$  we have directly

$$(2.6) \quad \lambda(z) y^T(z) x(z) = [\psi^T(z) y(z)]^T x(z).$$

Since the eigenvalues of  $\psi^T(z)$  coincide with those of  $\psi(z)$ , we can take  $y(z)$  as follows:

$$(2.7) \quad \lambda_0(z) y(z) = \psi^T(z) y(z),$$

and

$$y(z) > 0.$$

$y(z) > 0$  and  $x(z) \geq 0$  (or  $\leq 0$ ) by the above argument (i) imply

$$y^T(z) x(z) \neq 0.$$

Therefore, inserting (2.7) into (2.6) we have

$$\lambda(z) = \lambda_0(z).$$

(iii) Finally, we will show that  $\xi_0(w)$  is an increasing function of  $w$  and the equation  $z - w\lambda_0(z) = 0$  has a unique root  $\xi_0(w)$  in  $0 < z < 1$ , for any  $0 < w < 1$ .

By the relations

$$\psi(0) = [p_{ij} \delta_{j0}] \quad \text{and} \quad \psi(1) = P.$$

we have

$$(2.8) \quad \lambda_0(0) = p_{00} \quad \text{and} \quad \lambda_0(1) = 1,$$

respectively. Hence,

$$(2.9) \quad \lim_{v \rightarrow -\infty} e^{-v} \lambda_0(e^v) = \infty, \quad \lim_{v \rightarrow 0} e^{-v} \lambda_0(e^v) = 1.$$

Thus, for any  $0 < w < 1$ , there exists a unique root  $v_0(w)$  of the equation

$$(2.10) \quad e^{-v} \lambda_0(e^v) = \frac{1}{w} \quad \text{for } v \leq 0,$$

and  $v_0(w)$  is an increasing function of  $w$ , because the left hand of (2.10) is a strictly convex function by Lemma 6.

This means that  $\xi_0(w) = e^{v_0(w)}$  is an increasing function of  $w$  and a unique root of  $z - w\lambda_0(z) = 0$  in  $0 < z < 1$ , and completes the proof of the lemma.

Since the maximal eigenvalue  $\lambda_0(z)$  is a simple root of  $|\lambda I - \psi(z)| = 0$ , the Jordan decomposition theorem shows that there exists a regular matrix  $T(z) = [t_{ij}(z)]$  such that

$$(2.11) \quad T^{-1}(z) \psi(z) T(z) = \begin{pmatrix} \lambda_0(z) & & & & \\ & J_1(z) & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & J_q(z) \end{pmatrix},$$

where

$$J_i(z) = \begin{pmatrix} \lambda_i(z) & 1 & & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & & \lambda_i(z) \end{pmatrix}$$

is a square matrix on the diagonal,  $\lambda_i(z)$  ( $i=1, 2, \dots, q$ ) are eigenvalues of  $\psi(z)$  and  $q$  is the number of distinct eigenvalues of  $\psi(z)$  excluding  $\lambda_0(z)$ . The first column of  $T(z)$ ,

$$t_0(z) = [t_{00}(z), \dots, t_{m0}(z)]^T,$$

is the eigenvalue associated with  $\lambda_0(z)$ , so we can assume  $t_0(z) > 0$ . For convenience  $t_0(z)$  is normalized to  $\sum_{i=0}^m t_{i0}(z) = 1$ .

*Lemma 8.* Let  $\xi_0(w)$  be a unique root of  $z - w\lambda_0(z) = 0$  in  $0 < z < 1$ , then we have

$$(2.12) \quad w \sum_{k=0}^m \psi_{0k}(\xi_0(w)) t_{k0}(\xi_0(w)) = \xi_0(w) t_{00}(\xi_0(w))$$

*Proof.* It follows from (2.11) that

$$\psi(z) T(z) = T(z) \begin{pmatrix} \lambda_0(z) & & & \\ & J_1(z) & & \\ & & \ddots & \\ & & & J_q(z) \end{pmatrix}$$

Then comparing (0, 0) components of both sides, we have

$$\sum_{k=0}^m \psi_{0k}(z) t_{k0}(z) = t_{00}(z) \lambda_0(z).$$

By the relation  $\xi_0(w) = w \lambda_0(\xi_0(w))$ , (2.12) holds.

Now define a function  $h_i(r)$  by

$$\begin{aligned} h_i(r) &= 1 && \text{if } i \leq r, \\ &= 0 && \text{if } i > r. \end{aligned}$$

We may state the following theorem:

*Theorem 1.* Under Condition I, for  $0 < z \leq 1$  and  $0 < w < 1$  we have

$$(2.13) \quad \begin{aligned} A(r; z, w) &= [z^{r+1} I_{(r)} - (z-1) w A(r; 0, w) \psi(z)] \\ &\quad \times [zI - w\psi(z)]^{-1}, \end{aligned}$$

where

$$(2.14) \quad \begin{aligned} I_{(r)} &= [\delta_{ij} h_i(r)], \\ A_{i0}(r; 0, w) &= \frac{\xi_0^r(w) t_{i0}(\xi_0(w))}{(1 - \xi_0(w)) t_{00}(\xi_0(w))} h_i(r) \end{aligned}$$

and

$$(2.15) \quad A_{ij}(r; 0, w) = 0 \quad \text{if } j \geq 1.$$

*Proof.* (2.15) is easily seen from (1.7).

