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COMPLEMENTARY CONVEX PROGRAMMING

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Abstract

This paper is concerned with the complementary convex programming problem:

$$P: \quad \text{maximize} \quad z = a_{00} + \sum_{j=1}^{n} a_{0j}(-x_j)$$

$$\text{subject to} \quad a_{i0} \geq \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \cdots, m_1$$

$$g_i(x) \leq 0, \qquad i = 1, 2, \cdots, m_2$$

$$x_p x_q = 0, \qquad \text{for } (x_p, x_q) \in C,$$

where $g_i(x)$ are convex functions, and C is a given set of pairs of variables.

A cutting plane method for solving P is proposed. It consists of three types of cuts: Kelley's cuts developed for convex constraints, C-cuts derived from the condition $x_p x_q = 0$, and Δ-cuts useful for guaranteeing the finite Δ-mesh convergence. The Δ-mesh convergence is a newly introduced concept that the solution sequence converges to a Δ-mesh optimal solution of P(i.e., a point that maximizes z in the Δ-lattice of the feasible region). It is also proved that Δ-mesh optimal solutions come arbitrarily close to an optimal solution of P as Δ approaches to 0.

Although a Δ-mesh optimal solution is in general a suboptimal solution of P, it is shown that Δ can be selected so that the resulting Δ-mesh

optimal solution is an exact optimal solution in case P has no convex constraint (i.e., $m_2=0$).

1. Introduction

In this paper, we will deal with complementary convex programming problems (CCP problems), which are convex programming problems with one more constraint, the complementarity condition (see (2) below), added. Several application areas of the complementarity condition are discussed in [6], [7].

A cutting plane method for solving a CCP problem will be developed. It is guaranteed to converge to a Δ-mesh optimal solution in a finite number of steps. The Δ-mesh optimal solution is a lattice point of the Δ-mesh in the feasible region of the problem, which maximizes the objective value. It is in practice a good approximate solution to an optimal solution of the given problem. There are cases in which Δ-mesh optimal solution is also a real optimal solution. One is when an additional constraint that makes only lattice points of Δ-mesh feasible (consider the integer programming problem which assumes $\Delta=1$) is imposed. The other is a complementary linear programming problem (i.e., LP problem with the complementarity condition) for which it is known that Δ can be selected so that the resulting Δ-mesh optimal solution is a real optimal solution.

The cutting plane method presented in this paper consists of three types of cuts: Kelley's cut [9] for the convex constraints, C-cuts for the complementarity condition and Δ-cuts for guaranteeing the finite Δ-mesh convergence. Δ-cuts are developed by extending the concept of Gomory's cuts [3], [4], [5] for integer constraints.

First let us define a convex programming (CVP) problem* by

$$Q: \quad \text{maximize} \quad z(x)=a_{00}+\sum_{j=1}^{n} a_{0j}(-x_j)$$

* Although the second constraints $x_j=-(-x_j)$ are not really restrictions, they must be here to carry out the simplex method based on the lexicographical order, discussed in Section 3.

$$(1) \qquad \text{subject to} \quad \begin{aligned} & x_{n+i} = a_{i0} + \sum_{j=1}^{n} a_{ij}(-x_j), && i = 1, 2, \cdots, m_1 \\ & x_j = -(-x_j), && j = 1, 2, \cdots, n \\ & x_{r+i} = -g_i(x), && i = 1, 2, \cdots, m_2 \\ & x_k \geq 0, && k = 1, 2, \cdots, n+m_1+m_2 \end{aligned}$$

where $r=n+m_1$, $x=(x_1, x_2, .., x_n) \in R^n$, $a_{ij} \in R$, and $g_i(x)$ are convex functions of x. $x_{n+i} \in R$, $i=1, 2, .., m_1+m_2$ are slack variables. Note that the objective function is linear. Q has m_1 linear constraints (constraints given by linear inequalities) and m_2 convex constraints.

Let C be a set of pairs of variables $x_1, x_2, .., x_r$. The *complementarity condition* is defined for C by

$$(2) \qquad x_p x_q = 0 \text{ for every } (x_p, x_q) \in C .$$

A *complementary convex programming* (CCP) *problem* P is a CVP problem Q restricted further by the complementarity condition (2).

For problem $S(=P, Q$ or anything else), solutions $(x_1, x_2, .., x_n) \in R^n$ are *feasible* if they satisfy S's constraints. The feasible region of S, $F(S)$, is the set of all feasible solutions. An *optimal solution* $\bar{x}$ of S is a feasible solution satisfying $z(\bar{x}) \geq z(x)$ for all $x \in F(S)$. The set of all optimal solutions of S is denoted by $O(S)$.

2. Δ-Mesh Optimality

The Δ-mesh $R_\Delta^n \subset R^n$ is defined by the set of all vectors $x \in R^n$ such that their components are all multiples of Δ, where $\Delta > 0$ and $\Delta \in R$. In other words, $x \in R_\Delta^n$ if and only if $x = \Delta x'$ where x' is an integer vector. Let

$$F_\Delta(S) = F(S) \cap R_\Delta^n ,$$

where S is P, Q or anything else. $\bar{x} \in F_\Delta(S)$ is a *Δ-mesh optimal solution* of S if $\bar{x}$ satisfies $z(\bar{x}) \geq z(x)$ for all $x \in F_\Delta(S)$. We denote the set of all Δ-mesh optimal solutions by $O_\Delta(S)$.

