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AN INTERMITTENTLY USED SYSTEM WITH PREVENTIVE MAINTENANCE

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Abstract

An intermittently used system with preventive maintenance is discussed. The Laplace-Stieltjes transform of the distribution and the mean time of the disappointment time are derived by using Markov renewal processes. The effect of the preventive maintenance policy is also discussed.

1. Introduction

A system can be classified into one of two types according as whether it is always used or it is used intermittently. In this paper the reliability analysis of an intermittently used system is discussed. In particular, a model with preventive maintenance is discussed.

Gaver [2] defined the concept of reliability for an intermittently used system, which is called the "disappointment time." That is, the disappointment time is the time of system failure during a usage period, or of occurrence of a demand during a system inoperative period, whichever occurs first. We note that a system is alternatively operative and inoperative (or under repair). Gaver [2] obtained the Laplace-Stieltjes (LS) transform of the distribution of the disappointment time and the mean of the disappointment time for a one-unit system. In this paper we shall discuss a one-unit system with preventive maintenance. The

LS transform of the distribution of the disappointment time and the mean time of the disappointment time will be derived for a one-unit system with preventive maintenance. The effect of preventive maintenance is discussed. In the subsequent discussions we shall consider two models, Model I and Model II, in which the preventive maintenance policies are different from each other.

2. Model I

Consider a one-unit system in which the system is alternatively operative and under repair. The failure time of the unit is assumed to obey some distribution $F(t)$, and the repair time of the unit another distribution $G(t)$. Each switchover is assumed to be instantaneous. A unit functions perfectly upon repair.

Consider the preventive maintenance policy for the unit. The time of the beginning of the preventive maintenance, measured from the instant that the unit begins to be operative, is assumed to obey distribution $A(t)$, i.e., we consider a random preventive maintenance policy. The holding time of the preventive maintenance (or the preventive repair time) is assumed to obey distribution $B(t)$. We assume that the preventive maintenance time is stochastically shorter than the repair time, i.e., $B(t)>G(t)$. A unit functions perfectly upon preventive maintenance.

Finally consider the behavior of occurrence of a need (or a use). The occurrence time of a need is assumed to obey an exponential distribution $\alpha(t)=1-\exp(-\lambda t)$, and the holding time of a need a distribution $\beta(t)$. Then we define the probability $P(t)$ that the system is not used at time t, given that the system was not used at $t=0$. We have

$$P(t)=\sum_{n=0}^{\infty}[\alpha(t)_{*}\beta(t)]^{n*}*\bar{\alpha}(t)=[1-\alpha(t)_{*}\beta(t)]^{(-1)}*\bar{\alpha}(t)\,, \tag{1}$$

where we define that, in general, $[\gamma(t)]^{0*}=E(t)$ (i.e., a unit function), $[\gamma(t)]^{n*}=[\gamma(t)]^{(n-1)*}*\gamma(t)$ $(n\geq 1)$, $[1-\gamma(t)]^{(-1)}=\sum_{n=0}^{\infty}[\gamma(t)]^{n*}$, and $\bar{\gamma}(t)=1-\gamma(t)$. Further, the probability $\bar{P}(t)$ that the system is used at time t, given that the system was not used at $t=0$, is given by

$$(2)\qquad \bar{P}(t) = \sum_{n=0}^{\infty} [\alpha(t) * \beta(t)]^{n*} * \alpha(t) * \bar{\beta}(t)$$
$$= [1-\alpha(t) * \beta(t)]^{(-1)} * \alpha(t) * \bar{\beta}(t) .$$

If we assume that $\beta(t) = 1 - \exp(-\mu t)$, we have

$$(3)\qquad P(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} .$$

For our model, we define the following four states (which are the regeneration points except state s_3):

State s_0: The unit begins to be operative and the need does not occur.

State s_1: The unit begins to get repaired, and the need does not occur.

State s_2: The preventive maintenance begins, and the need does not occur.

State s_3: The disappointment time, the time of system failure during a usage period, or of occurrence of a demand during a system inoperative period, whichever occurs first.