On the other hand, we have

$$p_{ij}^{(0)}(r, s) = \delta_{ij} \delta_{rs} h_i(r),$$

$$p_{ij}^{(n+1)}(r, s) = \sum_{k=0}^m \sum_{\alpha=0}^{\infty} p_{ik}^{(n)}(r, \alpha) p_{kj}(\alpha, s)$$

Taking generating functions in the above equations and using (1.5) to (1.8),

$$\sum_{s=0}^{\infty} p_{ij}^{(0)}(r, s) z^s = z^r \delta_{ij} h_i(r),$$

$$\sum_{s=0}^{\infty} p_{ij}^{(n+1)}(r, s) z^s = \left(1 - \frac{1}{z}\right) p_{i0}^{(n)}(r, 0) \psi_{0j}(z) + \frac{1}{z} \sum_{k=0}^m \sum_{\alpha=0}^{\infty} p_{ik}^{(n)}(r, \alpha) z^{\alpha} \psi_{kj}(z),$$

and hence,

$$zA_{ij}(r; z, w) - w \sum_{k=0}^m A_{ik}(r; z, w) \psi_{kj}(z) = z^{r+1} \delta_{ij} h_i(r) + (z-1) w A_{i0}(r; 0, w) \psi_{0j}(z),$$

where  $A_{i0}(r; 0, w) = \sum_{n=0}^{\infty} p_{i0}^{(n)}(r, 0) w^n.$

Using (2.15), this leads to

$$A(r; z, w) [zI - w\psi(z)] = [z^{r+1} I_{(r)} - (z-1) wA(r; 0, w) \psi(z)],$$

which is equivalent to (2.13).

Next, we must determine  $A_{i0}(r; 0, w)$  explicitly. From (2.11) we get

$$(2.16) \quad T(z) [zI - w\psi(z)] T^{-1}(z) = \begin{pmatrix} z-w\lambda_0(z) & & & & 0 \\ & J_1(z, w) & & & \\ & & \ddots & & \\ 0 & & & J_q(z, w) & \end{pmatrix}$$

where

$$\mathbf{J}_i(z, w) = \begin{pmatrix} z-w\lambda_i(z) & & -w & & & \\ & & & & & 0 \\ & 0 & & & & \\ & & \cdot & \cdot & \cdot & \\ & & & & & -w \\ & & & & & z-w\lambda_i(z) \end{pmatrix}$$

If we denote by  $\mathbf{U}(z, w)=[u_{ij}(z, w)]$  the matrix

$$\begin{pmatrix} \mathbf{J}_1(z, w) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \mathbf{J}_q(z, w) \end{pmatrix},$$

we get

$$|\mathbf{U}(z, w)| = (z-w\lambda_1(z)) \cdots (z-w\lambda_q(z)).$$

Now let  $U_{ij}(z, w)$  be the cofactor of  $u_{ij}(z, w)$  and put  $\mathbf{T}^{-1}(z)=[\tau_{ij}(z)]$ . Then we have

$$\begin{aligned} (2.17) \quad A_{ij}(r; z, w) &= \sum_{k=0}^m (z^{r+1} \delta_{ik} h_i(r) + (z-1) w A_{i0}(r; 0, w) \psi_{0k}(z)) \\ &\times \left( \frac{t_{k0}(z) \tau_{0j}(z)}{z-w\lambda_0(z)} + \sum_{\beta=0}^m \sum_{l=0}^m \frac{t_{kl}(z) U_{\beta l}(z, w) \tau_{\beta j}(z)}{|\mathbf{U}(z, w)|} \right), \end{aligned}$$

which is easily seen by inserting

$$[zI-w\psi(z)]^{-1} = \mathbf{T}(z) \begin{bmatrix} z-w\lambda_0(z) & & 0 \\ & & \mathbf{U}(z, w) \\ & 0 & \end{bmatrix}^{-1} \mathbf{T}^{-1}(z)$$

into (2.13).

It follows from Lemma 7 that there exists  $[zI-w\psi(z)]^{-1}$  for all  $0 < z \leq 1$  and  $0 < w < 1$  except for the root  $z = \xi_0(w)$ . Since  $A_{ij}(r; z, w)$  is an analytic function of  $z$  for  $0 < z \leq 1$  and  $|\mathbf{U}(z, w)| \neq 0$  for all  $0 < z \leq 1$  and  $0 < w < 1$  by Lemma 7,  $z = \xi_0(w)$  must be a zero of the coefficient of  $(z-w\lambda_0(z))^{-1}$  in the right-hand side of (2.17). Furthermore there exists  $j$  satisfying  $\tau_{0j}(\xi_0(w)) \neq 0$ , because of the regularity of  $\mathbf{T}^{-1}(z)$ . Thus,

$$\begin{aligned}
 & (1 - \xi_0(w)) w A_{i_0}(r; 0, w) \sum_{k=0}^m \psi_{0k}(\xi_0(w)) t_{k0}(\xi_0(w)) \\
 & = \xi_0^{r+1}(w) t_{i_0}(\xi_0(w)) h_i(r)
 \end{aligned}$$

This equation and (2.12) lead to (2.14)

### 3. Equilibrium Behavior of the Process $N_n, W_n$

In this section, we consider the limiting behavior of the transition probability,  $P_{ij}^{(*)}(r, s)$ .