Throughout this paper, we assume that z and x_{n+i}, $i=1, 2, .., m_1$, also take on multiples of Δ only for all $x \in R_\Delta{}^n$. This assumption is for example accomplished if

$$(3) \quad \begin{aligned} &a_{00}=0, \quad a_{0j}\text{: integers}, \quad j=1,2,\cdots,n \\ &a_{i0}\in R_\Delta, \quad a_{ij}\text{: integers}, \quad i=1,2,\cdots,m_1 \\ &\qquad\qquad\qquad\qquad\qquad\quad j=1,2,\cdots,n \end{aligned}$$

hold in (1). (In practice this does not lose any generality because if original coefficients and Δ are all rational numbers, we can multiply a positive number with those coefficients so that (3) may hold true.)

The last half of the condition (3) may be deleted from the assumption if we regard each linear constraint

$$x_{n+i}=a_{i0}+\sum_{j=1}^{n}a_{ij}(-x_j)\ ,\ x_{n+i}\geq 0$$

as a special case of the convex constraint

$$x_{r+i}=-g_i(x)\ ,\ x_{r+i}\geq 0$$

(i.e., m_1 is set to 0 and m_2 is set to m_1+m_2). In this case, however, it should be noticed that variables x_p and x_q of the complementarity condition cannot be variables x_{n+i}, $i=1, 2, .., m_1$.

Δ-mesh optimal solutions of a CCP problem and a CVP problem are illustrated in Fig. 1. $F(Q)$ of the given CVP problem Q is the region surrounded by x_1 and x_2 axes, and the curve given by $g(x)=0$. Δ-mesh optimal solution of Q is also indicated in Fig. 1. On the other hand, $F(P)$ is given by two line segments on the x_1 axis and x_2 axis, because of the complementarity condition $x_1x_2=0$. Thus the optimal solution and Δ-mesh optimal solution of P are on the x_1 axis as shown in Fig. 1.

Consider a sequence of solutions

$$x^1, x^2, \cdots, x^k, \cdots .$$

If the sequence $\{x^k\}$ converges to a Δ-mesh optimal solution of S, it is said to have the property of *Δ-mesh convergence*. In particular, if

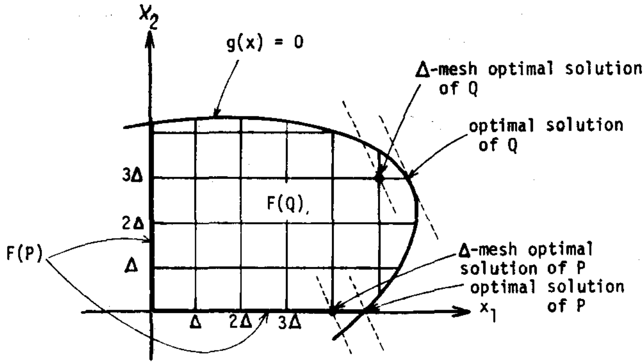

Fig. 1. The optimal solution and Δ-mesh optimal solution of CVP problem Q and CCP problem P.

$$x^k = x^{k+1} = \cdots \quad \text{and} \quad x^k \in O_\Delta(S)$$

for a finite k, the sequence has the property of *finite Δ-mesh convergence.*

3. Cutting Plane Method and the Manipulation of Tableaus

The cutting plane method for CCP problem P consists of the generation of a sequence of LP (linear programming) problems $P_1, P_2, \ldots, P_k, \ldots$, such that each P_k has the same objective function as P and satisfies

(4) $$F(P_k) \supset F_\Delta(P).$$

Theorem 1: Let P_k be an LP problem as defined above. Then if $\bar{x} \in O(P_k)$ is also in $F_\Delta(P)$, $\bar{x} \in O_\Delta(P)$ follows.

Proof: Let $x' \in O_\Delta(P)$ and let $z(\bar{x})$ and $z(x')$ be the objective values of $\bar{x}$ and x'. Then $z(\bar{x}) \geq z(x')$ holds because of (4). On the other hand, $z(\bar{x}) \leq z(x')$ from the fact that $\bar{x} \in F_\Delta(P)$. Thus, $z(\bar{x}) = z(x')$ follows.

QED.

Let us now assume that $\bar{x} \in O(P_k)$ obtained in the computation is not in $F_\Delta(P)$. Then, an LP problem P_{k+1} is generated from P_k by adding a new constraint, called *cut*, which excludes $\bar{x}$ from $F(P_k)$ but excludes none of points in $F_\Delta(P)$. Thus P_{k+1} also satisfies (4) with k replaced by $k+1$. This process is repeated until a Δ-mesh optimal solution of P is obtained.

Specifically, the above process is carried out by using *modified simplex tableaus* (or simply *tableaus*):

$$
(5)\qquad
\begin{aligned}
&z = a_{00} + \sum_{j=1}^{n} a_{0j}(-t_j) \\
&x_{n+i} = a_{i0} + \sum_{j=1}^{n} a_{ij}(-t_j)\,, \qquad i = 1, 2, \cdots, m_1 \\
&x_l = a_{m_1+l0} + \sum_{j=1}^{n} a_{m_1+lj}(-t_j)\,, \quad l = 1, 2, \cdots, n \\
&x_{r+i} = a_{r+i0}, \qquad i = 1, 2, \cdots, m_2 \\
&\left(\begin{array}{l} t_j \geq 0\,, \quad j = 1, 2, \cdots, n \\ x_k \geq 0\,, \quad k = 1, 2, \cdots, n+m_1+m_2 \end{array}\right)
\end{aligned}
$$

where

$$
a_{r+i0} = -g_i\,(a_{m_1+10},\ a_{m_1+20}, \cdots, a_{m_1+n0})\,, \qquad i = 1, 2, \cdots, m_2\,.
$$

In (5), t_j, $j=1, 2, .., n$, are nonbasic variables of the current LP problem.

