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ON THE BUSY PERIOD IN THE QUEUEING SYSTEM WITH FINITE CAPACITY

ON HASHIDA

Nippon Telegraph and Telephone Public Corporation

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Abstract

In this paper, the busy period in the queueing system $M/G/1$ with finite capacity (waiting room) is studied. First, the equations of the state including supplementary variables are solved by taking the Laplace transform with respect to time, and the Laplace transform of the busy period distribution is obtained. Next, using the above results, the steady-state queue length distribution, the probability of overflow and the waiting time distribution under the steady state are considered. The expressions for the mean length of the busy period and for the mean waiting time are also obtained. Some tables are presented as the numerical examples for the 2–Erlang and 4–Erlang service time distributions.

1. Introduction

The queueing systems with finite capacity (waiting room) have been discussed in numerous articles (Cohen [1], Erlang [2], Finch [3], [4], Jain [5], Keilson [7], Riordan [8] and others). This paper deals with the busy period and the transient distribution of the number of customers in the queueing system $M/G/1$ with finite capacity. First, we discuss the transient distribution during the busy-period process using the supplementary variable technique. The transient characteristics during the general process are then studied in terms of the busy-period and the idle-period processes through renewal-theoretic arguments which were applied by Jaiswal [6].

The problem of the buffer storage in a data communication system motivated consideration of finite capacity in the $M/G/1$ system. In designing the capacity of the buffer storage, the probability of overflow, i.e. the probability that an arriving customer is forced to go away, is one of the important performance measures.

We assume that:

(1) customers arrive in accordance with a Poisson process of mean arrival rate λ.

(2) the service times are identically and independently distributed with the distribution function $F(x)$ which is absolutely continuous and has the probability density $f(x)$. The mean service time h is given by

$$h=\int_0^\infty F^c(x)dx$$

where $F^c(x)=1-F(x)$.

(3) the capacity of the system is N, i.e., if a customer finds N customers in the system including the one being served on his arrival, the arriving customer is not admitted to the system and does not influence the development of the queueing process.

2. Busy-period Process

Let Z_t denote the number of customers present in the system at time t and X_t denote the elapsed service time of the customer served at time t if the server is busy. It is obvious that the process $\{(Z_t, X_t), t \in [0, \infty)\}$ is a Markov process with the state space $\{0, 1, \ldots, N\} \times [0, \infty)$. If $F(x)$ is absolutely continuous and if

$$\frac{1}{F^c(x)} \cdot \frac{dF(x)}{dx} \equiv \eta(x) \tag{2.1}$$

is bounded for $x>0$, then the transition probabilities:

$$\begin{aligned} &\hat{p}_j(0,t) \equiv Pr\{Z_t = 0 \mid Z_0 = j\} \\ &\hat{p}_j(m,x,t)dx \equiv Pr\{Z_t = m, x<X_t<x+dx \mid Z_0 = j\} \\ &\qquad\qquad m = 1, 2, \cdots, N; j = 0, 1, \cdots, N \end{aligned} \tag{2.2}$$

are well defined for $x>0$, $t \geq 0$ and are the only solution of the forward Kolmogorov equations for the process $\{(Z_t, X_t)\}$ [1].

Now, we also denote by Z_t the number of customers present at time t during the busy period starting at $t=0$ with $j(>0)$ customers, one of which enters service. We investigate the following transition probabilities:

$$\begin{aligned} p_j(m,x,t)dx \equiv Pr\{&Z_t = m, x<X_t<x+dx, \\ &Z_\tau>0 \text{ for all } \tau(0<\tau<t) \mid Z_0 = j\} \quad (m \geq 1). \end{aligned} \tag{2.3}$$

The forward Kolmogorov equations are for $t>0$

$$\begin{cases} \left[\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial x} + \lambda + \eta(x)\right] p_j(m,x,t) \\ \qquad = (1-\delta_{m,1})\lambda p_j(m-1,x,t) \\ \qquad\qquad 1 \leq m \leq N-1, \quad x>0 \\ \left[\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial x} + \eta(x)\right] p_j(N,x,t) \\ \qquad = \lambda p_j(N-1,x,t) \\ \qquad\qquad x>0 \end{cases} \tag{2.4}$$

where $\delta_{m,1}$ is the Kronecker's delta function.

The initial condition is

(2.5) $$p_j(m, x, 0) = \delta(x)\delta_{m,j}$$

where $\delta(x)$ is the Dirac delta function.

Considering the transitions at $x=0$, we get the boundary conditions for $t>0$:

(2.6) $$\begin{cases} p_j(m, 0, t) = \int_0^\infty p_j(m+1, x, t)\eta(x)dx \\ \qquad\qquad\qquad\qquad 1 \le m \le N-1 \\ p_j(N, 0, t) = 0 . \end{cases}$$

Moreover, if $b_j(t)dt$ denotes the probability that the busy period which started at $t=0$ with j customers terminates between t and $t+dt$, we have

(2.7) $$b_j(t) = \int_0^\infty p_j(1, x, t)\eta(x)dx .$$

Denoting by $\pi_j(m, x, s)$ the Laplace transform of $p_j(m, x, t)$ and taking the Laplace transform of (2.4) and (2.6), we have

(2.8) $$\begin{cases} \frac{\partial}{\partial x}\pi_j(m, x, s) + \{s+\lambda+\eta(x)\}\pi_j(m, x, s) \\ \qquad = (1-\delta_{m,1})\lambda\pi_j(m-1, x, s) + \delta(x)\delta_{m,j} \\ \qquad 1 \le j \le N-1 ,\ 1 \le m \le N-1 ,\ x \ge 0 \\ \frac{\partial}{\partial x}\pi_j(N, x, s) + \{s+\eta(x)\}\pi_j(N, x, s) \\ \qquad = \lambda\pi_j(N-1, x, s) \\ \qquad 1 \le j \le N-1 ,\ x \ge 0 \end{cases}$$

$$(2.9)\qquad \begin{cases} \pi_j(m,0,s)=\int_0^\infty \pi_j(m+1,x,s)\,\eta(x)\,dx \\ \qquad\qquad\qquad 1\leq m\leq N-1 \\ \pi_j(N,0,s)=0\,. \end{cases}$$