Under Condition I. 1 the Markov chain with the matrix  $P$  is recurrent. Hence, there exists a stationary distribution  $\Pi = [\pi_0, \dots, \pi_m]^T$  satisfying  $P^T \Pi = \Pi$ . If we define the following notation:

$$(i) \quad \left| \begin{array}{cccc} a_{00} & a_{01} & \cdots & a_{0m} \\ a_{10} & a_{11} & \cdots & a_{1m} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m0} & a_{m1} & \cdots & a_{mm} \end{array} \right| = \left| \begin{array}{ccc} a_{00} & a_{01} & \cdots a_{0m} \\ a_{10} & a_{11} & \cdots a_{1m} \\ \vdots & \vdots & \vdots \\ a_{i-10} & a_{i-11} & \cdots a_{i-1m} \\ a_{i+10} & a_{i+11} & \cdots a_{i+1m} \\ \vdots & \vdots & \vdots \\ a_{m0} & a_{m1} & \cdots a_{mm} \end{array} \right|$$

then by wellknown techniques we have

$$(3.1) \quad (i) \quad \pi_i = \left| \begin{array}{cccc} \phi_{01} & \phi_{02} & \cdots & \phi_{0m} \\ \phi_{11}-1 & \phi_{12} & \cdots & \phi_{1m} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \phi_{m1} & \phi_{m2} & \cdots & \phi_{mm}-1 \end{array} \right| \left| \begin{array}{ccc} 1 & \phi_{01} & \cdots \phi_{0m} \\ 1 & \phi_{11}-1 & \cdots \phi_{1m} \\ \vdots & \vdots & \vdots \\ 1 & \phi_{m1} & \cdots \phi_{mm}-1 \end{array} \right|.$$

Let  $N$  be a random variable to which  $N_n$  converges in law. Put  $\rho = EN/\mu$ , where  $1/\mu$  is the mean service time.

Lemma 9.

$$(3.2) \quad \lambda_0'(1) = \rho$$

*Proof.* Differentiate (2.7) in  $z$ . Then we get

$$(3.3) \quad [\lambda_0(z)I - \psi^T(z)] \mathbf{y}'(z) + [\lambda_0'(z)I - \psi^{T'}(z)] \mathbf{y}(z) = \mathbf{0}.$$

It is obvious that  $\lambda_0(z)I - \psi^T(z)$ ,  $\psi^{T'}(z)$  and  $\mathbf{y}(z)$  are all finite for  $0 < z < \infty$ . Further, by Lemma 6

$$\begin{aligned} \frac{d}{dv} e^{-v} \lambda_0(e^v) &= -e^{-v} \lambda_0(e^v) + \frac{d}{de^v} \lambda_0(e^v) < \infty \\ &\text{for } -\infty < v < \infty, \end{aligned}$$

i.e.,

$$\lambda'(z) < \infty \quad \text{for } 0 < z < \infty.$$

Hence, by (3.3) we have

$$\mathbf{y}'(z) < \infty \quad \text{for } 0 < z < \infty.$$

Using the relation  $\lambda_0(1)=1$ ,  $\psi^T(1)=P^T$  and (2.7),

$$\begin{aligned} \mathbf{y}(1) &= c\Pi, \\ \psi^{T'}(1) &= [j\hat{p}_{ij}g'(1)]^T = \left[ \frac{j}{\mu} \hat{p}_{ij} \right]^T, \\ \mathbf{e} [\lambda_0(1)I - \psi^T(1)] &= \mathbf{0}, \end{aligned}$$

where  $c$  is a constant number and  $\mathbf{e}$  is a row vector  $[1, 1, \dots, 1]$ . Therefore, multiplying the vector  $\mathbf{e}$  to (3.3) and putting  $z=1$ , we get

$$\mathbf{e} \left[ [\lambda_0'(1)I - \left[ \frac{j}{\mu} \hat{p}_{ij} \right]^T ] \Pi = \mathbf{0}, \right.$$

and hence

$$\lambda'_0(1) = \rho.$$

Since the root  $\xi_0(w)$  of  $z-w\lambda_0(z)=0$  is an increasing function of  $w$  by Lemma 7, we may put

$$(3.4) \quad \lim_{w \uparrow 1} \xi_0(w) = \xi_0(1-) \equiv \xi_0.$$

*Lemma 10.* Under Condition I we have

$$(3.5) \quad \xi_0 = \lambda_0(\xi_0)$$

and

- (i) if  $\rho > 1$ , then  $0 < \xi_0 < 1$ ;
- (ii) if  $\rho \leq 1$ , then  $\xi_0 = 1$ .