Let (5) be a tableau of LP problem P_k. We call the column

$$
a_j = (a_{0j}, a_{1j}, \cdots, a_{rj})^T
$$

lexicographically positive if the topmost nonzero entry of a_j is positive. This is denoted by $a_j \succ 0$. If $a_i - a_j \succ 0$, we write $a_i \succ a_j$. Now introduce the following definitions:

Lexicographically dual feasible (LDF) *condition* $\Leftrightarrow a_j \succ 0$, $j=1, 2, .., n$.

Primal feasible (PF) *condition* $\Leftrightarrow (a_{10}, a_{20}, .., a_{r0}) \geq 0$.

Δ-mesh condition $\Leftrightarrow a_{k0} \in R_\Delta$, $k=0, 1, .., r$.

Convex feasible (CF) *condition* $\Leftrightarrow a_{r+i0} \geq 0$, $i=1, 2, .., m_2$.

Next consider the solution given by

$$(6)\qquad \begin{aligned}\bar{z} &= a_{00}\\ \bar{x}_j &= a_{m_1+j0},\ j=1,2,\cdots,n\end{aligned}$$

which is obtained from tableau (5) by putting $t_1=t_2=\cdots=t_n=0$.

If tableau (5) of P_k satisfies the LDF condition and the PF condition, it is known (cf. the simplex method) that $\bar{x}$ given by (6) is in $O(P_k)$. Furthermore, we can easily prove that if the tableau satisfies

(i) the LDF condition and the PF condition,

(ii) the CF condition,

(iii) the complementarity condition of P,

(iv) the Δ-mesh condition,

then $\bar{x}$ given by (6) is in $F_\Delta(P)$ and hence in $O_\Delta(P)$.

It is convenient to start the cutting plane method with the initial tableau (corresponding to P_1) obtained by setting $t_1=x_1,\ t_2=x_2, \ldots, t_n=x_n$:

$$(7)\qquad \begin{aligned}z &= a_{00}+\sum_{j=1}^{n} a_{0j}(-x_j)\\ x_{n+i} &= a_{i0}+\sum_{j=1}^{n} a_{ij}(-x_j)\,, && i=1,2,\cdots,m_1\\ x_j &= -(-x_j)\,, && j=1,2,\cdots,n\\ x_{r+i} &= -g_i(0,0,\cdots,0)\,, && i=1,2,\cdots,m_2\,.\end{aligned}$$

We assume that (7) is LDF. In case (7) is not LDF, it is possible under a moderate assumption to transform the problem so that the LDF condition may be satisfied. For example if $F(P)$ is bounded, the technique of [1] may be used to obtain an LDF tableau.

Now if (5) is LDF but has row i with the negative first entry a_{i0} $(1\leq i\leq r)$ (i.e., condition (i) above is violated), pivot operations can be applied so that resulting tableaus may be all LDF and furthermore column a_0 of the tableau may strictly decrease in the lexicographical sense (i.e., the dual simplex method. See [4] [5]). In particular, we have

(8) $$z^{(1)} \geq z^{(2)} \geq \cdots \geq z^{(k)} \geq \cdots,$$

where $z^{(k)}$ is α_{00} in the k-th tableau. After a finite number of pivot operations a tableau satisfying the PF condition results if it is feasible.

In the cutting plane method, a cut is generated if one of (ii), (iii) and (iv) discussed above is not satisfied. The cut is written by

(9) $$s = \beta_0 + \sum_{j=1}^{n} \beta_j (-t_j)$$

$$s \geq 0$$

where $$\beta_0 < 0,$$

in terms of the nonbasic variables t_j, $j=1, 2, .., n$. After (9) added, a pivot operation is again incurred because (9) has the negative first entry β_0. Now it is time to discuss three types of cuts corresponding to cases (ii), (iii) and (iv).

4. Kelley's Cut for CVP Problems

Kelley's cut (abbreviated by *K-cut*) developed for CVP problems [9] is outlined in this section. Consider (5) and assume that

(10) $$x_{r+i} = -g_i(\alpha_{m_1+10}, \alpha_{m_1+20}, \cdots, \alpha_{m_1+n0}) < 0,$$

for some i. Namely, by $\bar{x}$ given by (6) the constraint $g_i(x) \leq 0$ is not satisfied. If $g_i(x)$ is differentiable at $\bar{x}$, then the constraint

(11) $$-(g_i(\bar{x}) - u^T\bar{x}) - u^T x \geq 0$$

where $$u = \nabla g_i(x)|_{x=\bar{x}}$$

works as a cut corresponding to the CF condition.

Theorem 2 [9]: Let P_k satisfy (4) and let $\bar{x} \in O(P_k)$ be given by (6). If (10) holds, then K-cut defined by (11) excludes $\bar{x}$ from $F(P_k)$ but excludes no point in $F(P)$ (and hence no point in $F_\Delta(P)$).

