

# ALTERNATIVE APPROACHES TO THE ANALYSIS OF THE $M/G/1$ AND $G/M/1$ QUEUES

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## Introduction

In the imbedded Markov chain analysis of queues we notice the following features.

- (1) We can consider imbedding the chain either before or after regeneration points.
- (2) We can use either the forward or backward Kolmogorov equations but the few authors who have used the backward equations have been dealing with the  $G/M/1$  queue [1], [2].

The natural extension of these ideas to the supplementary variable queue length analysis appears to be

- (1) Define the additional variable as the time to either the previous arrival (departure for  $M/G/1$ ) or the subsequent arrival.
- (2) Apply the forward equations when dealing with the  $M/G/1$  queue and the backward equations when considering the  $G/M/1$  queue.

Previous work has used only the forward equations and has always defined the variable as the time to the earlier regeneration point. We develop simple techniques to handle the alternative equations generated by applying the above concepts.

**Part I.  $M/G/1$**

Consider the standard analysis of the  $M/G/1$  queue by the supplementary variable method. Since the choice of the additional variable is not unique we propose to define it as the time to service completion instead of the time since the beginning of service. The effect of this is twofold in that we do not have to deal with integral boundary equations and that we never have products of functions of the supplementary variable. Accordingly, we have found this approach useful in dealing with more complicated queueing problems. For instance, all the standard priority queues have been analysed using this technique [3], two of which had previously yielded only to an imbedding approach. We therefore suggest that this method could possibly be fruitfully employed in other queues with the basic  $M/G/1$  property.

*Definitions and Basic Equations*

Let  $S(x)$  be the service time probability density distribution (independent and identically distributed random variables) and assume arrivals occur in a Poisson stream with parameter  $\lambda$ .

Define  $P_i(x, t)dx$  as the probability that at time  $t$  there are  $i$  customers in the queue and the customer in service has time between  $x$  and  $x+dx$  to service completion,

and  $P_0(t)$  as the probability that the queue is empty at time  $t$ .

We will assume that the queue is empty at  $t=0$  and define

$$P_i(x, s) = \int_0^\infty P_i(x, t) e^{-st} dt \quad \text{Re}(s) > 0$$

$$P_i(\hat{p}, s) = \int_0^\infty P_i(x, s) e^{-px} dx \quad \text{Re}(p) > 0$$

and 
$$P(z, \hat{p}, s) = \sum_{i=1}^\infty P_i(\hat{p}, s) z^i \quad |z| < 1.$$

The forward Kolmogorov Equations become

$$i > 1$$

$$(1) \quad \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) P_i(x, t) = -\lambda P_i(x, t) + \lambda P_{i-1}(x, t) + P_{i+1}(0, t) S(x)$$

$$(2) \quad \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) P_i(x, t) = -\lambda P_i(x, t) + \lambda P_0(t) S(x) + P_2(0, t) S(x)$$

$$(3) \quad \frac{\partial}{\partial t} P_0(t) = -\lambda P_0(t) + P_1(0, t).$$

*Solution*

Forming the function  $P(z, \hat{p}, \hat{s})$  on equations (1) and (2) gives

$$[s - \hat{p} + \lambda(1 - z)] P(z, \hat{p}, \hat{s}) = - \left[ 1 - \frac{U(\hat{p})}{z} \right] P(z, 0, \hat{s}) + \lambda z \hat{p}_0(\hat{s}) U(\hat{p}) - U(\hat{p}) P_1(0, \hat{s})$$

where 
$$U(\hat{p}) = \int_0^\infty e^{-\hat{p}x} S(x) dx.$$

We can substitute from equation (3) for  $P_1(0, \hat{s})$  to give

$$(4) \quad [s - \hat{p} + \lambda(1 - z)] P(z, \hat{p}, \hat{s}) = - \left[ 1 - \frac{U(\hat{p})}{z} \right] P(z, 0, \hat{s}) + U(\hat{p}) - [s + \lambda(1 - z)] U(\hat{p}) P_0(\hat{s})$$

Now let  $\hat{p} = s + \lambda(1 - z) = Q$  say and let  $T(s)$  be the zero in  $z$  with smallest absolute value of  $z = U(Q)$ . This root can be shown to exist uniquely inside the unit circle by an application of Rouché's Theorem.

