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PRIORITY QUEUES WITH GENERAL INPUT AND MIXTURES OF ERLANG SERVICE TIME DISTRIBUTIONS

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Abstract

Single server priority queuing systems with general independent input and mixtures of Erlang service time distributions are examined and the probability generating functions which characterize the equilibrium joint queue length distribution are determined. Some waiting time distributions are derived from the queue length results.

Introduction

Although many results have been published in the field of priority queues (in particular we note the work of Keilsen [1] and Gaver [2]) a notable common feature of all these publications is that they deal only with the case of Poisson input. The form of the equilibrium queue length distribution for priority queues with more general arrival processes is still unknown except for some results by the author [3] on the pre-emptive priority discipline in a queue with general recurrent input and negative exponential services. There are two limitations to the latter work which must be overcome if the results are to be of any practical use.

(1) The service time distribution for all customers had to be identical irrespective of priority class.

(2) Only the pre-emptive discipline was considered.

In Part I of this paper we extend the pre-emptive case so that each priority class has a different finite mixture of Erlang service time distributions with a common parameter.

That is, we let the service time take the form

$$\sum_{l=1}^{m} J_l E_l(x)$$

where E_l is the Erlang distribution of order l, m is an arbitrary integer and the J_l's are positive constants subject to the constraint $\sum_{l=1}^{m} J_l=1$.

For practical purposes a distribution of this form can be closely approximated to almost any real life distribution and consequently we hope the results of such an analysis will prove useful in practical situations.

In Part II we derive results for the postponable priority system.

The interarrival time distribution function is denoted by $G(x)$.

Part I. The Pre-emptive Model

Because an Erlang distribution can be considered as a convolution of negative exponential distributions the model considered is equivalent to bunches of customers arriving and being served negative exponentially. In the following discussion we will consider the latter discipline and confine our attention to two priority classes since the results so obtained can easily be extended to any finite number of classes (cf. [3]).

Definitions and Basic Equations

We denote by

Q_t the probability that an arriving customer is a class t customer $t=1,\ 2$

J_r the probability that the priority batch size is r, $r=1, 2, \cdots, k$

I_r the probability that the nonpriority batch size is $r, r=1, 2, \cdots, m$

P_{ij} the probability that just before an arrival instant there are i priority and j nonpriority customers in the queue

P_i the probability that just before an arrival instant there are a total of i customers in the queue

$$R_i(z) = \sum_{j=0}^{\infty} P_{ij} z^j .$$

We use the imbedded Markov chain technique to compare the state of the queue at subsequent arrival points and bearing in mind the following two points we are able to write down the steady state equations.

(1) Since we are considering the left hand side of our equations to always have priority customers present (the equations resulting from P_{0j} being redundant as in [3]) only priority customers will receive service during an inter-arrival period. The number of possible arrivals in a non-priority batch can only serve to fill the queue up to size j and can therefore never exceed either j or m.

(2) In equation (1.2) (following) the combined total of priority customers present and priority arrivals must be at least i since only departures can occur in the interarrival interval. Consequently the number of customers present must be the maximum of i-r and 0.

Since the state of the queue at successive arrival points form an aperiodic irreducible Markov chain the steady state equations can be written down as follows.

$i \geq k$

$$P_{ij} = Q_1 \sum_{r=1}^{k} J_r \sum_{q=0}^{\infty} P_{q+i-r,j} \int_0^{\infty} \frac{(\mu x)^q}{q!} e^{-\mu x}\, dG(x) \tag{1.1}$$

$$+ Q_2 \sum_{r=1}^{\min(m,j)} I_r \sum_{q=0}^{\infty} P_{q+i,j-r} \int_0^{\infty} \frac{(\mu x)^q}{q!} e^{-\mu x}\, dG(x)$$

$1 \leq i \leq k-1$

$$P_{ij} = Q_1 \sum_{r=1}^{k} J_r \sum_{q=\max(i-r,0)}^{\infty} P_{q,j} \int_0^{\infty} \frac{(\mu x)^{q+r-i}}{(q+r-i)!} e^{-\mu x}\, dG(x) \tag{1.2}$$

$$+ Q_2 \sum_{r=1}^{\min(m,j)} I_r \sum_{q=0}^{\infty} P_{q+i,j-r} \int_0^{\infty} \frac{(\mu x)^q}{q!} e^{-\mu x}\, dG(x)$$