Note that (11) can be expressed in terms of the present nonbasic variables $t_1, t_2, .., t_n$ of (5) (by substituting (5) into (11)) as follows:

$$(12)\qquad s=\beta_0+\sum_{j=1}^{n}\beta_j(-t_j)$$
$$s\geq 0$$

where
$$\beta_0=\alpha_{r+i0}$$
$$\beta_j=-\sum_{k=1}^{n}\alpha_{m_1+k}\frac{\partial g_i}{\partial x_k}\bigg|_{(x_k=\alpha_{k0})}.$$

Other generalized versions of Kelley's cut for CVP problems (for example, see [2] [10]) can also be used for our purpose, though we omit the detailed discussion.

5. Complementarity Cut for CCP Problem

Let us consider a pair $(x_p, x_q)\in C$ expressed as follows in (5):

$$(13)\qquad \begin{aligned} x_p&=\alpha_{p0}+\sum_{j=1}^{n}\alpha_{pj}(-t_j)\\ x_q&=\alpha_{q0}+\sum_{j=1}^{n}\alpha_{qj}(-t_j)\,. \end{aligned}$$

If the next holds,

$$(14)\qquad \alpha_{p0}>0\,,\quad \alpha_{q0}>0\,,$$

then the solution (6) obtained from (5) (i.e., (13)) does not satisfy the complementarity condition. For this pair, the following *complementarity cut* (abbreviated by *C-cut*) is obtained:

$$(15)\qquad s=\beta_0+\sum_{j=1}^{n}\beta_j(-t_j)$$
$$s\geq 0$$

where
$$\beta_0=-1$$
$$\beta_j=\begin{cases}0, \text{ if } \alpha_{kj}\leq 0 \text{ for } k=p,q\\ \min\{-\alpha_{kj}/\alpha_{k0}\,|\,\alpha_{kj}>0,\ k=p,q\}\,, & \text{otherwise.}\end{cases}$$

Theorem 3: Consider a tableau (5) of LP problem P_k having rows (13) satisfying (14). Then, C-cut (15) excludes $\bar{x}$, obtained by (6), from $F(P_k)$ but excludes no point in $F(P)$ (and hence no point in $F_\Delta(P)$).

Proof: First note that $\bar{x}$ given by (6) is obtained by letting $t_j=0$, $j=1, 2,..,n$, in (5). But letting $t_j=0$, $j=1, 2,..,n$, in (15) results in

(16) $$s=-1 \ngeq 0 .$$

This means that $\bar{x}$ is excluded by C-cut (15). The second part of the theorem is proved by showing that every point t excluded by (15) does not satisfy $x_p x_q=0$. For that, note that such t must be in the region

(17) $$\sum_{j=1}^{n} \beta_j (-t_j) < 1$$

$$t_j \geq 0 , \quad j=1,2,\cdots,n .$$

Since $a_{p0}>0$, (17) leads to

(18) $$a_{p0}-\sum_{j=1}^{n} a_{p0}\beta_j (-t_j) > 0 .$$

By the definition of β_j, we have

$$a_{pj} > 0 \Rightarrow a_{pj} \leq -a_{p0}\beta_j$$

$$a_{pj} \leq 0 \Rightarrow a_{pj} \leq 0 \leq -a_{p0}\beta_j .$$

Hence

$$x_p = a_{p0}+\sum_{j=1}^{n} a_{pj}(-t_j) \geq a_{p0}-\sum_{j=1}^{n} a_{p0}\beta_j (-t_j) > 0 .$$

In a similar manner, $x_q>0$ is proved. Thus $x_p x_q>0$ holds for any t in (17). QED.

6. Δ-Cuts

Let us assume that x_i, $0\leq i\leq r$, be expanded as follows:

(19) $$x_i=a_{i0}+\sum_{j=1}^{n} a_{ij}(-t_j) . \quad \text{(Note } x_i \text{ may be } x_0=z) .$$

Whenever $a_{i0} \notin R_\Delta$, the next *Δ-cut* can be generated:

$$(20) \qquad s = \beta_0 + \sum_{j=1}^{n} \beta_j (-t_j)$$
$$s \geq 0 .$$

where

$$\beta_0 = -f_0$$
$$(21) \qquad \beta_j = \begin{cases} (f_0/(\Delta - f_0))\, a_{ij}, & \text{if } a_{ij} < 0 \\ -a_{ij}, & \text{if } a_{ij} \geq 0 \end{cases}$$
$$j = 1, 2, \cdots, n$$
$$f_0 = a_{i0} - [a_{i0}/\Delta]\, \Delta .$$

$[A]$ is the integer part of A. Δ-cut may be considered as an extension of Gomory's cut [3] for mixed-integer programs. Although other Gomory's cuts [4], [5] developed for all-integer programs can also give rise to cuts satisfying the next theorem, we will use only (20) for simplicity.

Theorem 4: Let (5) be a tableau of P_k in which x_i be represented by (19). If $a_{i0} \notin R_\Delta$, then Δ-cut (20) excludes the present solution (6) from $F(P_k)$ but excludes no point in $R_\Delta{}^n$.

The proof is omitted since it is similar to that of Gomory's [3].