We therefore have

$$(5) \quad P_0(\hat{s}) = \frac{1}{s + \lambda(1 - T)}$$

and

$$(6) \quad P(z, \hat{p}, \hat{s}) = \frac{z [U(\hat{p}) - U(Q)] [1 - Q P_0(\hat{s})]}{[Q - \hat{p}] [z - U(Q)]}.$$

We could use the result for  $P(z, 0, \hat{s})$  from (4) to integrate the  $x$  derivative from the generating function formed on (1) and (2). This would give the result for  $P(z, x, \hat{s})$ .

The busy period distribution can be found by a similar method using zero avoiding transition probabilities and the virtual waiting time result follows directly from equations (5) and (6),

i.e., if  $W(x, t)dx$  denotes the probability that a customer arriving at time  $t$  has to wait between  $x$  and  $x+dx$  before service initiation and  $W(\hat{p}, \hat{s})$  defines the associated Laplace Transforms, we note that

$$W(x, t) = \sum_{i=1}^{\infty} P_i(x, t) * [S(x)]^{*i-1} + P_0(t) \delta(x-0)$$

where  $*$  denotes convolution.

Laplace Transforms on  $x$  and  $t$  gives

$$W(\hat{p}, \hat{s}) = \frac{P[U(\hat{p}), \hat{p}, \hat{s}]}{U(\hat{p})} + P_0(\hat{s})$$

and the result follows from (6).

### Part II. *G/M/1*

We give, in this section, two simple approaches to the supplementary variable analysis of the standard *G/M/1* queue. Work on more complicated queues with a general arrival pattern has been hindered in the past partly due to the lack of a simple procedure. The approaches given here are the analogues of the standard *M/G/1* supplementary variable method and the analysis in Part I and we believe they should prove useful in handling other queues with a basic *G/M/1* structure. We have used this approach to give a result for the *G/M/1* queue with randomly arriving breakdowns [3], i.e. for a system which is difficult to handle by any other method.

The usual method of obtaining queue length results is to begin by setting up the Forward Kolmogorov Equations. Takacs [1] and Connolly [4] use the Backward Equations in an imbedded Markov chain approach

to the  $G/M/1$  queue and we use the same beginning here for the supplementary variable analysis.

The only difference between the two methods worked here is that the additional variable can be defined as either the time since the previous arrival or the time to the next arrival.

The interarrival times we assume to be independent and identically distributed random variables with probability density

$$A(x) = \lambda(x) e^{-\int_0^x \lambda(t) dt}$$

The service time has a negative exponential distribution with parameter  $\mu$ .

#### Method A

Define

$P_{ij}(x, y, t) dx dy$  as the probability that the queue goes from state  $(i, x)$  to state  $(j, y)$  in time  $t$  where  $i(j)$  is the number of customers present and the time to the last arrival lies between  $x$  and  $x+dx$  ( $y$  and  $y+dy$ ).

Since we use the Backward equations, the state  $(j, y)$  enters the equations only through boundary conditions and therefore for simplicity we suppress it in the notation

i.e.  $P_i(x, t) \equiv P_{ij}(x, y, t)$  in the equations.

The Laplace Transforms can be defined as in Part I and the Backward equations become

$$(7) \quad \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) P_i(x, t) = -[\lambda(x) + \mu \varepsilon_i] P_i(x, t) + \lambda(x) P_{i+1}(0, t) + \mu \varepsilon_i P_{i-1}(x, t)$$

where  $\varepsilon_i = 0$  if  $i = 0$   
 $= 1$  otherwise.

Define the function  $R_i(x, t) = P_i(x, t) \exp \left\{ - \int_0^x \lambda(t) dt \right\}$  and multiply throughout (7) by  $\exp \left\{ - \int_0^x \lambda(t) dt \right\}$

$$(8) \quad \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) R_i(x, t) = -\mu \varepsilon_i R_i(x, t) + A(x) R_{i+1}(0, t) + \mu \varepsilon_i R_{i-1}(x, t)$$

Solution

Laplace Transforms on  $x$  and  $t$  give

$$(9) \quad [s - \rho + \mu \varepsilon_i] R_i(\hat{p}, \hat{s}) = -R_i(0, \hat{s}) + A(\hat{p}) R_{i+1}(0, \hat{s}) + \delta_{ij} \exp \left\{ -\rho y - \int_0^y \lambda(t) dt \right\} + \mu \varepsilon_i R_{i-1}(\hat{p}, \hat{s}) .$$