Solution

We form the generating function $R_i(z)$ on equation (1.1) giving for $i \geq k$

$$R_i(z) = Q_1 \sum_{r=1}^{k} J_r \sum_{q=0}^{\infty} R_{q+i-r}(z) \int_0^\infty \frac{(\mu x)^q}{q!} e^{-\mu x} dG(x)$$

$$+ Q_2 \sum_{r=1}^{m} I_r z^r \sum_{q=0}^{\infty} R_{q+i}(z) \int_0^\infty \frac{(\mu x)^q}{q!} e^{-\mu x} dG(x)$$

which has solution

$$(1.3) \qquad R_i(z) = \sum_{r=1}^{k} A_r \delta_r^{\,i}$$

where $A_r = A_r(z)$ and $\delta_r = \delta_r(z)$ is one of the k zeros with smallest absolute value of the equation

$$\frac{V^k}{Q_1 \sum_{r=1}^{k} J_r V^{k-r} + Q_2 V^k \sum_{r=1}^{m} I_r z^r} = \phi(\mu(1-V)) \, ,$$

where $\phi(s) = \int_0^\infty e^{-sx} dG(x)$.

These zeros can be shown to be those which lie in the interior of the unit circle by an application of Rouche's Theorem.

The A_r's can be evaluated by substituting this solution into equation (1.2) after generating functions have been formed for $1 \leq i \leq k-1$

$$R_i(z) = Q_1 \sum_{r=1}^{k} J_r \sum_{q=\max(i-r,0)}^{\infty} R_q(z) \int_0^\infty \frac{(\mu x)^{q+r-i}}{(q+r-i)!} e^{-\mu x} dG(x)$$

$$+ Q_2 \sum_{r=1}^{m} I_r z^r \sum_{q=0}^{\infty} R_{q+i}(z) \int_0^\infty \frac{(\mu x)^q}{q!} e^{-\mu x} dG(x)$$

For solution (1.3) to conform we must have

$$Q_1 \sum_{r=i+1}^{k} J_r \sum_{b=1}^{k} A_b \delta_b^{\,i-r} \sum_{t=0}^{r-i-1} \int_0^\infty \frac{(\mu x \delta_b)^t}{t!} e^{-\mu x} dG(x) = 0$$

i.e.

$$\sum_{b=1}^{k} A_b \sum_{r=i+1}^{k} J_r \sum_{t=0}^{r-i-1} \delta_b^{\,i-r+t} \zeta_t = 0 \tag{1.4}$$

where
$$\zeta_t = \int_0^\infty \frac{(\mu x)^t}{t!} e^{-\mu x}\, dG(x)\,.$$

Let $B_b = \dfrac{A_b}{\sum_{b=1}^{k} A_b}$ and note from (1.3) that $\sum_{b=1}^{k} A_b = R_0(z)$.

Then equations (1.4) can be written in matrix form in terms of B_b as

$$\begin{pmatrix} J_2 & J_3 & \cdots\cdots & J_k & 0 \\ J_3 & J_4 & \cdots & J_k\ \ 0 & 0 \\ \vdots & & & & \\ J_k & 0 & & \cdots\ 0 & 0 \\ 0 & 0 & \cdots\cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \cdots\cdots & 0 & \zeta_0 & 0 \\ 0 & \cdots\cdots & \zeta_0 & \zeta_1 & 0 \\ \vdots & & & & \\ \zeta_0 & \zeta_1 \cdots & & \zeta_{k-2} & 0 \\ 0 & 0 & & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega_1^{k-1} & \omega_2^{k-1} & \cdots & \omega_k^{k-1} \\ \omega_1^{k-2} & \omega_2^{k-2} & \cdots & \omega_k^{k-2} \\ \vdots & \vdots & & \vdots \\ \omega_1 & \omega_2 & & \omega_k \\ 1 & 1 & & 1 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ \\ B_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

where $\omega_b = 1/\delta_b$.