7. Statement of the Cutting Plane Method and its Finite Δ-mesh Convergence

It was shown in Sections 4—6 that if the CF condition, the complementarity condition or the Δ-mesh condition is violated in a tableau of P_k (satisfying the LDF and PF conditions), K-cut (12), C-cut (15) or Δ-cut (20) can be generated respectively. In addition, each LP problem P_k can be solved (i.e., a tableau satisfying the LDF and PF conditions is obtained) in a finite number of pivot operations according to the dual simplex method. Consequently, the cutting plane method for a given CCP problem P and $\Delta > 0$, which was outlined in Section 3, may be summarized as follows.

Cutting plane method:

Step 1: Start with P_1 (see (7)). P_1 is assumed to be LDF. Solve P_1. If P_1 is infeasible, $F_\Delta(P)=\phi$ holds. Terminate. Otherwise let $k=1$ and go to Step 2.

Step 2: If an optimal solution of P_k obtained is in $F_\Delta(P)$, terminate; it is also in $O_\Delta(P)$. Otherwise, go to Step 3.

Step 3: Generate a cut (9) according to which of the CF, complementarity and Δ-mesh conditions, are violated in the optimal tableau of P_k (see the discussion below), and obtain P_{k+1} where

(22) $$P_{k+1}:\ P_k \text{ and cut (9)}.$$

Solve P_{k+1}. If P_{k+1} is infeasible, $F_\Delta(P)=\phi$ holds. Terminate the computation. If feasible, after deleting the cut row annexed to P_k and increasing k by 1, return to Step 2.

Now consider the following three conditions imposed on the selection of cuts in Step 3 of the cutting plane method.

(a) If $a_{r+i}<0$, $1\leq i\leq m_2$, in the successive optimal tableaus for P_k, $k=k_0, k_0+1, \ldots$, then K-cut is generated after a finite number of pivot operations.

(b) If $a_{p0}>0$, $a_{q0}>0$ for some $(x_p, x_q)\in C$ in the successive optimal tableaus for P_k, $k=k_0, k_0+1, \ldots$, then C-cut is generated after a finite number of pivot operations.

(c) If $a_{i0}\notin R_\Delta$ for some i, $(0\leq i\leq r)$ in the successive optimal tableaus for P_k, $k=k_0, k_0+1, \ldots$, then Δ-cut is generated after a finite number of pivot operations.

As implicitly proved in the proof of the next theorem, the following selection rule, for example, guarantees that the above three conditions are satisfied.

Cut selection rule: Choose the topmost row which violates at least one of the CF, complementarity, and Δ-mesh conditions. If the row violates more than one condition, generate a cut according to the following priority order.

(23) Δ-cut first, C-cut second, and K-cut third
or Δ-cut first, K-cut second, and C-cut third.

Under the above conditions, the finite Δ-mesh convergence is proved by using the argument similar to that of Gomory's [3], [4].

Theorem 5: The cutting plane method for CCP problem P has the property of finite Δ-mesh convergence provided that the cut selection conditions (a), (b), (c) are satisfied, that $F(P)$ (and hence $F_\Delta(P)$) be compact (i.e., bounded), and that $F_\Delta(P)$ be nonempty.

Proof: From the assumptions that $F_\Delta(P)$ is bounded and not empty, there exists a Δ-mesh optimal solution of P whose objective value is denoted by $z^0(<\infty)$. From (8) we have

$$z^{(1)} \geq z^{(2)} \geq \cdots \geq z^{(k)} \geq \cdots \geq z^0 \tag{24}$$

where $z^{(k)}$ is the value of z (i.e., a_{00}) in the k-th tableau. We first prove that there exists a finite k such that

$$\begin{aligned} &z^{(k)} = z^{(k+1)} = \cdots \\ &z^{(k)} \in R_\Delta . \end{aligned} \tag{25}$$

For that, assume that no such k exists. Let $z^{(k_1)} > z^{(k_2)}$, then there exists $z^{(k_3)}$ such that $z^{(k_2)} \geq z^{(k_3)}$ and $z^{(k_3)} \in R_\Delta$, since Δ-cut is generated from row z after a finite number of pivot operations (Condition (c)) and the new value of a_{00} after the pivot operation is in R_Δ as easily proved (if $z^{(k_2)} \in R_\Delta$, let $k_2=k_3$). By assumption, there exists $z^{(k_4)}$ such that $z^{(k_3)} > z^{(k_4)}$. Then by a similar argument we can prove the existence of $z^{(k_5)}$ such that $z^{(k_4)} > z^{(k_5)}$ and $z^{(k_5)} \in R_\Delta$. Thus $z^{(k_3)} - z^{(k_5)} \geq \Delta$. This process cannot continue infinitely because the sequence $z^{(k)}$ is bounded below by z^0 and $\Delta > 0$. Thus (25) is proved.

Next we turn to the second row x_{n+1} of tableaus. We have the sequence

$$x_{n+1}{}^{(k)} \geq x_{n+1}{}^{(k+1)} \geq \cdots \geq x_{n+1}{}^{(l)} \geq \cdots \geq x_{n+1}{}^0$$

where k is the one given by (25), and $x_{n+1}{}^0$ is a lower bound of x_{n+1} ($x_{n+1}{}^0$ exists because $F(P)$ is bounded). By an argument similar to that of z

we can prove that after a finite number l,

$$x_{n+1}^{(l)} = x_{n+1}^{(l+1)} = \cdots$$

$$x_{n+1}^{(l)} \in R_\Delta$$

is satisfied.