Define  $R(z, \hat{p}, \hat{s}) = \sum_{i=0}^{\infty} R_i(\hat{p}, \hat{s}) z^i \quad |z| < 1$

and (9) becomes

$$(10) \quad [s - \rho + \mu(1-z)] R(z, \hat{p}, \hat{s}) = - \left[ 1 - \frac{A(\hat{p})}{z} \right] R(z, 0, \hat{s}) - \frac{A(\hat{p})}{z} R_0(0, \hat{s}) + \mu R_0(\hat{p}, \hat{s}) + z^j \exp \left\{ -\rho y - \int_0^y \lambda(t) dt \right\} .$$

Equation (9) with  $i=0$  and  $\hat{p}=s$  gives

$$(11) \quad A(s) R_0(0, \hat{s}) = R_0(0, \hat{s}) - \delta_{0j} \exp \left\{ -s y - \int_0^y \lambda(t) dt \right\}$$

and hence

$$A(s) [s - \rho] R_0(\hat{p}, \hat{s}) = [A(\hat{p}) - A(s)] R_0(0, \hat{s}) + \delta_{0j} \exp \left\{ - \int_0^y \lambda(t) dt \right\} [A(s) e^{-\rho y} - A(\hat{p}) e^{-s y}] .$$

We can now substitute into equation (10) and then let  $\hat{p}=s+\mu \cdot (1-z)=N$  say and choose  $T$  to be that zero with smallest absolute value of  $z=A(N)$ .  $T$  can be shown to be the only zero of  $z=A(N)$  inside the unit circle by an application of Rouché's Theorem.

We now have

$$(12) \quad R_0(0, \hat{s}) \frac{[1-A(s)]T}{A(s)(1-T)} = T^j \exp\left\{-[s + \mu(1-T)]y - \int_0^y \lambda(t) dt\right\} \\ - \delta_{0j} \exp\left\{-s y - \int_0^y \lambda(t) dt\right\} \{A(s) e^{-\mu(1-T)y} - T\}.$$

We note that

- (1)  $R_i(0, \hat{s}) = P_i(0, \hat{s})$
- (2)  $P_0(0, \hat{s})$  is an unrealistic starting point i.e. an empty queue just after an arrival and therefore  $R_1(0, \hat{s})$  appears more useful, and can be found from equation (11). In fact any initial start can be found by substituting back into (10).
- (3) If we integrate  $y$  out of (12) we have Takacs' [1] equation (27) p. 121.

#### Method B

Define  $P_{ij}(x, y, t) dx dy$  as in Method A except that  $x(y)$  denotes the time to the subsequent arrival.

We again suppress the state  $(j, y)$  in the notation.

The Backward differential equations are

$i > 0$

$$(13) \quad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) P_i(x, t) = -\mu P_i(x, t) + \mu P_{i-1}(x, t)$$

$$(14) \quad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) P_0(x, t) = 0$$

$$(15) \quad P_i(0, t) = \int_0^\infty A(x) P_{i+1}(x, t) dx.$$

$$\text{Define } P(z, x, \hat{s}) = \sum_{i=1}^{\infty} P_i(x, \hat{s}) z^{i-1} \quad |z| \leq 1$$

where  $P_i(x, \hat{s}) = \mathcal{L} P_i(x, t)$ .

Forming  $P(z, x, \hat{s})$  on equation (13) gives

$$(16) \quad \frac{\partial}{\partial x} P(z, x, \xi) = - [s + \mu(1-z)] P(z, x, \xi) + \mu P_0(x, \xi) + z^{j-1} \delta(x-y)(1-\delta_{j0}),$$

(14) becomes

$$(17) \quad \frac{\partial}{\partial x} P_0(x, \xi) = -s P_0(x, \xi) + \delta_{j0} \delta(x-y)$$

and (17) yields

$$(18) \quad P_0(x, \xi) = e^{-sx} \delta_{j0} \int_0^x \delta(t-y) e^{st} dt + P_0(0, \xi) e^{-sx}.$$

Substituting (18) into (16) we obtain

$$(19) \quad P(z, x, \xi) e^{[s+\mu(1-z)]x} = z^{j-1} \int_0^x \delta(t-y) e^{[s+\mu(1-z)]t} dt (1-\delta_{j0}) - \frac{P_0(0, \xi)}{1-z} [1 - e^{\mu(1-z)x}] + P(z, 0, \xi) + \mu \delta_{j0} \int_0^x e^{\mu(1-z)u} \int_0^u \delta(t-y) e^{st} dt du.$$

At this stage it is more convenient to integrate  $y$  out of equation (19). We define

$$N_{ij}(x, t) = \int_0^\infty P_{ij}(x, y, t) dy$$

with the Laplace Transform and generating function definitions on  $N$  identical to those on  $P$ .