By following the argument of Wishart [4] noting that the first two matrices are of the form

$$C \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad \text{and hence} \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = C^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

we conclude that [cf. [4]] the solution is

$$A_b = R_0(z) \prod_{j \neq b}^{k} \left(\frac{\delta_b}{\delta_b - \delta_j} \right)$$

i.e.

$$(1.5) \qquad R_i(z) = R_0(z) \sum_{b=1}^{k} \delta_b{}^i \prod_{\substack{j=1 \\ j \neq b}}^{k} \left(\frac{\delta_b}{\delta_b - \delta_j} \right)$$

To evaluate $R_0(z)$ we first solve the boundary equations for P_i in the same fashion resulting in

$$P_i = \sum_{b=1}^{\max(k,m)} C_b \, \varepsilon_b{}^i$$

where $$C_b = P_0 \prod_{\substack{j=1 \\ j \neq b}}^{\max(k,m)} \left(\frac{\varepsilon_b}{\varepsilon_b - \varepsilon_j} \right)$$

and the ε_b's are the max (k, m) zeros with smallest absolute value of the equation

$$\frac{V^{\max(k,m)}}{\sum_{r=1}^{\max(k,m)} (Q_1 J_r + Q_2 I_r) \, V^{\max(k,m)-r}} = \phi(\mu(1-V)) \, .$$

By using the normalising condition that $\sum_{i=0}^{\infty} P_i = 1$, we find P_0 as

$$(1.6) \qquad P_0 = \left[\sum_{b=1}^{\max(k,m)} \frac{1}{1-\varepsilon_b} \prod_{\substack{j=1 \\ j \neq b}}^{\max(k,m)} \left(\frac{\varepsilon_b}{\varepsilon_b - \varepsilon_j} \right) \right]^{-1}$$

and

$$(1.7) \qquad P_i = P_0 \sum_{b=1}^{\max(k,m)} \varepsilon_b{}^i \prod_{\substack{j=1 \\ j \neq b}}^{\max(k,m)} \left(\frac{\varepsilon_b}{\varepsilon_b - \varepsilon_j} \right) .$$

We use (1.7) to find $R_0(z)$ by using the relationship

$$\sum_{i=0}^{\infty} P_i \, z^i = \sum_{i=0}^{\infty} R_i(z) \, z^i$$

i.e.

$$R_0(z)\sum_{b=1}^{k}\frac{1}{(1-\delta_b z)}\prod_{\substack{j=1\\j\neq b}}^{k}\left(\frac{\delta_b}{\delta_b-\delta_j}\right) \tag{1.8}$$

$$=P_0\sum_{b=1}^{\max(k,m)}\frac{1}{1-\varepsilon_b z}\prod_{\substack{j=1\\j\neq b}}^{\max(k,m)}\left(\frac{\varepsilon_b}{\varepsilon_b-\varepsilon_j}\right).$$

Thus substituting (1.8) and (1.6) into (1.5) gives the required result for the partial generating function.

To revert back from this generating function to the probability generating function for the original "singly arriving customers with mixtures of Erlang service times" (labelling these customers as "old customers" and the appropriate generating function by $M(z_1, z_2)$) we note that

$$P_{l,n}=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}Pr\{a \text{ priority old customers, } b \text{ non-priority old customers}\}\ J_l^{a*}\,I_n^{b*},$$

where * denotes convolution.

Using this and forming generating functions we see immediately that

$$R(z_1, z_2)=\sum_{l=1}^{\infty}\sum_{n=0}^{\infty}P_{l,n}\,z_1^{\,l}\,z_2^{\,n}=M[J(z_1), I(z_2)]$$

where $J(z_1)=\sum_{l=1}^{\infty}J_l\,z_1^{\,l}$

$$I(z_2)=\sum_{l=1}^{\infty}I_l\,z_2^{\,l}\,.$$

Waiting Time Distribution

Define

$B_k(t)$ as the busy period distribution initiated by a single arrival and involving customers with priority labelling $1, 2, \cdots, k$

$C_k(t)$ as the total time a customer spends either in service or in a pre-empted state

$W_k(t)$ as the waiting time of a k customer.