Applying this argument to $x_{n+2}, \ldots, x_{n+m_1}, x_1, \ldots, x_n$ successively, we will see that after a finite number of pivot operations,

$$x^{(s)} = x^{(s+1)} = \cdots$$

$$x^{(s)} \in R_\Delta{}^n$$

holds. $x^{(s)}$ must be in $F_\Delta(P)$, since otherwise K-cut or C-cut must be generated after a finite number of pivot operations and it provides a new different solution, contradicting the fact that x has converged to $x^{(s)}$. Also, $x^{(s)}$ must be in $O_\Delta(P)$ because $F(P_u) \supset F_\Delta(P)$ from the definition of cuts, where the s-th tableau is generated for LP problem P_u.
QED.

Although conditions (a), (b) and (c) are introduced to guarantee the finite convergence, other selection rules sometimes may prove effective in speeding up the convergence. For example the following rule appears reasonable that selects C-cuts or K-cuts first, until the improvement in the objective values becomes very small, and then relies on Δ-cuts. After an appropriate number of Δ-cuts, C-cuts or K-cuts are again generated, and the process is repeated.

8. Example

Let us solve the following CCP problem by the cutting plane method.

$$P: \text{maximize} \quad z = 2(-x_1) + (-x_2)$$

$$\text{subject to} \quad x_3 = -14 + 2x_1 + 7x_2$$

$$x_1 = -(-x_1)$$

$$x_2 = -(-x_2)$$

$$x_4 = -\frac{9}{4}\exp\left[\frac{2}{15}x_1 - \frac{2}{5}x_2 + \frac{4}{5}\right] + \frac{93}{10}x_1 + \frac{31}{10}x_2 - \frac{279}{20} \quad (= -g_1(x))$$

$$x_5 = 1 - x_1^2/64 - x_2^2/36 \quad (= -g_2(x))$$

$$x_1 x_2 = 0$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0 .$$

Δ is assumed to be 1.

The feasible region of P without the complementarity condition is the shaded region of Fig. 2. If we take into consideration the complementarity condition, only the region given by the two bold line segments on x_1 axis and x_2 axis are feasible. Point A is apparently the optimal solution of P. Any point in R_Δ^n is an integer lattice point (since $\Delta=1$ in this case). Four dots in the line segments of x_1 axis and x_2 axis (which are $F(P)$) are points in $F_\Delta(P)$. Thus, the Δ-mesh optimal solution of P is obviously B which is located very close to A.

Table 1 shows the initial tableau of P_1(see (7)). It is LDF. Since row x_3 has the negative first entry, one pivot operation is applied to Table 1 according to the dual simplex method. The resulting tableau is Table 2. A K-cut is generated in Table 2, since x_4 is negative.

$$\left.\frac{\partial g_1}{\partial x_1}\right|_{\substack{x_1=0\\x_2=2}} = -9 , \qquad \left.\frac{\partial g_1}{\partial x_2}\right|_{\substack{x_1=0\\x_2=2}} = -4 .$$

By (12) we obtain the K-cut annexed to Table 2. After one pivot operation, Table 3 (optimal tableau of P_2) results.

Table 3 does not satisfy the complementarity condition $x_1x_2=0$. From the pair of rows x_1 and x_2, C-cut is then generated according to (15):

$$\beta_0 = -1$$

$$\beta_1 = -(2/55)(11/18) = -(1/45)$$

$$\beta_2 = -(11/14)(4/55) = -(2/35) .$$

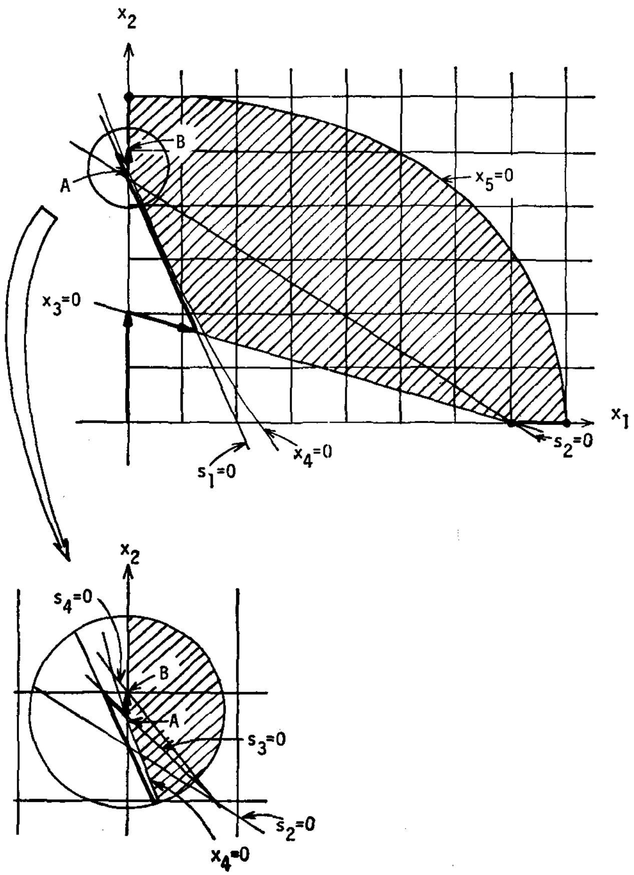

Fig. 2. Application of the cutting plane method to a CCP problem.