The state transition diagram (which corresponds to the signal flow graph) for our model is shown in Fig. 1. We shall obtain each branch gain in Fig. 1.

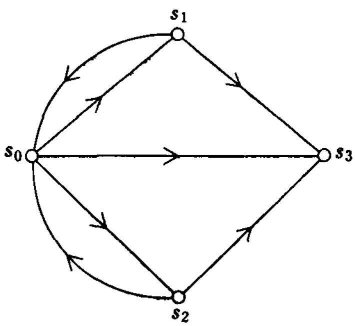

Fig. 1. The state transition diagram.

From state s_0, three transitions can be considered: The transition to state s_1 is an event that the system fails before the preventive maintenance, and at that time the system is not used, the transition to state s_2 is an event that the preventive maintenance begins before the system failure, and at that time the system is not used, and the transition to state s_3 is an event that the system fails or the preventive maintenance begins, and at that time the system is used. Thus we have the following three branch gains:

$$q_{01}(s)=\int_0^\infty e^{-st}P(t)\bar{A}(t)\,dF(t)\,, \tag{4}$$

$$q_{02}(s)=\int_0^\infty e^{-st}P(t)\bar{F}(t)\,dA(t)\,, \tag{5}$$

$$q_{03}(s)=\int_0^\infty e^{-st}\bar{P}(t)\bar{A}(t)\,dF(t)+\int_0^\infty e^{-st}\bar{P}(t)\bar{F}(t)\,dA(t)\,. \tag{6}$$

From state s_1, two transitions can be considered: The transition to state s_0 is an event that the repair is completed before the occurrence of a need. The transition to state s_3 is an event that a need occurs before the completion of repair. Thus we have

$$q_{10}(s)=\int_0^\infty e^{-st}e^{-\lambda t}\,dG(t)=\hat{G}(s+\lambda)\,, \tag{7}$$

$$q_{13}(s)=\int_0^\infty e^{-st}\bar{G}(t)\lambda e^{-\lambda t}\,dt=\frac{\lambda}{s+\lambda}[1-\hat{G}(s+\lambda)]\,, \tag{8}$$

where $\hat{G}(s)$ denotes the LS stransform of $G(t)$.

From state s_2, two transitions can be considered. Using the similar techniques in (7) and (8), we have

$$q_{20}(s)=\int_0^\infty e^{-st}e^{-\lambda t}\,dB(t)=\hat{B}(s+\lambda)\,, \tag{9}$$

(10) $$q_{23}(s)=\int_0^\infty e^{-st}\,\bar{B}(t)\,\lambda\,e^{-\lambda t}\,dt=\frac{\lambda}{s+\lambda}\,[1-\hat{B}(s+\lambda)]\,.$$

Defining that state s_0 is a source and state s_3 is a sink in Fig. 1, and applying Mason's gain formula [3], we have the following system gain:

(11) $$\varphi_0(s)=\frac{q_{03}(s)+q_{01}(s)\,q_{13}(s)+q_{02}(s)\,q_{23}(s)}{1-q_{01}(s)\,q_{10}(s)-q_{02}(s)\,q_{20}(s)}\,,$$

which is the LS transform of the distribution of the disappointment time. The mean time of the disappointment time is given by using Mason's gain formula [3]:

(12) $$T_0=\frac{\xi_0+q_{01}(0)\,\xi_1+q_{02}(0)\,\xi_2}{1-q_{01}(0)q_{10}(0)-q_{02}(0)q_{20}(0)}\,,$$

where each branch gain $q_{ij(s)}$ is replaced by $q_{ij(0)}$ except that $q_{i3(s)}$ ($i=$ 0, 1, 2) is replaced by ξ_i, the unconditional mean in state s_i, i.e.,

(13) $$\xi_i=-\sum_{j=1}^{3}\frac{dq_{ij}(s)}{ds}\Big|_{s=0}\,.$$

In our model, each ξ_i ($i=0$, 1, 2) is given by

(14) $$\xi_0=\int_0^\infty t\,\bar{A}(t)\,dF(t)+\int_0^\infty t\,\bar{F}(t)\,dA(t)\,,$$