Then

$$W_k(t) = \sum_{l=0}^{\infty}\sum_{r=0}^{\infty} Pr\begin{bmatrix} r(1,2,\cdots,k-1)\text{-customers in the queue;} \\ l\ k\text{-customers in the queue at the} \\ \text{time of arrival of the } k\text{-customer} \end{bmatrix}$$

$$\times B_{k-1}{}^{r*}(t) * C_k{}^{l}(t)\,.$$

We note that since all classes have the same service distribution

$$B_{k-1}(t) \equiv C_k(t)$$

$$\therefore\quad W_k(t) = \sum_{r=0}^{\infty} Pr\begin{bmatrix} r(1,2,\cdots,k)\text{-customers on queue} \\ \text{at the time of arrival of the} \\ k\text{-customer} \end{bmatrix} B_{k-1}{}^{r*}(t).$$

Take Laplace transforms with

$$W_k(\hat{s}) = \int_0^{\infty} e^{-st}\, W_k(t)\, dt \quad \text{and} \quad B_{k-1}(\hat{s}) = \int_0^{\infty} e^{-st}\, B_{k-1}(t)\, dt\,,$$

then

$$W_k(\hat{s}) = R^{(\kappa)}\,[B_{k-1}(\hat{s})]$$

where $$R^{(k)}(z) = \sum_{j=0}^{\infty} Pr(j \text{ customers in first } k \text{ priority classes})\ z^j.$$

The case of singly arriving customers and a finite number of priority classes

$$B_{k-1}(\hat{s}) = 1 - \frac{\dfrac{s}{\mu+s}}{1-\dfrac{\mu}{\mu+s}\,\mathcal{E}_{k-1}(s)} \quad \text{[Takacs[5], p. 137 Equ. 37]}$$

where $\mathcal{E}_{k-1}(s)$ is the root in z which lies inside the unit circle of the equation

$$z = \mathscr{L}_{k-1}\,[s+\mu(1-z)]$$

$\mathscr{L}_{k-1}(s)$ is the Laplace transform of the interarrival time distribution of $(1, 2, \cdots, k-1)$-customers.

We can easily show that

$$\mathscr{L}_{k-1}(s)=\frac{\sum_{t=1}^{k-1} Q_t\,\varphi(s)}{1-\sum_{t=k}^{r} Q_t\,\varphi(s)}$$

$R^{(k)}(z)$ is derived from [1] by letting $z_1=z_2=\cdots=z_k=z$ and $z_{k+1}=z_{k+2}=\cdots=z_l=1$ in the result

$$R(z_1, z_2, \cdots, z_l)=\prod_{r=1}^{l}\frac{1-\delta_r\, z_{r+1}}{1-\delta_r\, z_r} \quad \text{with} \quad z_{l+1}\equiv 1\,.$$

Part II. The Postponable Model

In this section we derive analogous results for the single server postponable model with two priority classes. We will assume that the service time distribution is negative exponential with mean $1/\mu$ for both classes. The natural extension to mixtures of Erlang service time distributions for priority customers serves only to further complicate the form of the basic equations and since the approach used in Part I combined with the following analysis is sufficient to provide a result for such a model we see no advantage in reiterating the previous analysis.

Definitions and Basic Equations

Let

$P_{kij}{}^{(n)}$=The probability that just before the nth arrival there are i priority and j non priority customers in queue and a kth priority customer is in service for $k=1, 2,$ and $i+j>0$

$P_l{}^{(n)}$=The probability that l customers are found in the queue by the nth arriving customer.

For convenience of notation we write

$P_{100}^{(n)}=P_0^{(n)}=$The probability that the queue is empty when the nth arrival occurs.

$Q_i=$The probability that an arrival is a class i customer.

Since the state of the queue at the arrival points forms a Markov chain we can set up the following basic equations.

To have a non priority customer is service, and yet have priority customers present just before the $(n+1)$th arrival point means that no service could have been completed during the previous interarrival period—any service termination would mean that a priority customer would enter the service bay i.e. for $i>0,\ j>0$

$$P_{2ij}^{(n+1)}=[Q_1 P_{2,i-1,j}^{(n)}+Q_2 P_{2,i,j-1}^{(n)}]\int_0^\infty e^{-\mu x}\,dG(x) \tag{2.1}$$

and for $i>0,\ j=0$

$$P_{2i0}^{(n+1)}=0\ . \tag{2.2}$$

On the other hand the situation of a priority customer's being in service at the $(n+1)$th arrival point could have developed in a number of mutually exclusive ways:

(1) A priority customer was in service at the nth regeneration point; either a priority or a non-priority arrival occurred and the number of services completed during the interarrival period were sufficient to bring the priority queue size down to i. In such a situation the non-priority queue can never have any departures during the considered period since a priority customer must always occupy the service bay.