The resulting row is annexed to Table 3. After one pivot operation, Table 4 results (optimal tableau of P_3). Although it is also possible to generate K-cut from x_4 (note x_4 is negative), we generate Δ-cut from row x_2, since $\alpha_{30} \notin R_\Delta$. Following (21),

$$f_0{}' = 1/2$$

Table 1. Initial tableau; * shows the pivot element.

	1	$-x_1$	$-x_2$
$z=$	0	2	1
$x_3=$	−14	−2	−7*
$x_1=$	0	−1	0
$x_2=$	0	0	−1

$x_4=-6.93,\quad x_5=1$

Table 2. Optimal tableau of P_1 and K-cut.

	1	$-x_1$	$-x_3$
$z=$	−2	12/7	1/7
$x_3=$	0	0	−1
$x_1=$	0	−1	0
$x_2=$	2	2/7	−1/7
$x_4=-10,\quad x_5=8/9$			
$s_1=$	−10	−55/7*	−4/7

Table 3. Optimal tableau of P_2 and C-cut.

	1	$-s_1$	$-x_3$
$z=$	−46/11	12/55	1/55
$x_3=$	0	0	−1
$x_1=$	14/11	−7/55	4/55
$x_2=$	18/11	2/55	−9/55
$x_4=-0.05,\quad x_5=0.61$			
$s_2=$	−1	−1/45	−2/35*

$$\beta_0=-f_0$$
$$\beta_1=-\alpha_{31}=-(1/10)$$
$$\beta_2=(f_0/(1-f_0))\,\alpha_{32}=-(63/22)$$

are obtained, and the resulting cut is annexed to Table 4.

Table 4. Optimal tableau of P_3 and Δ-cut.

	1	$-s_1$	$-s_2$
$z=$	$-9/2$	$19/90$	$7/22$
$x_3=$	$35/2$	$7/18$	$-35/2$
$x_1=$	0	$-7/45$	$14/11$
$x_2=$	$9/2$	$1/10$	$-63/22$

$x_4=-0.823$, $x_5=0.56$

$s_3=$	$-1/2$	$-1/10$	$-63/22$*

Table 5. Optimal tableau of P_4 and Δ-cut.

	1	$-x_1$	$-s_3$
$z=$	$-43/9$	1	$5/9$
$x_3=$	$175/9$	5	$-35/9$
$x_1=$	0	-1	0
$x_2=$	$43/9$	1	$-5/9$

$x_4=0.117$, $x_5=0.37$

$s_4=$	$-7/9$	-1	$-35/18$*

Table 6. Optimal tableau of P_5: Δ-mesh optimal solution of P.

	1	$-x_1$	$-s_4$
$z=$	-5	$5/7$	$2/7$
$x_3=$	21	7	-2
$x_1=$	0	-1	0
$x_2=$	5	$9/7$	$-2/7$

$x_4=0.97$, $x_5=0.306$

In this case, after two pivot operations, the optimal tableau of P_4 (Table 5) is reached. Since this tableau still does not satisfy the condition $\alpha_{30} \in R_\Delta$, Δ-cut is again generated from x_2 (row s_4). One pivot operation now yields Table 6, in which all the conditions are satisfied. Thus, Table 6 provides a Δ-mesh optimal solution of P:

$$z = -5$$

$$x_1 = 0\,, \qquad x_2 = 5\,.$$

The trajectory of solutions obtained for $P_1, P_2, \ldots, P_5$ is also illustrated in Fig. 2.

Note that, however, the solution of Table 5 is also incidentally feasible in P, though $a_{30} \notin R_\Delta$. Thus if we terminated the computation at this stage, a better solution

$$z = -(43/9)$$

$$x_1 = 0\,, \qquad x_2 = 43/9$$

would have been obtained. Although this situation does not always occur, it will provide a better solution than the Δ-mesh optimal solution.

9. Relation between Δ-mesh Optimal Solutions and Optimal Solutions

The selection of Δ plays a crucial role in the cutting plane method. If the value of Δ is too large, the displacement from the real optimal solution of P could be intolerable, whereas if Δ is too small, the convergence speed would be slow. The next theorem relates the size of Δ with the accuracy of Δ-mesh optimal solution: Δ-mesh optimal solution can come arbitrarily close to some optimal solution of P if an appropriate Δ is chosen.

Let $\bar{x} \in O(P)$ and $J(\bar{x}) \subset \{1, 2, \cdots, r\}$ be such that

(26)
(a) $j \in J(\bar{x})$ implies $\bar{x}_j = 0$, and
(b) either p or $q \in J(\bar{x})$ for every $(x_p, x_q) \in C$.

We define that CCP problem P has *C-interior* if P has $\bar{x} \in O(P)$, $J(\bar{x})$ and $\hat{x} \in R^n$ such that

$$\hat{x} \neq \bar{x}$$

$$\hat{x}_k \begin{cases} = 0, \text{ if } k \in J(\bar{x}) \\ \geq 0, \text{ if } k \notin J(\bar{x}), \quad k = 1, 2, \cdots, n \end{cases}$$

$$\hat{x}_{n+i} = a+_{i0} \sum_{j=1}^{n} a_{ij}(-\hat{x}_j) \begin{cases} = 0, & \text{if } n+i \in J(\bar{x}) \\ \geq 0, & \text{if } n+i \notin J(\bar{x}), \end{cases} \tag{27}$$

$$i = 1, 2, \cdots, m_1$$

$$\hat{x}_{r+i} = -g_i(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n) > 0, \quad i = 1, 2, \cdots, m_2.$$

Theorem 6: Let us assume that coefficients of CCP problem P are all integers, $F(P)$ is compact and P has C-interior. Let x_Δ stand for a Δ-mesh optimal solution of P. Then for any $\varepsilon>0$, there exists Δ such that $\|\bar{x}-x_\Delta\| \leq \varepsilon$. Furthermore there exists a sequence $\Delta_1, \Delta_2, .., \Delta_k, ..$ such that $\lim_{k\to\infty} x_{\Delta_k} \in O(P)$.

