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DUALITY BETWEEN OBJECTS AND CONSTRAINTS IN VECTOR OPTIMUM PROBLEMS

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Abstract

The vector maximum problem studied in [1], [2] and [3] has several objective functions, $f_1(x), f_2(x), \ldots, f_m(x)$, to be maximized and constraint set $\{x \in X;\ g_j(x) \leqq z_j\ (j=1, 2, \ldots, n)\}$ where $f_i(x)$ $(i=1, 2, \ldots, m)$ and $g_j(x)$ $(j=1, 2, \ldots, n)$ are real valued functions defined on $X \subset R^l$ and $z_j(j=1, 2, \ldots, n)$ are real numbers. The relationship between the above vector maximum problem and the vector minimum problem which has the objective functions, $g_1(x), g_2(x), \ldots, g_n(x)$, to be minimized and the constraint set $\{x \in X;\ f_i(x) \geqq y_i\ (i=1, 2, \ldots, m)\}$ where y_i $(i=1, 2, \ldots, m)$ are real numbers has been investigated.

1. Introduction

Suppose that we have n different kinds of resources from which we make l different kinds of products. To make x_i units of the i-th product $(i=1, 2, \ldots, l)$ z_j units of the j-th resource $(j=1, 2, \ldots, n)$ are required and the following relations are satisfied.

$$z_j = g_j(x_1, x_2, \cdots, x_l) \qquad (j=1, 2, \cdots, n),$$

or equivalently

$$z = g(x)$$

where $x=(x_1, x_2, \ldots, x_l)$, $z=(z_1, z_2, \ldots, z_n)$ and $g(x)=(g_1(x), g_2(x), \ldots, g_n(x))$. At the same time we obtain $y=f(x)$ dollars as profit where $f(x)$ is a real valued function defined on R^l. Here we consider the following mathematical programming problem:

Problem 1–1. Let $X^1=\{x\in R^l;\ x\geqq 0,\ g(x)\leqq z_*\}$[1]. Find an $x^*\in X^1$, if it exists, such that $f(x^*)\geqq f(x)$ for all $x\in X^1$.

Let x^* be a solution of Problem 1–1 and define

$$X^2=\{x\in R^l;\ x\geqq 0, f(x)\geqq y^*\}$$

where y^* is a real number such that $y^*=f(x^*)$. If there exists no $x\in X^2$ such that $g(x^*)\geq g(x)$[2], then the solution x^* of Problem 1–1 also gives the efficient use of the resources. But if there exists an $\tilde{x}\in X^2$ such that $g(\tilde{x})\leq g(x^*)$, since x^* is a solution of Problem 1–1, we have $f(\tilde{x})=f(x^*)$, which implies that $\tilde{x}$ is also a solution of Problem 1–1. The difference between $\tilde{x}$ and x^* is shown by $g(\tilde{x})\leq g(x^*)$. Therefore it is significant to consider the following vector minimum problem:

Problem 1–2. Find an $\tilde{x}\in X^2$, if it exists, such that there exists no $x\in X^2$ satisfying the relation $g(x)\leq g(\tilde{x})$.

The solution of this problem gives the most efficient use of resources under the condition that we obtain more than or equal to y^* dollars as profit. In the subsequent sections, we shall investigate the relationship between Problem 1–1 and Problem 1–2 in more general form. We call this relationship "duality between objects and constraints".

2. Vector Optimum Problems

Problem 1–1 is a usual mathematical programming problem which has one real objective function, but Problem 1–2 is a vector minimum problem which has several real objective functions. Hence they have no symmetric relation. In order to deal with them symmetrically,

1) For any two vectors $a=(a_1,a_2,\ldots,a_k)$, $b=(b_1,b_2,\ldots,b_k)$ in R^k, $a\leqq b$ (or $b\geqq a$) implies $a_i\leqq b_i$ $(i=1,2,\ldots,k)$.

2) For any two vectors $a=(a_1,a_2,\ldots,a_k)$, $b=(b_1,b_2,\ldots,b_k)$ in R^k, $a\leq b$ (or $b\geq a$) implies $a\leqq b$ and $a\neq b$.

we shall introduce a vector maximum problem instead of Problem 1–1 and set the vector minimum problem corresponding to it. We shall first define three kinds of concepts of optimality under the partial ordering $\geqq$ in R^m, and then consider vector optimum problems precisely by using them. In the following three definitions, S is a given nonempty subset of R^m.