(15) $$\xi_1=[1-\hat{G}(\lambda)]/\lambda\,,$$

(16) $$\xi_2=[1-\hat{B}(\lambda)]/\lambda\,.$$

In particular, if we assume that

(17) $$A(t)=\begin{cases}0 & \text{for}\quad t<t_0\\ 1 & \text{for}\quad t\geq t_0\,,\end{cases}$$

i.e., if we assume the constant time preventive maintenance policy, we have

$$\xi_0 = \int_0^{t_0} \bar{F}(t)\, dt, \tag{18}$$

and the mean time T_0 in (12) is a function of t_0. In the remaining part of this section, we shall discuss the effect of the preventive maintenance policy.

To simplify the discussion we consider the equilibrium alternating renewal process instead of the alternating renewal process. Cox [1], p. 85 discussed three possibilities to obtain the equilibrium alternating renewal process from the alternating renewal process. If the behavior of the needs obeys the equilibrium alternating renewal process, we have

$$P(t) = \frac{\mu}{\lambda+\mu} \tag{19}$$

$$\bar{P}(t) = \frac{\lambda}{\lambda+\mu} \tag{20}$$

which are independent of time t.

We shall discuss the effect of the preventive maintenance policy assuming the equilibrium alternating renewal process for the behavior of the needs. First consider the mean time of the disappointment time given in (12). Using (19) and (20), we have

$$T_0 = \frac{\int_0^{t_0} \bar{F}(t)\, dt + pF(t_0)\dfrac{1-\hat{G}(\lambda)}{\lambda} + p\bar{F}(t_0)\dfrac{1-\hat{B}(\lambda)}{\lambda}}{1-pF(t_0)\,\hat{G}(\lambda) - p\bar{F}(t_0)\,\hat{B}(\lambda)}, \tag{21}$$

where $p=\mu/(\lambda+\mu)$. Next consider the mean time of the disappointment time without preventive maintenance. As $t_0 \to \infty$ in (21), we have

$$T_0 = \frac{E(X) + p\dfrac{1-\hat{G}(\lambda)}{\lambda}}{1-p\hat{G}(\lambda)} \tag{22}$$

where

(23) $$E(X)=\int_0^\infty t\,dF(t)=\int_0^\infty \bar{F}(t)\,dt\,.$$

Comparing two equations (21) and (22), we have

Theorem. If (i) $B(t)>G(t)$, (ii) the failure rate $r(t)$ of the failure time distribution $F(t)$ is strictly increasing, and (iii)

(24) $$r(\infty)>\frac{1-p\hat{G}(\lambda)}{p\,[\hat{B}(\lambda)-\hat{G}(\lambda)]\left[E(X)-\frac{1}{\lambda+\mu}\right]},$$

then we can adopt a suitable interval length t_0 and the mean time (21) for the model with preventive maintenance is greater than that of (22) for the model without preventive maintenance.

Proof. We shall show that (21) is greater than (22) under the above assumptions. It is clear that the denominators of (21) and (22) are positive. Then the numerator of subtracting (22) from (21) is given by

(25) $$N(t_0)=\left[\int_0^{t_0}\bar{F}(t)\,dt+pF(t_0)\,\frac{1-\hat{G}(\lambda)}{\lambda}+p\bar{F}(t_0)\,\frac{1-\hat{B}(\lambda)}{\lambda}\right]$$
$$\times[1-p\hat{G}(\lambda)]-[1-pF(t_0)\,\hat{G}(\lambda)-p\bar{F}(t_0)\,\hat{B}(\lambda)]$$
$$\times\left[E(X)+p\,\frac{1-\hat{G}(\lambda)}{\lambda}\right].$$

We have

(26) $$N(0)=-\frac{p(1-p)}{\lambda}\,[\hat{B}(\lambda)-\hat{G}(\lambda)]-[1-p\hat{B}(\lambda)]\,E(X)<0,$$