Proof: Let $\bar{x} \in O(P)$, $J(\bar{x})$ and $\hat{x}$ be defined by (26) and (27). Let H be the set of $x \in R^n$ such that

$$x_j \begin{cases} = 0, & \text{if } j \in J(\bar{x}) \\ \geq 0, & \text{if } j \notin J(\bar{x}), \quad j = 1, 2, \cdots, n \end{cases}$$

$$x_{n+i} \begin{cases} = 0, & \text{if } n+i \in J(\bar{x}) \\ \geq 0, & \text{if } n+i \notin J(\bar{x}), \quad i = 1, 2, \cdots, m_1 \end{cases}$$

$$x_{r+i} = -g_i(x) > 0, \qquad i = 1, 2, \cdots, m_2.$$

H is nonempty (by assumption), convex and $H \subset F(P)$. Now, for $\hat{x} \in H$, $x(\lambda) = \lambda\hat{x} + (1-\lambda)\bar{x}$, $1 \geq \lambda > 0$, is also in H since H is convex.

Let $B_\varepsilon(x(\lambda)) = \{y \in H \mid \|y - x(\lambda)\| < \varepsilon\}$. Then for each λ and ε it is possible to select Δ such that

$$B_\varepsilon(x(\lambda)) \cap R_\Delta{}^n \neq \phi \text{ i.e., } B_\varepsilon(x(\lambda)) \cap F_\Delta(P) \neq \phi. \tag{28}$$

(By assumption on P, there exists $x \in F(P) \cap B_\varepsilon(x(\lambda))$, whose components x_i, $i=1, 2, .., n$, are all rational numbers. Then let Δ be any positive number satisfying $1/\Delta$: integer and x_i/Δ: integer for $i=1, 2, .., n$.) Obviously any $x' \in B_\varepsilon(x(\lambda)) \cap R_\Delta{}^n$ satisfies $z(\bar{x}) \geq z(x_\Delta) \geq z(x')$. Now for each λ_k, $k=1, 2, ..$, such that $\lim_{k\to\infty} \lambda_k = 0$, we can choose ε_k and Δ_k satisfying (28) and $\lim_{k\to\infty} \varepsilon_k=0$. Thus $\lim_{k\to\infty} x(\lambda_k)=\bar{x}$ and hence $\lim_{\lambda_k\to 0}\lim_{\varepsilon_k\to 0} x'=\bar{x}$ imply that $z(\bar{x})=\lim_{k\to\infty} z(x_{\Delta_k})$. $\lim_{k\to\infty} x_{\Delta_k}\in F(P)$ follows from the compactness of $F(P)$. (Even if this limit does not exist, we can choose a subsequence $\{\Delta_j\}$ of $\{\Delta_k\}$ so that x_{Δ_j} has a limit, since $F(P)$ is compact. Then $\{\Delta_j\}$ may be regarded as the sequence $\{\Delta_k\}$.) This proves the second half of the theorem. The first half immediately follows from the second half.

QED.

From this theorem, we see that the sequence of Δ_j, $j=1, 2,\ldots$, for example such that $\Delta_j=1/j$ always contains a subsequence $\{\Delta_k\}$ having the property $\lim_{k\to\infty} x_{\Delta_k} \in O(P)$.

10. Discussion

When problem P has no convex constraint (i.e., $m_2=0$), P is called a complementary linear programming (CLP) problem. In this case, the cutting plane method is considerably simplified; only C-cuts and Δ-cuts are required to carry out the computation. This is a great saving because K-cut requires the differentiation of g_i which is quite time consuming.

Another advantage is that an *exact* optimal solution can be obtained by the cutting plane method if we choose an appropriate value of Δ. As proved in [6], there exists Δ such that

$$(29) \qquad F_{\Delta}(P) \cap O(P) \neq \phi$$

for each CLP problem, under the assumption that coefficients are all rational numbers (This follows from the fact that there exists $\bar{x}\in O(P)$ which is also a basic solution of Q and that there exists only a finite number of basic solutions). Therefore if we use Δ satisfying (29) and apply the cutting plane method, the resulting Δ-mesh optimal solution is also an optimal solution of P. The calculation of Δ of (29), however, is not easy for most of CLP problems. Thus it is difficult to consider this approach as a practical method applicable to all CLP problems. In some cases such as CLP problems with unimodular coefficient matrices, however, it is trivial to calculate Δ satisfying (29) and the cutting plane method may be used to obtain an exact optimal solution.

Finally, note that our method can also be applied to CVP problems which do not have the complementarity condition. It still has the property of finite Δ-mesh convergence. This makes a contrast with the conventional cutting plane methods for CVP problems [9], [10], none of which has the property of finite convergence.

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