And if $\gamma_j(s)$ denotes the Laplace transform of $b_j(t)$, we have from (2.7)

$$(2.10)\qquad \gamma_j(s)=\int_0^\infty \pi_j(1,x,s)\,\eta(x)\,dx\,.$$

Now, assuming that $j\neq 1$, we solve the first equation of (2.8) in case of $m=1$ and have

$$(2.11)\qquad \pi_j(1,x,s)=a_j(1,s)e^{-(s+\lambda)x}F^c(x)$$

where, in general, we put

$$(2.12)\qquad a_j(m,s)=\lim_{x\to 0}\pi_j(m,x,s)\,.$$

If we solve the first equation of (2.8) in case of $m=2$ assuming that $j\neq 2$, we have

$$(2.13)\qquad \pi_j(2,x,s)=\{\lambda a_j(1,s)x+a_j(2,s)\}e^{-(s+\lambda)x}F^c(x)$$

where $a_j(2, s)$ is expressed by $a_j(1, s)$ from the boundary condition. Substituting (2.12) and (2.13) in (2.9), we get

$$(2.14)\qquad a_j(2,s)=\frac{1+\lambda\varphi^{(1)}(s+\lambda)}{\varphi(s+\lambda)}\,a_j(1,s)$$

where $\varphi(s)$ is the Laplace transform of $f(x)$ and

$$(2.15)\qquad \varphi^{(n)}(s)=\frac{d^n}{ds^n}\varphi(s)\,.$$

Thus we obtain

$$(2.16)\qquad \pi_j(2,x,s)=\left\{\lambda x+\frac{1+\lambda\varphi^{(1)}(s+\lambda)}{\varphi(s+\lambda)}\right\}a_j(1,s)e^{-(s+\lambda)x}F^c(x)\,.$$

Now, to write (2.16) in a tractable form, we define

$$\sigma(s,\zeta)=\zeta-\frac{\lambda}{s+\lambda}\varphi\{(s+\lambda)(1-\zeta)\}. \tag{2.17}$$

Then, using Goursat's formula, we obtain the following expression:

$$\left\{\lambda x+\frac{1+\lambda\varphi^{(1)}(s+\lambda)}{\varphi(s+\lambda)}\right\}e^{-(s+\lambda)x}=\left(\frac{\lambda}{s+\lambda}\right)\sigma(s,0)\frac{1}{2\pi i}\times\int_{C}\frac{e^{-(s+\lambda)(1-\zeta)x}}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta^{2}} \tag{2.18}$$

where the contour C is a circle having a center at the origin of the complex domain, and $1/\sigma(s,\zeta)$ is analytic within and on C. Hence we have

$$\begin{cases}\pi_j(1,x,s)=a_j(1,s)\sigma(s,0)\dfrac{F^c(x)}{2\pi i}\displaystyle\int_C\frac{e^{-(s+\lambda)(1-\zeta)x}}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta}\\ \pi_j(2,x,s)=a_j(1,s)\sigma(s,0)\left(\dfrac{\lambda}{s+\lambda}\right)\\ \qquad\times\dfrac{F^c(x)}{2\pi i}\displaystyle\int_C\frac{e^{-(s+\lambda)(1-\zeta)x}}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta^2}.\end{cases} \tag{2.19}$$

On the analogy of (2.19), the following relations are obtained for $1\leq m\leq j-1$:

$$\begin{cases}\pi_j(m,x,s)=a_j(1,s)\sigma(s,0)\left(\dfrac{\lambda}{s+\lambda}\right)^{m-1}\\ \qquad\times\dfrac{F^c(x)}{2\pi i}\displaystyle\int_C\frac{e^{-(s+\lambda)(1-\zeta)x}}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta^m}\\ a_j(m,s)=a_j(1,s)\sigma(s,0)\left(\dfrac{\lambda}{s+\lambda}\right)^{m-1}\dfrac{1}{2\pi i}\displaystyle\int_C\frac{1}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta^m}\end{cases} \tag{2.20}$$

This can be proved by mathematical induction as follows: Assume that (2.20) holds when $m=n-1$ ($<j-2$). If we solve the first equation of (2.8) in case of $m=n$, we get

$$\pi_j(n,x,s)=e^{-(s+\lambda)x}F^c(x)\left[\lambda\int_0^x \pi_j(n-1,y,s)\right. \tag{2.21}$$

$$\left.\times e^{(s+\lambda)y}\frac{dy}{F^c(y)}+a_j(n,s)\right].$$

Substituting (2.20), where $m=n-1$, into (2.21), we have

$$\pi_j(n,x,s)=a_j(1,s)\sigma(s,0)\left(\frac{\lambda}{s+\lambda}\right)^{n-1} \tag{2.22}$$

$$\times\frac{F^c(x)}{2\pi i}\int_c\frac{-e^{-(s+\lambda)x}+e^{-(s+\lambda)(1-\zeta)x}}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta^n}$$

$$+e^{-(s+\lambda)x}F^c(x)a_j(n,s).$$

Substituting (2.22) into the boundary condition (2.9), we have

$$a_j(1,s)\sigma(s,0)\left(\frac{\lambda}{s+\lambda}\right)^{n-2}\frac{1}{2\pi i}\int_c\frac{1}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta^{n-1}} \tag{2.23}$$