*Proof.* (3.5) is a direct result of the continuity of  $\lambda_0(z)$ . If we put  $z=e^v$ , by  $\lambda_0(1)=1$  and (3.2) we get

$$\frac{d}{dv} e^{-v} \lambda_0(e^v) \Big|_{v=0} = -1 + \rho.$$

(i) If  $\rho > 1$ , the gradient of  $e^{-v}\lambda_0(e^v)$  at  $v=0$  is strictly positive. Hence, there exists a unique  $v_0 < 0$  satisfying

$$e^{-v_0} \lambda_0(e^{v_0}) = 1$$

by Lemma 6 and (2.9). If we set  $\xi_0=e^{v_0}$ , then we have  $0 < \xi_0 < 1$ .

(ii) If  $\rho \leq 1$ , we can see by using the same argument in (i) that for  $v \leq 0$   $e^{-v}\lambda_0(e^v)=1$  if and only if  $v=0$ . Hence, we have  $\xi_0=1$ .

*Lemma 11.* Under Condition I we have  $t_{i_0}(1)=t_{00}(1)$  for  $i=1, \dots, m$ , where  $t_0(z)=[t_{00}(z), \dots, t_{m_0}(z)]^T$  is the normalized eigenvector of  $\psi(z)$  associated with  $\lambda_0(z)$ .

*Proof.* For convenience we put  $t_0(1)=t_0=[t_{00}, \dots, t_{m_0}]^T$ . The relations  $P t_0=t_0$  and  $\sum_{i=0}^m t_{i_0}=1$  lead to

$$t_{i_0} = (-1)^i \begin{pmatrix} \dot{p}_{10} & \dot{p}_{20} & \cdots & \dot{p}_{m_0} \\ \dot{p}_{11}-1 & \dot{p}_{21} & \cdots & \dot{p}_{m_1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \dot{p}_{1m} & \dot{p}_{2m} & \cdots & \dot{p}_{mm}-1 \end{pmatrix} \Bigg| \Bigg| \Bigg|$$

$$\begin{pmatrix} 1 & \dot{p}_{10} & \cdots & \dot{p}_{m_0} \\ 1 & \dot{p}_{11}-1 & \cdots & \dot{p}_{m_1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \dot{p}_{1m} & \cdots & \dot{p}_{mm}-1 \end{pmatrix}.$$



If we add to the first row of the determinant in the numerator all the other rows, rewrite the first row using  $\sum_{j=1}^m p_{ij}=1$  and permute the row, then the numerator is equal to the determinant

$$\begin{vmatrix} p_{11}-1 & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22}-1 & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm}-1 \end{vmatrix}.$$

Therefore, we have  $t_{i0}=t_{00}$  since  $t_{i0}$  is independent of  $i$ .

We shall assume  $i \leq r$ , or  $h_i(r)=1$ .

*Theorem 2.* Under Condition I we have that

- (i) if  $\rho \geq 1$ , then  $\lim_{n \rightarrow \infty} p_{ij}^{(n)}(r, s) = 0$  for all  $j$  and  $s$ ;
- (ii) if  $\rho < 1$ , then there exist  $p_j(s) = \lim_{n \rightarrow \infty} p_{ij}^{(n)}(r, s)$  for  $j=0, 1, \dots, m$

which are independent of the initial state. Furthermore, the generating function  $A_j(z) = \sum_{s=0}^{\infty} p_j(s)z^s$  is given by

$$(3.6) \quad A(z) = (1-\rho)(z-1) \psi_0(z) [zI - \psi(z)]^{-1} \quad \text{for } 0 < z \leq 1$$

where  $A(z) = [A_0(z), \dots, A_m(z)]$  and

$$\psi_0(z) = [\psi_{00}(z), \dots, \psi_{0m}(z)].$$

*Proof.* It is obvious that

$$\lim_{w \uparrow 1} [zI - w\psi(z)] = [zI - \psi(z)]$$

Define  $\phi_{ij}(z, w)$  and  $\phi_{ij}(z)$  by

$$[zI - w\psi(z)]^{-1} = [\phi_{ij}(z, w)] \quad \text{for } 0 < w < 1 \text{ and } z \neq \xi_0(w),$$

and

$$\lim_{w \uparrow 1} \phi_{ij}(z, w) = \phi_{ij}(z, 1) = \phi_{ij}(z) \quad \text{if } z \neq \xi_0.$$

From (2.13) and (2.15) we get

$$\begin{aligned}
 (3.7) \quad & \lim_{w \uparrow 1} (1-w) A_{ij}(\boldsymbol{r}; z, w) \\
 &= (z-1) \sum_{k=0}^m \psi_{0k}(z) \phi_{kj}(z) \lim_{w \uparrow 1} (1-w) A_{i0}(\boldsymbol{r}; 0, w) \\
 & \hspace{15em} \text{if } z \neq \xi_0
 \end{aligned}$$