Definition 2–1. $y^* \in R^m$ is said to be a *strong upper bound* of S if

$$y^* \geqq s \qquad \text{for all } s \in S.$$

$y_* \in R^m$ is said to be a *strong lower bound* of S if

$$y_* \leqq s \qquad \text{for all } s \in S.$$

The set of all strong upper bounds of S and the set of all strong lower bounds of S are denoted by SU[S] and SL[S] respectively. Especially, if y^* and y_* satisfy

$$y^* \in \mathrm{SU}[S] \cap S$$

and

$$y_* \in \mathrm{SL}[S] \cap S$$

respectively they are said to be a *strong maximal point* of S and a *strong minimal point* of S respectively.

Definition 2–2. $y^* \in R^m$ is said to be a *Pareto upper bound* of S if there exists no $s \in S$ such that $s \geq y^*$. $y_* \in R^m$ is said to be a *Pareto lower bound* of S if there exists no $s \in S$ such that $s \leq y_*$. The set of all Pareto upper bounds of S and the set of all Pareto lower bounds of S are denoted by PU[S] and PL[S] respectively. Especially, if y^* and y_* satisfy

$$y^* \in \mathrm{PU}[S] \cap S$$

and

$$y_* \in \mathrm{PL}[S] \cap S$$

respectively they are said to be a *Pareto maximal point* of S and a *Pareto minimal point* of S respectively.

Definition 2-3. $y^* \in R^m$ is said to be a *weak (Pareto) upper bound* of S if there exists no $s \in S$ such that $s > y^*$[3]. $y_* \in R^m$ is said to be a *weak (Pareto) lower bound* of S if there exists no $s \in S$ such that $s < y_*$. The set of all weak upper bounds of S and the set of all weak lower bounds of S are denoted by WU[S] and WL[S] respectively. Especially, if y^* and y_* satisfy

$$y^* \in \mathrm{WU}[S] \cap S$$

and

$$y_* \in \mathrm{WL}[S] \cap S$$

respectively they are said to be a *weak (Pareto) maximal point* of S and a *weak (Pareto) minimal point* of S respectively.

For convenience, we assume the following equalities:

$$\mathrm{SU}[\emptyset] = \mathrm{SL}[\emptyset] = \mathrm{PU}[\emptyset] = \mathrm{PL}[\emptyset] = \mathrm{WU}[\emptyset] = \mathrm{WL}[\emptyset] = R^m .$$

From the above definitions we have the following properties:

Property 2-1. For every $S \subset R^m$

(i) $\mathrm{SU}[S] \subset \mathrm{PU}[S] \subset \mathrm{WU}[S]$,

(ii) $y^* \in \mathrm{WU}[S], \quad p > 0 \rightarrow (y^* + p) \in \mathrm{PU}[S]$,

(iii) SU[S] and WU[S] are closed,

(iv) $\mathrm{WU}[S] = \mathrm{CL}[\mathrm{PU}[S]]$[4].

Let S be a nonempty closed subset of R^m and have a finite strong upper bound. Then it is easily verified that S has at least one Pareto maximal point, which is also a weak maximal point (from (i) of Property 2-1). But S does not necessarily have a strong maximal point. Therefore we restrict our attention to Pareto upper bounds and weak upper bounds. (ii) and (iv) above show the topological relation between them. Practically, Pareto upper bounds are more useful than weak upper

3) For any two vectors $a = (a_1, a_2, \ldots, a_k)$, $b = (b_1, b_2, \ldots, b_k)$ in R^k, $a < b$ (or $b > a$) implies $a_i < b_i$ $(i = 1, 2, \ldots, k)$.

4) For any $A \subset R^k$, CL[A] represents the closure of A.

bounds. Mathematically, however, it may be more easy to deal with weak upper bounds than Pareto upper bounds, because WU[S] is closed for every $S \subset R^m$ but PU[S] is not necessarily closed. We can also obtain the similar properties of the three kinds of lower bounds.

Now we state a vector maximum problem and a vector minimum problem corresponding to it in terms of the above concepts.

Problem 2-p. Let z^* be a vector in R^n, X be a nonempty subset of R^l, and $f_i(x)$ $(i=1, 2, \ldots, m)$ and $g_j(x)$ $(j=1, 2, \ldots, n)$ be real valued continuous functions. Let

$$f(x) = (f_1(x), f_2(x), \ldots, f_m(x)),$$

$$g(x) = (g_1(x), g_2(x), \ldots, g_n(x))$$

and

$$X[g \leqq z^*] = \{x \in X; \ g(x) \leqq z^*\}.$$

Find an $x^* \in X[g \leqq z^*]$, if it exists, such that $f(x^*) \in \mathrm{WU}[f(X[g \leqq z^*])]$.