$$=a_j(1,s)\sigma(s,0)\left(\frac{\lambda}{s+\lambda}\right)^{n-1}\frac{1}{2\pi i}$$

$$\times\int_c\frac{\varphi\{(s+\lambda)(1-\zeta)\}-\varphi(s+\lambda)}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta^n}$$

$$+a_j(n,s)\varphi(s+\lambda)$$

$$=a_j(1,s)\sigma(s,0)\left[\left(\frac{\lambda}{s+\lambda}\right)^{n-2}\frac{1}{2\pi i}\int_c\frac{1}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta^{n-1}}\right.$$

$$\left.-\left(\frac{\lambda}{s+\lambda}\right)^{n-1}\frac{1}{2\pi i}\int_c\frac{\varphi(s+\lambda)}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta^n}\right]$$

$$+a_j(n,s)\varphi(s+\lambda), \qquad n\geq 2\,.$$

From this relation, we obtain the second equation of (2.20), where $m=n$, and substituting it into (2.22), we obtain the first equation of (2.20), where $m=n$. Thus, the result (2.20) is proved by mathematical induction.

When $m=j$, we solve the first equation of (2.8) and have

$$\pi_j(j,x,s)=e^{-(s+\lambda)x}F^c(x) \times\Big[\lambda\int_0^x \pi_j(j-1,y,s)\,e^{(s+\lambda)y}\frac{dy}{F^c(y)} +1+\pi_j(j,0,s)\Big], \quad x>0\,. \tag{2.24}$$

Hence, we obtain

$$a_j(j,s)=1+\pi_j(j,0,s) \tag{2.25}$$

while, for $m\neq j$,

$$a_j(m,s)=\pi_j(m,0,s)\,. \tag{2.26}$$

Using (2.25), we obtain the same expression for $m=j$ as (2.20).

Next, we consider the case of $m>j$. If we solve the equation (2.8) in case of $m=j+1$, we get

$$\pi_j(j+1,x,s)=e^{-(s+\lambda)x}F^c(x) \times\Big[\lambda\int_0^x \pi_j(j,y,s)\,e^{(s+\lambda)y}\frac{dy}{F^c(y)} +a_j(j+1,s)\Big]. \tag{2.27}$$

Substituting (2.20) with $m=j$ into (2.27), we have

$$\pi_j(j+1,x,s)=a_j(1,s)\sigma(s,0)\left(\frac{\lambda}{s+\lambda}\right)^j \tag{2.28}$$

$$\times \frac{F^c(x)}{2\pi i}\int_c \frac{-e^{-(s+\lambda)x}+e^{-(s+\lambda)(1-\zeta)x}}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta^{j+1}}$$

$$+e^{-(s+\lambda)x}F^c(x)a_j(j+1,s).$$

Substituting (2.28) in the boundary condition (2.9), where $m=j$, and using (2.25), we have

$$a_j(j+1,s)=a_j(1,s)\sigma(s,0)\left(\frac{\lambda}{s+\lambda}\right)^j\frac{1}{2\pi i}\int_c\frac{1}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta^{j+1}} \tag{2.29}$$

$$+\left(\frac{\lambda}{s+\lambda}\right)\frac{1}{\sigma(s,0)}$$

$$=a_j(1,s)\sigma(s,0)\left(\frac{\lambda}{s+\lambda}\right)^j\frac{1}{2\pi i}\int_c\frac{1}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta^{j+1}}$$

$$+\left(\frac{\lambda}{s+\lambda}\right)\frac{1}{2\pi i}\int_c\frac{1}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta}.$$

From (2.28) and (2.29), we obtain

$$\pi_j(j+1,x,s)=a_j(1,s)\sigma(s,0)\left(\frac{\lambda}{s+\lambda}\right)^j\frac{F^c(x)}{2\pi i} \tag{2.30}$$

$$\times\int_c\frac{e^{-(s+\lambda)(1-\zeta)x}}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta^{j+1}}$$

$$+\left(\frac{\lambda}{s+\lambda}\right)\frac{F^c(x)}{2\pi i}\int_c\frac{e^{-(s+\lambda)(1-\zeta)x}}{\sigma(s,\zeta)}\frac{d\zeta}{\zeta}.$$

By mathematical induction as in case $1\leq m\leq j-1$, we can prove the similar relations for $j<m\leq N-1$. Therefore, combining (2.20) and these relations by the properties of the integral in the complex domain, we obtain the following results:

$$\pi_j(m,x,s)=a_j(1,s)\sigma(s,0)\left(\frac{\lambda}{s+\lambda}\right)^{m-1} \tag{2.31}$$

$$\times \frac{F^c(x)}{2\pi i} \int_c \frac{e^{-(s+\lambda)(1-\zeta)x}}{\sigma(s,\zeta)} \frac{d\zeta}{\zeta^m}$$

$$+\left(\frac{\lambda}{s+\lambda}\right)^{m-j} \frac{F^c(x)}{2\pi i} \int_c \frac{e^{-(s+\lambda)(1-\zeta)x}}{\sigma(s,\zeta)} \frac{d\zeta}{\zeta^{m-j}}$$

$$1 \leq j \leq N-1, \; 1 \leq m \leq N-1, \; x>0$$

$$a_j(m,s) = a_j(1,s)\sigma(s,0)\left(\frac{\lambda}{s+\lambda}\right)^{m-1} \frac{1}{2\pi i} \int_c \frac{1}{\sigma(s,\zeta)} \frac{d\zeta}{\zeta^m}$$

$$+\left(\frac{\lambda}{s+\lambda}\right)^{m-j} \frac{1}{2\pi i} \int_c \frac{1}{\sigma(s,\zeta)} \frac{d\zeta}{\zeta^{m-j}}$$

$$1 \leq j \leq N-1, \; 1 \leq m \leq N-1$$

which include the results (2.20) and the case of $m=j$.