(i) Suppose  $\rho > 1$ . From Lemma 10 (i), (2.14) and the assumption  $h_i(\boldsymbol{r})=1$ ,

$$\begin{aligned}
 (3.8) \quad & \lim_{w \uparrow 1} (1-w) A_{ij}(\boldsymbol{r}; z, w) \\
 &= \frac{(z-1) \xi_0' t_{i0}(\xi_0)}{(1-\xi_0) t_{00}(\xi_0)} \sum_{k=0}^m \psi_{0k}(z) \phi_{kj}(z) \lim_{w \uparrow 1} (1-w) = 0 \\
 & \hspace{15em} \text{if } z \neq \xi_0
 \end{aligned}$$

Since  $A_{ij}(\boldsymbol{r}; z, w)$  is monotone in  $z > 0$ , (3.8) also holds for  $z = \xi_0$ . Hence, by the definition of  $A_{ij}(\boldsymbol{r}; z, w)$  and Tauberian theorem we have  $\lim_{n \rightarrow \infty} \hat{p}_{ij}^{(*)}(\boldsymbol{r}, s) = 0$ .

Lemma 10 (ii) holds for  $\rho \leq 1$ , so (3.6) which is proved in (ii) is valid for  $\rho = 1$ . Therefore, inserting  $\rho = 1$  into (3.6) we have  $\lim_{n \rightarrow \infty} \hat{p}_{ij}^{(*)}(\boldsymbol{r}, s) = 0$  by Tauberian theorem. Thus,  $\lim_{n \rightarrow \infty} \hat{p}_{ij}^{(*)}(\boldsymbol{r}, s) = 0$  is valid for  $\rho \geq 1$ .

(ii) Suppose  $\rho < 1$ . Then  $\xi_0 = 1$  by Lemma 10 (ii). Hence, we get

$$\begin{aligned}
 (3.9) \quad & \lim_{w \uparrow 1} (1-w) A_{ij}(\boldsymbol{r}; z, w) = (1-\rho)(z-1) \sum_{k=0}^m \psi_{0k}(z) \phi_{kj}(z) \\
 & \hspace{15em} \text{for } 0 < z < 1
 \end{aligned}$$

by Lemma 11 and

$$\lim_{w \uparrow 1} \frac{1 - \xi_0(w)}{1 - w} = \lim_{w \uparrow 1} \xi_0'(w) = \frac{1}{1 - \rho},$$

where the last relation can be proved by differentiating the equation  $\xi_0(w) = w \lambda_0(\xi_0(w))$  in  $w$ , taking the limit  $w \rightarrow 1$  and using  $\xi_0(1) = 1$ ,  $\lambda_0(1) = 1$  and  $\lambda_0'(1) = \rho$ .

The equation (3.6) is the same expression of (3.9) in the matrix form for  $0 < z < 1$ . Since  $A_{ij}(r; z, w)$  is continuous and monotone in  $z > 0$ , (3.6) is valid at  $z = 1$ .

#### 4. Expectation of the Virtual Waiting-Time

Now, we shall determine the mean virtual waiting-time  $EW$  in the steady state. First of all we shall prepare the following lemma:

*Lemma 12.* Under Condition I we have

- (i)  $\psi_{ij}(1) = p_{ij}$  ;
- (ii)  $\psi_{ij}'(1) = \frac{j}{\mu} p_{ij}$  ;
- (iii)  $\psi_{ij}''(1) = \frac{j}{\mu} \left( \frac{j-1}{\mu} + \mu\nu_s - 1 \right) p_{ij}$  ;
- (iv)  $A_j(1) = \pi_j$ ,

where  $\nu_s$  is the second moment of service times.

*Proof.* (i) and (ii) have been said already. (iii) Differentiating  $\psi_{ij}(z)$  twice at  $z=1$ , from  $g'(1)=1/\mu$  and  $g''(1)=\sum_{i=1}^{\infty} (i-1)ia_i=\nu_s-1/\mu$  we get

$$\begin{aligned} \psi_{ij}''(z) &= j(j-1) p_{ij} g'(1)^2 + j p_{ij} g''(1) \\ &= \frac{j}{\mu} \left( \frac{j-1}{\mu} + \mu\nu_s - 1 \right) p_{ij} . \end{aligned}$$

(iv) Since  $A_j(1)=\sum_{s=0}^{\infty} p_j(s)$  is the probability that there are  $j$  arrivals at any time point,  $A_j(1)=\pi_j$  is valid.