(27) $$N(\infty)=0\,.$$

Differentiating $N(t_0)$ with respect to t_0, we have

(28) $$\frac{dN(t_0)}{dt_0}=\bar{F}(t_0)\,[1-p\hat{G}(\lambda)]$$
$$+pf(t_0)\,[\hat{B}(\lambda)-\hat{G}(\lambda)]\left[\frac{1}{\lambda+\mu}-E(X)\right],$$

where we assume that $F(t_0)$ has a density $f(t_0)$. Noting the failure rate $r(t_0)=f(t_0)/\bar{F}(t_0)$ and setting zero in (28), we have a $t_0{}^*$ such that

$$r(t_0{}^*)=\frac{1-p\hat{G}(\lambda)}{p\,[\hat{B}(\lambda)-\hat{G}(\lambda)]\left[E(X)-\dfrac{1}{\lambda+\mu}\right]}. \tag{29}$$

Using the assumptions that $B(t)>G(t)$, $r(t_0)$ is an increasing function of t_0, the inequality (24), and $N(0)<\infty$ and $N(\infty)=0$, we can show that $N(t_0)$ is a unimodal function and there exists a $\hat{t}_0$ such that $N(\hat{t}_0)=0$. Thus, if we choose a $t_0>\hat{t}_0$, we have $N(t_0)>0$, which completes the proof.

3. Model II

In the previous model we assumed that the preventive maintenance is made though the system is during a usage period, which causes the disappointment time. In this section, to avoid the disappointment time caused by the preventive maintenance, we consider a new model that the preventive maintenance is not made while the system is during a usage period. That is, we assume that the preventive maintenance is only made while the system is not in a usage period. The states for this new model are the same in the preceding section, and the signal flow graph for this new model is the same as Fig. 1. We shall, however, obtain each branch gain.

For states s_1 and s_2, the branch gains are the same as given by (7)–(10). Consider the branch gains from state s_0 to states s_1, s_2, and s_3. The transition from state s_0 to state s_1 is an event that (i) the system fails before the preventive maintenance and at that time the system is not in a usage period, or (ii) the preventive maintenance is not made because the system is in a usage period and that the system fails while the system is not in a usage period. Thus, we have

$$q_{01}(s)=\int_0^\infty e^{-st}P(t)\bar{A}(t)\,dF(t)+\int_0^\infty e^{-st}P(t)\left[\int_0^t \bar{P}(t)\,dA(t)\right]dF(t)\,. \tag{30}$$

If we consider a similar event just mentioned above except that the system fails while the system is in a usage period, we have the branch gain to state s_3:

$$(31) \qquad q_{03}(s)=\int_0^\infty e^{-st}\bar{P}(t)\,\bar{A}(t)\,dF(t) + \int_0^\infty e^{-st}\bar{P}(t)\left[\int_0^t \bar{P}(t)\,dA(t)\right]dF(t)\,.$$

The branch gain to state s_2 is the same one in (5). Thus we have all branch gains for this new model.

The LS transform of the distribution to the disappointment time is given in (11) and the mean time to the disappointment time is also given in (12), where branch gains $q_{0i}(s)$ $(i=1, 3)$ are given in (30) and (31), respectively, and the unconditional mean ξ_0 in state s_0 is given in (13).

In particular, if we consider the constant time preventive maintenance policy in (17), we have

$$(32) \qquad \xi_0=\int_0^{t_0}\bar{F}(t)\,dt+\bar{P}(t_0)\int_{t_0}^\infty \bar{F}(t)\,dt\,.$$

We can obtain a similar theorem that the preventive maintenance policy is effective in the sense of the mean time. We, however, omit the form of theorem because the result is complicated.

4. Conclusion

We have discussed an intermittently used system with preventive maintenance, and derived the LS transform of the distribution of the disappointment time and the mean time of the disappointment time. We have further shown the effect of the preventive maintenance policy. We believe that the preventive maintenance policy is important in actual situations.

We have described our model in a reliability context. We can, however, describe the same model in traffic, scheduling, and other contexts. So, our model is applicable in some fields.

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