Finally, we consider the case of $m=N$. If $j \neq N$, the solution of the second equation of (2.8) is given by

$$\text{(2.32)} \qquad \pi_j(N,x,s) = e^{-sx} F^c(x) \left[\lambda \int_0^x \pi_j(N-1,y,s)\, e^{sy} \frac{dy}{F^c(y)}\right].$$

Substituting (2.31) in (2.32) we obtain the following results:

$$\text{(2.33)} \qquad \pi_j(N,x,s) = a_j(1,s)\,\sigma(s,0)\left(\frac{\lambda}{s+\lambda}\right)^{N-1}$$

$$\times \frac{F^c(x)}{2\pi i} \int_c \frac{e^{-(s+\lambda)(1-\zeta)x} - e^{-sx}}{\sigma(s,\zeta)\left(\zeta - \frac{\lambda}{s+\lambda}\right)} \frac{d\zeta}{\zeta^{N-1}}$$

$$+\left(\frac{\lambda}{s+\lambda}\right)^{N-j} \frac{F^c(x)}{2\pi i} \int_c \frac{e^{-(s+\lambda)(1-\zeta)x} - e^{-sx}}{\sigma(s,\zeta)\left(\zeta - \frac{\lambda}{s+\lambda}\right)} \frac{d\zeta}{\zeta^{N-j-1}}$$

$$1 \leq j \leq N-1, \quad x>0.$$

Now, the results obtained so far include an unknown function

$a_j(1, s)$ which is determined by the boundary condition (2.9). If we put $m=N-1$ in (2.9) and substitute (2.31) and (2.33) in it, we have

$$a_j(1,s) = \frac{1}{\varphi(s+\lambda)} \times \frac{1+\left(\frac{\lambda}{s+\lambda}\right)^{N-j} \frac{1}{2\pi i} \times \int_c \frac{1-\varphi(s)}{\sigma(s,\zeta)\left(\zeta-\frac{\lambda}{s+\lambda}\right)} \frac{d\zeta}{\zeta^{N-j-1}}}{1+\left(\frac{\lambda}{s+\lambda}\right)^{N} \frac{1}{2\pi i} \times \int_c \frac{1-\varphi(s)}{\sigma(s,\zeta)\left(\zeta-\frac{\lambda}{s+\lambda}\right)} \frac{d\zeta}{\zeta^{N-1}}}, \quad 1 \leq j \leq N-1 . \tag{2.34}$$

Thus $\pi_j(m, x, s)$ $(m=1,..,N)$ are determined completely.

Next, we observe that $\{\pi_j(m, x, s)\}$ must satisfy the following normalizing condition:

$$\sum_{m=1}^{N} \int_0^\infty \pi_j(m, x, s)\, dx = \frac{1-\gamma_j(s)}{s}. \tag{2.35}$$

The right hand side of this equation is the Laplace transform of the probability that the busy period has not completed at time t yet. It is easily shown that $a_j(1, s)$ is also determined by this condition and coincides with (2.34). Let $\pi_j(m, s)$ denote unconditional $\pi_j(m, x, s)$, i.e.,

$$\pi_j(m, s) \equiv \int_0^\infty \pi_j(m, x, s)\, dx . \tag{2.36}$$

From (2.31) and (2.33), we have

$$\pi_j(m, s) = a_j(1, s)\, \sigma(s, 0) \left(\frac{\lambda}{s+\lambda}\right)^{m-1} \tag{2.37}$$

$$\times \frac{1}{2\pi i}\int_c \frac{1-\varphi\{(s+\lambda)(1-\zeta)\}}{\sigma(s,\zeta)(s+\lambda)(1-\zeta)}\frac{d\zeta}{\zeta^m}$$

$$+\left(\frac{\lambda}{s+\lambda}\right)^{m-j}\frac{1}{2\pi i}\int_c \frac{1-\varphi\{(s+\lambda)(1-\zeta)\}}{\sigma(s,\zeta)(s+\lambda)(1-\zeta)}\frac{d\zeta}{\zeta^{m-j}}$$

$$1\leq m\leq N-1$$

(2.38) $$\pi_j(N,s)=a_j(1,s)\,\sigma(s,0)\left(\frac{\lambda}{s+\lambda}\right)^{N-1}$$

$$\times\frac{1}{2\pi i}\int_c \frac{\dfrac{1-\varphi\{(s+\lambda)(1-\zeta)\}}{(s+\lambda)(1-\zeta)}-\dfrac{1-\varphi(s)}{s}}{\sigma(s,\zeta)\left(\zeta-\dfrac{\lambda}{s+\lambda}\right)}\frac{d\zeta}{\zeta^{N-1}}$$

$$+\left(\frac{\lambda}{s+\lambda}\right)^{N-j}\frac{1}{2\pi i}$$

$$\times\int_c \frac{\dfrac{1-\varphi\{(s+\lambda)(1-\zeta)\}}{(s+\lambda)(1-\zeta)}-\dfrac{1-\varphi(s)}{s}}{\sigma(s,\zeta)\left(\zeta-\dfrac{\lambda}{s+\lambda}\right)}\frac{d\zeta}{\zeta^{N-j-1}}.$$

Substituting (2.11) in (2.10), we obtain

(2.39) $$\gamma_j(s)=a_j(1,s)\,\varphi(s+\lambda).$$

Hence, substituting these relations in (2.35), we get the same results as (2.34).

When $j=1$ or $j=N-1$, we also obtain the same results as above.