*Theorem 3.* Under Condition I, for  $\rho < 1$

$$\begin{aligned} (4.1) \quad EW &= \rho + \frac{1}{2(1-\rho)} \left\{ \rho \left( 1 + \mu\nu_s - \frac{1}{\mu} \right) - \frac{\nu_N}{\mu^2} \right\} \\ &\quad + \frac{1}{1-\rho} \sum_{j=1}^m \left( \frac{j}{\mu} - 1 \right) \pi_j \frac{|R_j|}{|R_0|} , \end{aligned}$$

where  $\nu_N$  is the seconded moment of the random variable  $N$  and

$$(4.2) \quad R_j = \begin{pmatrix} 0 & p_{01} & \cdots & 1 & \cdots & p_{0m} \\ \frac{1}{\mu} & p_{11}-1 & \cdots & 1 & \cdots & p_{1m} \\ \vdots & \vdots & & \vdots & & \vdots \\ \frac{m}{\mu} & p_{m1} & \cdots & 1 & \cdots & p_{mm}-1 \end{pmatrix}.$$

*Proof.* The definition of  $A_j(z)$  implies that

$$(4.3) \quad EW = \sum_{j=0}^m A_j'(1).$$

On the other hand, (3.6) is equivalent to

$$(4.4) \quad zA_j(z) - \sum_{i=0}^m A_i(z) \psi_{ij}(z) = (1-\rho)(z-1) \psi_{0j}(z)$$

for  $j=0, \dots, m$ .

We shall evaluate the first and second derivatives of (4.4) at  $z=1$ . By Lemma 12 and the relation  $\sum_{i=0}^m \pi_i p_{ij} = \pi_j$  we have

$$(4.5) \quad A_j'(1) - \sum_{i=0}^m A_i'(1) p_{ij} = (1-\rho) p_{0j} + \pi_j \left( \frac{j}{\mu} - 1 \right),$$

and

$$(4.6) \quad 2A_j''(1) + A_j'''(1) - \sum_{i=0}^m A_i'''(1) p_{ij} - \frac{2j}{\mu} \sum_{i=0}^m A_i'(1) p_{ij} \\ = \frac{2j}{\mu} (1-\rho) p_{0j} + \frac{j}{\mu} \pi_j \left( \frac{j-1}{\mu} + \mu\nu_s - 1 \right).$$

If we insert  $\sum_{i=0}^m A_i'(1) p_{ij}$  of (4.5) into the sum of (4.6) with respect to  $j$ , we get

$$(4.7) \quad \sum_{j=0}^m \left( 1 - \frac{j}{\mu} \right) A_j'(1) = \frac{\rho}{2} \left( 1 + \mu\nu_s - \frac{1}{\mu} \right) - \frac{\nu_N}{2\mu^2}.$$

Rewriting (4.5) for  $j=1, \dots, m$  and (4.7) in a vector form,

$$(4.8) \quad A'(1) = K \cdot Q,$$

where

$$K = \left[ \frac{\nu_N}{2\mu^2} - \frac{\rho}{2} \left( 1 - \mu w_s - \frac{1}{\mu} \right), \pi_1 \left( 1 - \frac{1}{\mu} \right) \right. \\ \left. - (1-\rho) p_{01}, \dots, \pi_m \left( 1 - \frac{m}{\mu} \right) - (1-\rho) p_{0m} \right]$$

and

$$Q = [q_{ij}] = \begin{pmatrix} -1 & p_{01} & \dots & p_{0m} \\ \frac{1}{\mu} - 1 & p_{11} & \dots & p_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{m}{\mu} - 1 & p_{m1} & \dots & p_{mm} - 1 \end{pmatrix}^{-1}.$$

Multiplying both sides of (4.8) on the right by  $e=[1, \dots, 1]^T$ , it follows from (4.3) that

$$(4.9) \quad EW = K \cdot \left[ \sum_{j=0}^m q_{0j}, \sum_{j=0}^m q_{1j}, \dots, \sum_{j=0}^m q_{mj} \right]^T.$$

If we set  $L=Q^{-1}$  and denote by  $L_{ij}$  the cofactor of the  $(i, j)$  component of  $L$ , we have

$$\sum_{j=0}^m q_{ij} = \sum_{j=0}^m L_{ji} / |L| = |R_i| / |L|.$$

The definition of  $L$  leads to

$$(4.10) \quad |L| = \begin{vmatrix} 0 & p_{01} & \dots & p_{0m} \\ \frac{1}{\mu} & p_{11}-1 & \dots & p_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{m}{\mu} & p_{m1} & \dots & p_{mm}-1 \end{vmatrix} = \begin{vmatrix} 1 & p_{01} & \dots & p_{0m} \\ 1 & p_{11}-1 & \dots & p_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p_{m1} & \dots & p_{mm}-1 \end{vmatrix}.$$

The first term of the right hand side of (4.10) is equal to  $\rho|R_0|$  and the second term is  $|R_0|$  from (3.1) and (4.2). Hence, inserting

$\sum_{j=0}^m q_{ij} = |R_j|/(\rho-1)|R_0|$  into (4.9), we get

$$(4.11) \quad EW = \frac{1}{2(1-\rho)} \left\{ \rho \left( 1 + \mu\nu_s - \frac{1}{\mu} \right) - \frac{\nu_N}{\mu^2} \right\} \\ + \frac{1}{1-\rho} \sum_{j=1}^m \left( \frac{j}{\mu} - 1 \right) \pi_j \frac{|R_j|}{|R_0|} + \frac{1}{|R_0|} \sum_{j=1}^m \rho_{0j} |R_j|.$$