(2) A non-priority customer was in service, an arrival occurred, and at least one departure takes place so as to oust the ordinary customer from the service facility.

The Kolmogorov forward equations for such a situation are therefore:

For $i>0,\ j\geq 0$

$$P_{1,i,j}^{(n+1)}=Q_1\sum_{l=0}^{\infty}P_{1,i+l-1,j}^{(n)}\int_0^\infty \frac{e^{-\mu x}(\mu x)^l}{l!}\,dG(x) \tag{2.3}$$

$$+ Q_2 \sum_{l=0}^{\infty} P_{1,i+l,j-1}{}^{(n)} \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^l}{l!} dG(x)\, \varepsilon_{j0}$$

$$+ Q_1 \sum_{l=0}^{\infty} P_{2,i+l-1,j+1}{}^{(n)} \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^{l+1}}{(l+1)!} dG(x)$$

$$+ Q_2 \sum_{l=0}^{\infty} P_{2,i+l,j}{}^{(n)} \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^{l+1}}{(l+1)!} dG(x)$$

with boundary equations

(2.4) $$P_{10j}{}^{(n)} = 0 \quad \forall j > 0$$

where $$\varepsilon_{j0} = \begin{cases} 0 & \text{for } j = 0 \\ 1 & \text{for } j > 0 \,. \end{cases}$$

The final equations (equivalent, as in the pre-emptive model, to the case of an empty priority queue) can be found by considering the total queue length, imbedded chain probabilities.

For $i>0$

(2.5) $$P_i{}^{(n+1)} = \sum_{l=0}^{\infty} P_{i+l-1}{}^{(n)} \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^l}{l!} dG(x)$$

(2.6) $$P_0{}^{(n+1)} = \sum_{l=0}^{\infty} \sum_{j=l+1}^{\infty} P_l{}^{(n)} \int_0^{\infty} \frac{(\mu x)^j}{j!} e^{-\mu x} dG(x) \,.$$

We denote the steady state probabilities by P_{kij} for $k=1,2$ $i, j=0, 1, 2, \cdots$, and P_l for $l=0, 1, 2, \cdots$, as obvious extensions of the previous definitions. Then the field equations giving the relations between the P_{kij} and P_l are obtained simply by omitting the subscripts (n), $(n+1)$ in (2.1)–(2.6). We shall label these corresponding equations (2.1′)–(2.6′) and notice that they form a complete set of equations since for $j>0$

$$P_{20j} = P_j - \sum_{u=1}^{j-1} P_{2,j-u\ u} - \sum_{u=0}^{j} P_{1,j-u,u} \,.$$

Define $$R_{ki}(z) = \sum_{j=0}^{\infty} P_{kij} z^j \qquad k=1,2$$

and $$\Phi(s) = \int_0^{\infty} e^{-sx}\, dG(x)\ .$$

Solution

Formation of the generating function $R_{2i}(z)$ over equations (2.1′) and (2.2′) gives

$$R_{2i}(z) = Q_1\, \Phi(\mu)\, R_{2i-1}(z) + Q_2\, z\, \Phi(\mu)\, R_{2i}(z)$$

so that

$$R_{2i}(z) = \left[\frac{Q_1\, \Phi(\mu)}{1 - Q_2\, z\, \Phi(\mu)}\right]^i R_{20}(z) = A^i R_{20}(z) \text{ say} \tag{2.7}$$

where $$A = A(z)\ .$$

Utilizing equation (2.1′) we can rewrite equation (2.3′) as

$$\begin{aligned} P_{1,i,j} &= Q_1 \sum_{l=0}^{\infty} P_{1,i+l-1,j} \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^l}{l!}\, dG(x) \\ &+ Q_2 \sum_{l=0}^{\infty} P_{1,i+l,j-1} \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^l}{l!}\, dG(x) \\ &+ \frac{1}{\Phi(\mu)} \sum_{l=0}^{\infty} P_{2i+l,j+1} \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^{l+1}}{(l+1)!}\, dG(x) \end{aligned}$$