When $j=N$, the second equation of (2.8) and the second condition of (2.9) become as follows:

(2.40) $$\frac{\partial}{\partial x}\pi_N(N,x,s)+\{s+\eta(x)\}\,\pi_N(N,x,s)$$

$$= \lambda \pi_N(N-1, x, s) + \delta(x)$$

(2.41) $$\pi_N(N, 0, s) = 0 .$$

The second term of the right hand side in the first equation of (2.8) vanishes, so that its solution is given by only the first term of (2.31) where $j=N$. Substituting this solution in (2.40), we obtain

(2.42) $$\pi_N(N, x, s) = a_N(1, s)\, \sigma(s, 0) \left(\frac{\lambda}{s+\lambda}\right)^{N-1} \times \frac{F^c(x)}{2\pi i} \int_c \frac{e^{-(s+\lambda)(1-\zeta)x} - e^{-sx}}{\sigma(s, \zeta)\left(\zeta - \frac{\lambda}{s+\lambda}\right)} \frac{d\zeta}{\zeta^{N-1}} + e^{-sx} F^c(x) .$$

From the boundary condition and (2.42), the unknown function $a_N(1, s)$ is given by

(2.43) $$a_N(1, s) = \frac{\varphi(s)/\varphi(s+\lambda)}{1 + \left(\frac{\lambda}{s+\lambda}\right)^N \frac{1}{2\pi i} \int_c \frac{1-\varphi(s)}{\sigma(s, \zeta)\left(\zeta - \frac{\lambda}{s+\lambda}\right)} \frac{d\zeta}{\zeta^{N-1}}}$$

which coincides with the result from the normalizing condition. $\pi_N(m, s)$ is given by only the first term of (2.37) and $\pi_N(N, s)$ is given by

(2.44) $$\pi_N(N, s) = a_N(1, s)\, \sigma(s, 0) \left(\frac{\lambda}{s+\lambda}\right)^{N-1} \times \frac{1}{2\pi i} \int_c \frac{\dfrac{1-\varphi\{(s+\lambda)(1-\zeta)\}}{(s+\lambda)(1-\zeta)} - \dfrac{1-\varphi(s)}{s}}{\sigma(s, \zeta)\left(\zeta - \frac{\lambda}{s+\lambda}\right)} \frac{d\zeta}{\zeta^{N-1}} + \frac{1-\varphi(s)}{s} .$$

Now, substituting (2.34) or (2.43) in (2.39), we obtain the Laplace transform of the busy period density as follows:

$$\gamma_j(s) = \frac{1+\left(\frac{\lambda}{s+\lambda}\right)^{N-j}\frac{1}{2\pi i}\int_c \frac{1-\varphi(s)}{\sigma(s,\zeta)(\zeta-\lambda/s+\lambda)}\frac{d\zeta}{\zeta^{N-j-1}}}{1+\left(\frac{\lambda}{s+\lambda}\right)^{N}\frac{1}{2\pi i}\int_c \frac{1-\varphi(s)}{\sigma(s,\zeta)(\zeta-\lambda/s+\lambda)}\frac{d\zeta}{\zeta^{N-1}}} \tag{2.45}$$

$$1 \leq j \leq N-1$$

$$\gamma_N(s) = \frac{\varphi(s)}{1+\left(\frac{\lambda}{s+\lambda}\right)^{N}\frac{1}{2\pi i}\int_c \frac{1-\varphi(s)}{\sigma(s,\zeta)(\zeta-\lambda/s+\lambda)}\frac{d\zeta}{\zeta^{N-1}}}. \tag{2.46}$$

If we denote the mean length of the busy period by $\bar{\gamma}_j{}^N$ where N means the capacity of the system and j means the initial condition, we obtain

$$\bar{\gamma}_j{}^N = \frac{h}{2\pi i}\sum_{l=0}^{j-1}\int_c \frac{-1}{\sigma(0,\zeta)}\frac{d\zeta}{\zeta^{N-l-1}}. \tag{2.47}$$

Specially, when $j=1$,

$$\bar{\gamma}_1{}^N = \frac{h}{2\pi i}\int_c \frac{-1}{\sigma(0,\zeta)}\frac{d\zeta}{\zeta^{N-1}} \qquad (N>1). \tag{2.48}$$

Therefore

$$\bar{\gamma}_j{}^N = \sum_{l=0}^{j-1}\bar{\gamma}_1{}^{N-l} \qquad 1 \leq j \leq N-1. \tag{2.49}$$

If $i=N$, we have from (2.46)

$$\bar{\gamma}_N{}^N = h\left\{1-\frac{1}{2\pi i}\int_c \frac{1}{\sigma(0,\zeta)(1-\zeta)}\frac{d\zeta}{\zeta^{N-1}}\right\} \tag{2.50}$$

or

$$\bar{\gamma}_N{}^N = h + \sum_{k=0}^{N-2} \bar{\gamma}_1{}^{k+2}. \tag{2.51}$$

For example, if the distribution of service times is *n.e.d.*, we obtain

$$\bar{\gamma}_1{}^N = h\,\frac{1-\rho^N}{1-\rho}, \qquad \rho = \lambda h \tag{2.52}$$

$$\bar{\gamma}_j{}^N = h\,\frac{j(1-\rho)-\rho^{N+1-j}+\rho^{N+1}}{(1-\rho)^2}, \qquad 1\leq j\leq N. \tag{2.53}$$

3. General Process

Suppose that the general process starts at time $t=0$ with $j(>0)$ customers and a customer enters service. Let $\{t_1, t_2, \ldots\}$ be the sequence of epochs at which busy periods start, then it is easy to see that the sequence $\{\tau_k = t_k - t_{k-1},\ k\geq 1\}$ $(t_0=0)$ constitutes a modified renewal process. If we denote the Laplace transform of the renewal density of this process by $g_j(s)$, we have

$$g_j(s) = \frac{\gamma_j(s)\dfrac{\lambda}{s+\lambda}}{1-\gamma_1(s)\dfrac{\lambda}{s+\lambda}}. \tag{3.1}$$