(19) becomes

$$(20) \quad N(z, x, \xi) = z^{j-1} \left[ \frac{1 - e^{-[s+\mu(1-z)]x}}{[s + \mu(1-z)]} \right] (1-\delta_{j0}) + \frac{N_0(0, \xi)}{1-z} [e^{-sx} - e^{-[s+\mu(1-z)]x}]$$



$$+ N(z, 0, \hat{s}) e^{-[s+\mu(1-z)]x} + \frac{\mu \delta_{j_0}}{s} \left\{ \frac{1 - e^{-[s+\mu(1-z)]x}}{s + \mu(1-z)} - \frac{(e^{-sx} - e^{-[s+\mu(1-z)]x})}{\mu(1-z)} \right\}.$$

Using the generating function formed on the boundary equation (15) gives

$$(21) \quad \begin{aligned} N(z, 0, \hat{s}) [z - \lambda(s + \mu(1-z))] + N_0(0, \hat{s}) \\ = \frac{N_0(0, \hat{s})}{1-z} [\lambda(s) - \lambda(s + \mu(1-z))] \\ + z^{j-1} \frac{[1 - \lambda(s + \mu(1-z))]}{s + \mu(1-z)} (1 - \delta_{j_0}) \\ + \frac{\mu \delta_{j_0}}{s} \left[ \frac{1 - \lambda[s + \mu(1-z)]}{s + \mu(1-z)} - \frac{\lambda(s) - \lambda[s + \mu(1-z)]}{\mu(1-z)} \right] \end{aligned}$$

As in Method A let  $T$  be the smallest absolute zero of  $z = \lambda(s + \mu(1-z))$  and let  $z$  take the value  $T$  in (21)

$$(22) \quad \begin{aligned} N_0(0, \hat{s}) \frac{[1 - \lambda(s)]}{1 - T} = \frac{T^{j-1} [1 - T]}{s + \mu(1 - T)} (1 - \delta_{j_0}) + \frac{\mu \delta_{j_0}}{s} \\ \cdot \left[ \frac{1 - T}{s + \mu(1 - T)} + \frac{T - \lambda(s)}{\mu(1 - T)} \right]. \end{aligned}$$

This is equation (25) p. 120 of Takacs [1]. The difference in the result for Method A and Method B is that equation (12) has, as a starting point, an empty queue whereas (22) represents an empty queue just before an arrival and is therefore equivalent to an initial state of a single customer.

Method A is the same approach as that outlined in Part I and Method B is similar to the standard supplementary variable analysis of the  $M/G/1$  queue. We have therefore not just a single backward versus forward analogue but a dual analogue in which the forward equations

for the M/G/1 with the “backward looking” supplementary variable relates to the backward equations for the G/M/1 with the “forward looking” variable and vice versa.

An explanation of this duality can be seen by considering the forward equations with the “forward looking” variable for the M/G/1 queue. For  $j > 1$

$$(a) \quad P_j(x, t + \Delta) = P_j(x + \Delta, t) [1 - \lambda\Delta] + P_{j+1}(0, t) S(x)\Delta + P_{j-1}(x, t) \lambda\Delta.$$

If we now consider this queue reversed so that departures become arrivals and vice versa, equations (a) for  $j > 0$  are the backward equations for the G/M/1 queue.

The probability density  $P_j(x, t)$  becomes the probability that in time  $x$  no arrivals occur and that in the following time  $t$  the queue goes from state  $(j, x)$  to the “final” state where  $x$  is the time to the previous arrival. This density is  $R_j(x, t)$  as defined by equation (8), and consequently equations (1) and (8) are identical.

### References

- [ 1 ] Takacs, L., *Introduction to the Theory of Queues*, Oxford University Press, 1962.
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- [ 3 ] Henderson, W., *Some Contributions to the Theory of Queues with Priorities and Breakdowns*, Ph.D. Thesis, Sheffield, 1970.
- [ 4 ] Connolly, B.W., “A Difference Equation Technique Applied to a Simple Queue with Arbitrary Interarrival Distribution,” *Journal of the Royal Statistical Society*, Ser. B, **20** (1958), 168–175.