If we verify that

$$(4.12) \quad \frac{1}{|R_0|} \sum_{j=1}^m \rho_{0j} |R_j| = \rho,$$

the proof of the lemma is complete from (4.11). Expanding  $|R_j|$  with the first column,

$$\sum_{j=1}^m \rho_{0j} |R_j| = \frac{1}{\mu} \sum_{j=1}^m \sum_{i=1}^m (-1)^{i-1} i \rho_{0j} \\ \begin{matrix} (j) \\ \left| \begin{array}{cccc} \rho_{01} & \rho_{02} & \cdots & 1 & \cdots & \rho_{0m} \\ \rho_{11}-1 & \rho_{12} & \cdots & 1 & \cdots & \rho_{1m} \\ \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ \rho_{m1} & \rho_{m2} & \cdots & 1 & \cdots & \rho_{mm}-1 \end{array} \right| \end{matrix} \\ = \frac{1}{\mu} \sum_{i=1}^m (-1)^{i-1} i \begin{matrix} (i) \\ \left( \begin{array}{cccc} 1 & \rho_{01} & \rho_{02} & \cdots & \rho_{0m} \\ 1 & \rho_{01} & \rho_{02} & \cdots & \rho_{0m} \\ 1 & \rho_{11}-1 & \rho_{12} & \cdots & \rho_{1m} \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \rho_{m1} & \rho_{m2} & \cdots & \rho_{mm}-1 \end{array} \right) \end{matrix}$$

$$\begin{array}{c}
 \left( \begin{array}{cccc}
 \dot{p}_{01} & \dot{p}_{02} & \cdots & \dot{p}_{0m} \\
 \dot{p}_{11}-1 & \dot{p}_{12} & \cdots & \dot{p}_{1m} \\
 \vdots & \vdots & & \vdots \\
 \vdots & \vdots & & \vdots \\
 \dot{p}_{m1} & \dot{p}_{m2} & \cdots & \dot{p}_{mm}-1
 \end{array} \right)
 \end{array}$$

Since the first term in the parenthesis vanishes and the second term is equal to  $(-1)^i \pi_i |R_0|$  by (3.1), the equation (4.12) holds.

### 5. Example

In this section we shall give a simple example. Assume that the interarrival time  $V$  is distributed as follows:

$$(5.1) \quad P_r\{V=1\} = a, \quad P_r\{V=k\} = (1-a)b(1-b)^{k-2}$$

if  $k \geq 2$ , and no more than one customer may arrive at any time point.

Now, suppose that we have  $m$  independent input processes in operation simultaneously, all of which have the common interarrival time distribution (5.1). Our input process is the superposition of them. The  $(i, j)$  component of the transition matrix  $P$  is given by

$$p_{ij} = \sum_{k+h=j} \binom{i}{k} a^k (1-a)^{i-k} \binom{m-i}{h} b^h (1-b)^{m-i-h}.$$

Therefore, the stationary distribution of  $P$  and its first and second moments are given by

$$(5.2) \quad \pi_i = \frac{\binom{m}{i} (1-a)^{m-i} b^i}{(1-a+b)^m},$$

$$(5.3) \quad \lambda = \frac{m}{1-a+b},$$

and

$$(5.4) \quad \nu_N = \frac{mb(mb+1-a)}{(1-a+b)},$$

respectively. Furthermore we have

$$(5.5) \quad |R_j| = \frac{|R_0|}{(1-a+b)} \quad \text{if } j \geq 1.$$

Inserting (5.5) into (4.1),

$$EW = \rho + \frac{1}{2(1-\rho)} \left\{ \rho \left( 1 + \mu\nu_s - \frac{1}{\mu} \right) - \frac{\nu_N}{\mu^2} \right\} \\ + \frac{\frac{\nu_N}{\mu^2} - \rho}{(1-\rho)(1-a+b)},$$

which is determined only by the first and second moments of the random variable  $N$  and  $S$ .

If we fix the ratio

$$(5.6) \quad \frac{1-a}{b} = \alpha,$$

then (5.2) to (5.4) can be rewritten as follows:

$$\pi_i = \frac{\binom{m}{i} \alpha^{m-i}}{(1+\alpha)^m}, \quad \lambda = \frac{mb}{1+\alpha} \quad \text{and} \quad \nu_N = \frac{m(m+\alpha)}{(1+\alpha)^2}.$$

Therefore, under the condition (5.6) the stationary distribution  $\Pi$  is invariant of  $a$ .

In the case  $m=2$  for simplicity, we shall compare  $EW$  for various  $\rho$  and  $a$  numerically.