Now, if $\hat{p}_j(m, x, t)$ $(m=1, 2, \ldots, N)$ and $\hat{p}_j(m, t)$ $(m=0, 1, \ldots, N)$ denote the probabilities for the general process, then for $m>0$

$$\begin{cases} \hat{p}_j(m, x, t) = p_j(m, x, t) + g_j(t) * p_1(m, x, t) \\ \hat{p}_j(m, t) = p_j(m, t) + g_j(t) * p_1(m, t) \end{cases} \tag{3.2}$$

where $*$ means the convolution operation. Therefore, if $\hat{\pi}_j(m, x, s)$ denotes the Laplace transform of $\hat{p}_j(m, x, t)$ and $\hat{\pi}_j(m, s)$ denotes the Laplace transform of $\hat{p}_j(m, t)$, we have

$$\hat{\pi}_j(m, x, s) = \pi_j(m, x, s) + g_j(s)\,\pi_1(m, x, s) \tag{3.3}$$

(3.4) $$\hat{\pi}_j(m, s) = \pi_j(m, s) + g_j(s)\, \pi_1(m, s)$$

where $\pi_j(m, x, s)$ or $\pi_j(m, s)$ is given by (2.23) or (2.29).

Next, we shall consider the limiting probabilities of $\hat{p}_j(m, x, t)$ and $\hat{p}_j(m, t)$. Let $\hat{p}(m, x)$ and $\hat{p}(m)$ be the steady-state probabilities, then they are defined as follows:

(3.5) $$\begin{cases} \hat{p}(m, x) \equiv \lim_{s\to 0} s\, \hat{\pi}_j(m, x, s) \\ \hat{p}(m) \equiv \lim_{s\to 0} s\, \hat{\pi}_j(m, s)\ . \end{cases}$$

Since from (3.1)

(3.6) $$\lim_{s\to 0} s\, g_j(s) = \frac{\lambda}{1+\lambda \bar{\gamma}_1{}^N}$$

holds, we obtain from (3.3)

(3.7) $$\hat{p}(m, x) = \frac{\lambda}{1+\lambda\bar{\gamma}_1{}^N} \frac{F^c(x)}{2\pi i} \int_c \frac{(\zeta-1)\, e^{-\lambda(1-\zeta)x}}{\sigma(0, \zeta)} \frac{d\zeta}{\zeta^m}$$

$$1 \leq m \leq N-1$$

$$\hat{p}(N, x) = \frac{\lambda}{1+\lambda\bar{\gamma}_1{}^N} \frac{F^c(x)}{2\pi i} \int_c \frac{e^{-\lambda(1-\zeta)x}-1}{\sigma(0, \zeta)} \frac{d\zeta}{\zeta^{N-1}}\ .$$

These results coincide with those which have been obtained by Cohen [1]. From (3.4) and (3.6), we also obtain

(3.8) $$\hat{p}(m) = \frac{\bar{\gamma}_1{}^{m+1} - \bar{\gamma}_1{}^m}{h(1+\lambda\bar{\gamma}_1{}^N)}\ , \qquad 1 \leq m \leq N-1$$

$$\hat{p}(N) = \frac{h-(1-\rho)\, \bar{\gamma}_1{}^N}{h(1+\lambda\, \bar{\gamma}_1{}^N)}\ .$$

It must be noted that $\hat{p}(N)$ given by (3.8) yields the probability of overflow from the system.

For $m=0$, if $p_j(0, t)$ denotes the probability that the system is idle at time t after the busy period started at $t=0$ with j (>0) customers, we have

$$(3.9) \qquad p_j(0, t) = b_j(t) * e^{-\lambda t} .$$

Denoting the Laplace transform of $p_j(0, t)$ by $\pi_j(0, s)$ and taking the Laplace transform of (3.9),

$$(3.10) \qquad \pi_j(0, s) = \gamma_j(s) \frac{1}{s+\lambda} .$$

Let $\hat{p}_j(0, t)$ denote the probability that the system is idle at time t during the general process, then we have

$$(3.11) \qquad \hat{p}_j(0, t) = p_j(0, t) + g_j(t) * p_1(0, t) .$$

Taking the Laplace transform,

$$(3.12) \qquad \hat{\pi}_j(0, s) = \pi_j(0, s) + g_j(s) \pi_1(0, s) .$$

Therefore, we obtain the limiting probability $\hat{p}(0)$:

$$(3.13) \qquad \hat{p}(0) = \lim_{s \to 0} s\, \hat{\pi}_j(0, s) = \frac{1}{1+\lambda\, \bar{\gamma}_1^N} .$$

Next, we consider the steady-state probability of the number of customers present at the service completion epochs. Let u_j $(j=0,1,.., N-1)$ denote the steady-state probability that the number of customers present at the service completion epochs is j, and let $\hat{q}(m)$ represent the steady-state probability that a service has just completed at an arbitrary time and that the number of customers present at this time is m. Then we have

$$(3.14) \qquad \hat{q}(m) = \int_0^\infty \hat{p}(m+1, x)\, \eta(x)\, dx .$$

Substituting (3.7) in (3.14), we have

$$(3.15) \qquad \hat{q}(m) = \frac{\lambda}{1+\lambda\, \bar{\gamma}_1} \frac{\bar{\gamma}_1^{m+1} - \bar{\gamma}_1^m}{h} \qquad 0 \le m \le N-1$$

where $\bar{\gamma}_1^0 = 0$. Since $\{u_j\}$ are conditional probabilities at the service completion epochs, we can write

$$u_j = C\,\hat{q}(j) \tag{3.16}$$

where C is determined by the normalizing condition. That is,

$$C = \left\{\frac{\lambda}{1+\lambda\,\bar{\gamma}_1^N}\,\frac{\bar{\gamma}_1^N}{h}\right\}^{-1}. \tag{3.17}$$