from which formation of the generating functions $R_{1i}(z)$ and substitution for $R_{2i}(z)$ from equation (2.7) yields for $i>0$

$$\begin{aligned} R_{1i}(z) &= Q_1 \sum_{l=0}^{\infty} R_{1i+l-1}(z) \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^l}{l!}\, dG(x) \\ &+ Q_2\, z \sum_{l=0}^{\infty} R_{1i+l}(z) \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^l}{l!}\, dG(x) \\ &+ \frac{R_{20}(z)\, A^{i-1}}{\Phi(\mu)\, z} \left[\Phi(\mu(1-A)) - \Phi(\mu)\right] \end{aligned} \tag{2.8}$$

Equation (2.8) has a solution

(2.9) $$R_{1i}(z) = M_i(z) + C_i(z)$$

where $C_i(z)$ is a particular solution of the form $DA^i[D=D(z)]$ and serves to eliminate the final term from consideration.

By substitution it is apparent that D must be

$$\frac{R_{20}(z)\,[\Phi(\mu(1-A))-\Phi(\mu)]}{z\,\Phi(\mu)\,[A-(Q_1+Q_2zA)\,\Phi(\mu(1-A))]}\,.$$

This leaves the relation

(2.10) $$M_i(z) = Q_1 \sum_{l=0}^{\infty} M_{i+l-1}(z) \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^l}{l!}\, dG(x) + Q_2\, z \sum_{l=0}^{\infty} M_{i+l}(z) \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^l}{l!}\, dG(x)$$

from which we derive $M_i(z)=B\varepsilon^i$ with $B=B(z)$ and $\varepsilon=\varepsilon(z)$ the root with smallest absolute value of the equation

$$\frac{v}{Q_1+Q_2zv} = \Phi(\mu(1-v))\,.$$

This root can be shown to lie inside the unit circle by an application of Rouche's theorem. The solution can be shown to be unique by forming factorial generating functions over i in equation (2.10) and then paralleling the argument of Neuts [6].

Our solution is completed by finding B and D (or $R_{20}(z)$) from the boundary equations.

(2.5′) and (2.6′) are the standard equations for the $GI/M/1$ queue and have the well-known solution

(2.11) $$P_i = (1-\delta)\,\delta^i$$

where δ is the root in v with smallest absolute value of the equation $v=\Phi(\mu(1-v))$.

Equation (2.4′) can be expressed as

$R_{10}(z)=P_{100}=1-\delta$ (from (2.11)) and hence by substituting in (2.9)

$$B+D=1-\delta$$

giving

(2.12) $$R_{1i}(z)=(1-\delta)\,\varepsilon^i+D\,[A^i-\varepsilon^i].$$

To evaluate $R_{20}(z)$ and therefore D we have from (2.11)

$$R(z)=\sum_{i=0}^{\infty}P_i z^i=\frac{1-\delta}{1-\delta z}\,.$$

$R(z)$ can also be formed from the joint distributions as

$$R(z)=\sum_{i=0}^{\infty}[R_{1i}(z)+R_{2i}(z)]\,z^i$$

$$=\frac{1-\delta}{1-z\varepsilon}+\frac{D+R_{20}(z)}{1-zA}-\frac{D}{1-z\varepsilon}$$

i.e.

(2.13) $$R_{20}(z)=\frac{z(1-\delta)(\delta-\varepsilon)(1-zA)}{(1-z\delta)\left\{1-z\varepsilon+\dfrac{(A-\varepsilon)\,[\Phi(\mu(1-A))-\Phi(\mu)]}{\Phi(\mu)\,[A-(Q_1+Q_2zA)\,\Phi(\mu(1-A))]}\right\}}$$

(2.14) $$D=\frac{(1-\delta)(\delta-\varepsilon)(1-Az)}{(1-z\delta)\left\{\dfrac{(1-z\varepsilon)\,\Phi(\mu)\,[A-(Q_1+Q_2zA)\,\Phi(\mu(1-A))]}{[\Phi(\mu(1-A))-\Phi(\mu)]}+(A-\varepsilon)\right\}}.$$

The solution of the system is given by equations (2.7), (2.12) and either (2.13) or (2.14).

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