In below, we shall show  $P$  and  $EW$  for the cases

$$\rho = 0.6, 0.8, 0.9,$$

$$a = 0.2, 0.4, 0.8.$$



Throughout these cases,  $1/\mu=1.5$  and  $\nu_s=2.5$  are fixed (See Tables (I)-(III)). Furthermore, we shall attach a figure in order to illustrate the variation of  $EW$ . In this figure, a dotted line shows the case in which each input process is Bernoulli arrival (i.e., independent case).

Table. The matrix  $P$  and the expectation  $EW$  of §5.

(I)  $\rho=0.9$

(1)  $a=0.2, b=0.34, EW=4.88$

$$P = \begin{bmatrix} 0.4356 & 0.4488 & 0.1156 \\ 0.506 & 0.426 & 0.068 \\ 0.64 & 0.32 & 0.04 \end{bmatrix}$$

(2)  $a=0.4, b=0.26, EW=7.32$

$$P = \begin{bmatrix} 0.5476 & 0.3848 & 0.0676 \\ 0.444 & 0.45 & 0.106 \\ 0.36 & 0.48 & 0.16 \end{bmatrix}$$

(3)  $a=0.8, b=0.085, EW=27.32$

$$P = \begin{bmatrix} 0.837 & 0.156 & 0.007 \\ 0.183 & 0.749 & 0.068 \\ 0.04 & 0.32 & 0.64 \end{bmatrix}$$

(II)  $\rho=0.8$

(1)  $a=0.2, b=0.29, EW=2.63$

$$P = \begin{bmatrix} 0.5041 & 0.4118 & 0.0841 \\ 0.568 & 0.374 & 0.058 \\ 0.64 & 0.32 & 0.04 \end{bmatrix}$$

(2)  $a=0.4, b=0.22, EW=3.73$

$$P = \begin{bmatrix} 0.6084 & 0.3432 & 0.0484 \\ 0.468 & 0.444 & 0.088 \\ 0.36 & 0.48 & 0.16 \end{bmatrix}$$

(3)  $a=0.8, b=0.07, EW=12.53$

$$P = \begin{bmatrix} 0.8649 & 0.1302 & 0.0049 \\ 0.186 & 0.758 & 0.056 \\ 0.04 & 0.32 & 0.64 \end{bmatrix}$$

(III)  $\rho=0.6$

(1)  $a=0.2, b=0.2, EW=1.33$

$$P = \begin{bmatrix} 0.64 & 0.32 & 0.04 \\ 0.64 & 0.32 & 0.04 \\ 0.64 & 0.32 & 0.04 \end{bmatrix}$$

(2)  $a=0.4, b=0.15, EW=1.73$

$$P = \begin{bmatrix} 0.7225 & 0.2450 & 0.0225 \\ 0.51 & 0.43 & 0.06 \\ 0.36 & 0.48 & 0.16 \end{bmatrix}$$

(3)  $a=0.8, b=0.05, EW=4.93$

$$P = \begin{bmatrix} 0.9025 & 0.0950 & 0.0025 \\ 0.19 & 0.77 & 0.04 \\ 0.04 & 0.32 & 0.64 \end{bmatrix}$$

For fixed  $\rho$ ,  $\Pi$  is independent of  $a$  and  $b$ . Hence, we find that there exists a remarkable difference in  $EW$  in accordance with the forms of

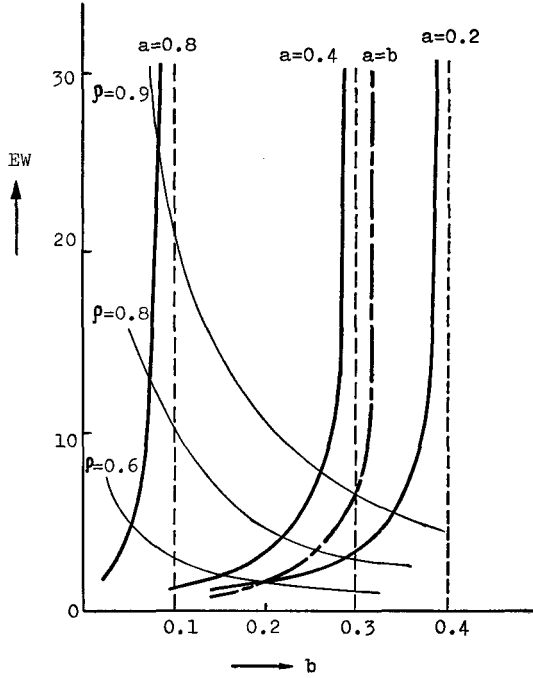


Figure. The mean virtual waiting-time  $EW$  of \$5.

Markov chain  $P$  all of which have the common distribution. For example, if we put  $\rho=0.9$  we can get that  $EW=4.88$  for  $a=0.2$  and  $EW=27.32$  for  $a=0.8$ . We shall discuss on the cause of this difference.

When  $a$  tends to 0, as we may infer from Tables (I)–(III), if the number of arrival customers at any time point is large, the number at the next time point will decrease with high probability, that is, the variance of interarrival times decreases with  $a$ .

This difference in the variance is one of the causes of the difference in  $EW$ .

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