Therefore, from (3.15), (3.16) and (3.17), we obtain

$$u_j = \frac{\bar{\gamma}_1^{j+1}-\bar{\gamma}_1^{j}}{\bar{\gamma}_1^N}\,, \qquad j=0,\cdots,N-1\,. \tag{3.18}$$

Using (3.8), we can also write

$$u_j = \frac{\hat{p}(j)}{1-\hat{p}(N)}\,, \qquad j=0,\cdots,N-1 \tag{3.19}$$

which has already been obtained by Cohen [1].

4. Waiting Time

We shall consider the probability density of the waiting time under the steady state, assuming that the service discipline is "first come, first served". Three alternatives may be considered as the definition for the probability density of the waiting time:

1) We assume that the waiting time of the customer who finds N customers in the system on arrival is equal to zero, so that waiting times are defined for all the arrived customers.

2) The waiting time is only defined for the customer who is admitted to the system.

3) If the waiting room becomes full, no further customer arrives, and the input process restarts when a service is completed, i.e., the case considered by Finch [3].

If we denote the probability densities for the above cases by $w_1(\tau)$, $w_2(\tau)$ and $w_3(\tau)$, respectively, using the steady-state probability $\hat{p}(m, x)$, we have

(4.1)
$$w_1(\tau)=\delta(\tau)\{\hat{p}(0)+\hat{p}(N)\}+\sum_{m=1}^{N-1}\int_{x=0}^{\infty}\int_{y=0}^{\tau}\hat{p}(m,x)\frac{f(x+y)}{F^c(x)}f^{(m-1)*}(\tau-y)\,dy\,dx$$

(4.2)
$$w_2(\tau)=\left[\sum_{m=1}^{N-1}\int_{x=0}^{\infty}\int_{y=0}^{\tau}\hat{p}(m,x)\frac{f(x+y)}{F^c(x)}f^{(m-1)*}(\tau-y)\,dy\,dx+\delta(\tau)\hat{p}(0)\right]\Big/\{1-\hat{p}(N)\}$$

(4.3)
$$w_3(\tau)=\delta(\tau)\hat{p}(0)+\sum_{m=1}^{N-1}\int_{x=0}^{\infty}\int_{y=0}^{\tau}\hat{p}(m,x)\frac{f(x+y)}{F^c(x)}f^{(m-1)*}(\tau-y)\,dy\,dx$$

where $f^{(n)*}(t)$ denotes the n-fold convolution of $f(t)$.

Denoting the Laplace transform of $w_i(\tau)$ $(i=1, 2, 3)$ by $\omega_i(\theta)$ and taking the Laplace transform, we have

(4.4)
$$\omega_1(\theta)=\hat{p}(N)+\hat{p}(0)\left[1+\frac{\lambda\varphi(\theta)^{N-1}}{2\pi i}\times\int_c\frac{(1-\zeta)[\varphi\{\lambda(1-\zeta)\}-\varphi(\theta)]}{\sigma(0,\zeta)\{\theta-\lambda(1-\zeta)\}\{\zeta-\varphi(\theta)\}}\frac{d\zeta}{\zeta^{N-1}}\right]$$

(4.5)
$$\omega_2(\theta)=\frac{\hat{p}(0)}{1-\hat{p}(N)}\left[1+\frac{\lambda\varphi(\theta)^{N-1}}{2\pi i}\times\int_c\frac{(1-\zeta)[\varphi\{\lambda(1-\zeta)\}-\varphi(\theta)]}{\sigma(0,\zeta)\{\theta-\lambda(1-\zeta)\}\{\zeta-\varphi(\theta)\}}\frac{d\zeta}{\zeta^{N-1}}\right]$$

(4.6)
$$\omega_3(\theta)=\hat{p}(0)\left[1+\frac{\varphi(\theta)^{N-1}}{2\pi i}\times\int_c\frac{(1-\zeta)[\varphi\{\lambda(1-\zeta)\}-\varphi(\theta)]}{\sigma(0,\zeta)\{\theta-\lambda(1-\zeta)\}\{\zeta-\varphi(\theta)\}}\frac{d\zeta}{\zeta^{N-1}}\right].$$

If $\bar{w}_i$ $(i=1, 2, 3)$ represents the mean waiting time for each case, we obtain

$$\bar{w}_1=\bar{w}_3=\frac{1}{\lambda}\frac{(N-1)(1+\lambda\bar{\gamma}_1{}^N)-\sum_{k=0}^{N-2}\bar{\gamma}_1{}^{k+2}/h}{1+\lambda\bar{\gamma}_1{}^N} \tag{4.7}$$

$$\bar{w}_2=\frac{1}{\lambda}\frac{(N-1)(1+\lambda\bar{\gamma}_1{}^N)-\sum_{k=0}^{N-2}\bar{\gamma}_1{}^{k+2}/h}{\bar{\gamma}_1{}^N/h}. \tag{4.8}$$

For example, if the service time distribution is *n.e.d.*, we find

$$\bar{w}_1=\bar{w}_3=\frac{\rho^2}{\lambda}\frac{1-N\rho^{N-1}+(N-1)\rho^N}{(1-\rho)(1-\rho^{N+1})} \tag{4.9}$$

$$\bar{w}_2=\frac{\rho^2}{\lambda}\frac{1-N\rho^{N-1}+(N-1)\rho^N}{(1-\rho)(1-\rho^N)}. \tag{4.10}$$

The formula (4.9) coincides with that by Finch.

5. Numerical Examples

First, for comparison, we calculate the steady-state probabilities of the number of customers in the system at the service completion epochs by using the imbedded Markov chain method. In the examples, only the case where the service time distribution is k-Erlang distribution is calculated. Table 1 shows the steady-state probabilities at the service completion epochs for the case of $k=2$ and Table 2 shows those for the case of $k=4$, while ρ is fixed to 0.8.

In the tables, the columns which correspond to $N=\infty$ show the probabilities for $M/E_k/1$ with infinite waiting room. Table 3 and Table 4 show the mean busy periods.

Using the values in Table 3 and Table 4, we can calculate the probabilities at an arbitrary time under the steady state, which are

Table 1. The probabilities at the completion epochs: $\{u_j\}$

j \ N	1	2	3	4	5	∞
0	1.0000	0.5102	0.3674	0.3031	0.2680	0.2000
1		0.4898	0.3527	0.2910	0.2572	0.1920
2			0.2798	0.2308	0.2041	0.1523
3				0.1751	0.1548	0.1155
4					0.1159	0.0865

$k=2,\ \rho=0.8.$

Table 2. The probabilities at the completion epochs: $\{u_j\}$

j \ N	1	2	3	4	5	∞
0	1.0000	0.4823	0.3428	0.2834	0.2524	0.2000
1		0.5177	0.3680	0.3043	0.2710	0.2147
2			0.2892	0.2391	0.2130	0.1688
3				0.1732	0.1542	0.1222
4					0.1094	0.0867

$k=4,\ \rho=0.8.$

Table 3. The mean busy periods: $\{\bar{\gamma}_1^N\}$

ρ \ N	1	2	3	4	5	∞
0.2	1.0000	1.2100	1.2441	1.2492	1.2499	1.2500
0.4	1.0000	1.4400	1.5936	1.6436	1.6594	1.6667
0.6	1.0000	1.6900	2.0761	2.2804	2.3866	2.5000
0.8	1.0000	1.9600	2.7216	3.2991	3.7317	5.0000
1.0	1.0000	2.2500	3.5625	4.8906	6.2227	∞

$k=2,\ h=1.0.$

Table 4. The mean busy periods: $\{\bar{\gamma}_1^N\}$

ρ \ N	1	2	3	4	5	∞
0.2	1.0000	1.2155	1.2459	1.2496	1.2500	1.2500
0.4	1.0000	1.4641	1.6112	1.6521	1.6629	1.6667
0.6	1.0000	1.7490	2.1465	2.3368	2.4250	2.5000
0.8	1.0000	2.0736	2.9174	3.5286	3.9621	5.0000
1.0	1.0000	2.4414	4.0073	5.6011	7.2001	∞

$k=4,\ h=1.0.$

Table 5. The probabilities at an arbitrary time: $\{\hat{p}(m)\}$

m \ N	1	2	3	4	5	∞
0	0.5556	0.3894	0.3147	0.2748	0.2509	0.2000
1	0.4444	0.3738	0.3021	0.2638	0.2409	0.1960
2		0.2368	0.2397	0.2093	0.1911	0.1523
3			0.1434	0.1587	0.1449	0.1155
4				0.0935	0 1085	0.0865
5					0.0636	0.0646

$k=2$, $\rho=0.8$.

Table 6. The probabilities at an arbitrary time: $\{\hat{p}(m)\}$

m \ N	1	2	3	4	5	∞
0	0.5556	0.3761	0.2999	0.2616	0.2398	0.2000
1	0.4444	0.4038	0.3220	0.2808	0.2575	0.2147
2		0.2201	0.2531	0.2207	0.2024	0.1688
3			0.1249	0.1599	0.1466	0.1222
4				0.0770	0.1040	0.0867
5					0.0498	0.0612

$k=4$, $\rho=0.8$.

shown in Table 5 and Table 6.

In practice, we have often estimated the capacity of the buffer storage on the assumption that the probability of overflow is nearly equal to the probability that a departing customer leaves behind more customers than the capacity. In the tables, the probability that a departing customer leaves behind more than 5 customers is 0.2537 ($k=2$) or 0.2076 ($k=4$) for the $M/E_k/1$ model with infinite capacity. On the other hand, the probability that an arriving customer finds the waiting room full is 0.0636 ($k=2$) or 0.0498 ($k=4$) for the $M/E_k/1$ model with finite capacity ($N=5$). Therefore, when N is small, we overestimate the capacity of the buffer storage by using the $M/E_k/1$ model with infinite capacity.

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References

[1] Cohen, J.W., *The Single Server Queue*, North-Holland, 1969.

[2] Brockmeyer, B., H.L. Halstrom and A. Jensen, "The Life and Works of A.K. Erlang," *Trans. Dan. Acad. Tech. Sci.*, No. 2 (1948).

[3] Finch, P.D., "The Effect of the Size of the Waiting Room on a Simple Queue," *J. Roy. Statist. Soc.*, **B–20** (1958), 182–186.

[4] Finch, P.D., "On the Transient Behavior of a Queueing System with Bulk Service and Finite Capacity," *Annals. Math. Statist.*, **33**, 3 (1962), 973–985.

[5] Jain, H.C., "Queueing Problem with Limited Waiting Space," *Nav. Res. Log. Quart.*, **9**, 3 & 4 (1962), 245–252.

[6] Jaiswal, N.K., *Priority Queues*, Academic Press, 1968.

[7] Keilson, J., "The Ergodic Queue Length Distribution for Queueing System with Finite Capacity," *J. Roy. Statist. Soc.*, **B–28**, 1 (1966), 190–201.

[8] Riordan, J., *Stochastic Service Systems*, John Wiley, 1962.