Problem 2-d. Let y^* be a vector in R^m, and X, $f(x)$, $g(x)$ be the same as those of Problem 2-p. Let

$$X[f \geqq y^*] = \{x \in X; \ f(x) \geqq y^*\}.$$

Find an $x^* \in X[f \geqq y^*]$, if it exists, such that $g(x^*) \in \mathrm{WL}[g(X[f \geqq y^*])]$.

We make a remark on the difference between the constructions of the above two problems. Problem 2-p has $f(x)$ as its objective function and $g(x)$ in its constraint set $X[g \leqq z^*]$, on the other hand Problem 2-d has $g(x)$ as its objective function and $f(x)$ in its constraint set $X[f \geqq y^*]$. Both of these problems have the constraint set in X. As it is arbitary to consider which problem is primal, we regard Problem 2-p as a primal problem and Problem 2-d as its dual one.

3. Fundamental Theorems

In this section, we shall investigate the relationship between Problem 2-p and Problem 2-d. First, we assume that there exists an $\bar{x}$ such that the set $X[g \leqq z^*] \cap X[f \geqq f(\bar{x})]$ is nonempty, closed and bounded. Then, there exists an $\hat{x} \in X[g \leqq z^*] \cap X[f \geqq f(\bar{x})]$ such that

$$f(\tilde{x}) \in \mathrm{PU}[f(X[g \leqq z^*] \cap X[f \geqq f(\bar{x})])] .$$

Hence $g(\tilde{x}) \leqq z^*$, $f(\tilde{x}) \geqq f(\bar{x})$ and $f(\tilde{x}) \in \mathrm{PU}[f(X[g \leqq z^*])]$. Furthermore, since the set $X[g \leqq z^*] \cap X[f \geqq f(\tilde{x})]$ is also nonempty, closed and bounded, there exists an $x^* \in X[g \leqq z^*] \cap X[f \geqq f(\tilde{x})]$ such that

$$g(x^*) \in \mathrm{PL}[g(X[g \leqq z^*] \cap X[f \geqq f(\tilde{x})])] .$$

Hence $g(x^*) \leqq z^*$, $f(x^*) \geqq f(\tilde{x})$ and $g(x^*) \in \mathrm{PL}[g(X[f \geqq f(\tilde{x})])]$. But from $f(\tilde{x}) \in \mathrm{PU}[f(X[g \leqq z^*])]$ it follows that $f(\tilde{x}) \geqq f(x^*)$. Consequently, we have

(1) $$x^* \in X[g \leqq z^*] ,$$

(2) $$f(x^*) \in \mathrm{PU}[f(X[g \leqq z^*])]$$

and

(3) $$g(x^*) \in \mathrm{PL}[g(X[f \geqq f(x^*)])] .$$

Thus we have proved the following theorem:

Theorem 3-1. If there exists an $\bar{x} \in R^l$ such that the set

$$X[g \leqq z^*] \cap X[f \geqq f(\bar{x})]$$

is nonempty, closed and bounded then there exists an $x^* \in R^l$ satisfying (1), (2) and (3).

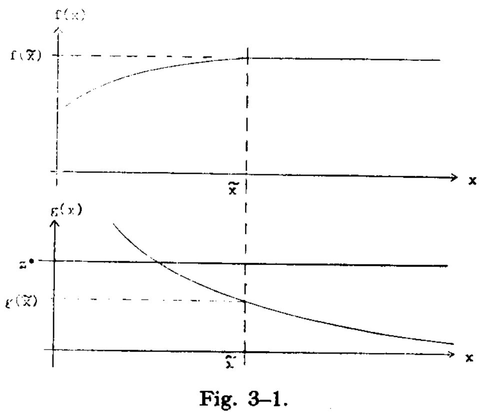

Fig. 3-1.

Corollary 3–1. Let $\tilde{x}\in R^l$ be a solution of Problem 2-p. If the set

$$X[g\leqq z^*]\cap X[f\geqq f(\tilde{x})]$$

is closed and bounded then there exists an x^* satisfying (1), (2) and (3). Especially, if

$$X[g\leqq z^*]\cap X[f\geqq f(\tilde{x})]=\{\tilde{x}\}\ ,$$

then $x^*=\tilde{x}$ satisfies (1), (2) and (3).

It should be noted that if the assumption of Corollary 3–1 does not hold then there does not necessarily exist an $x^*\in R^l$ satisfying (1), (2) and (3). An example is given by Fig. 3–1. In this example, we take $l=m=n=1$ and $X=\{x\in R;\ x\geqq 0\}$ in Problem 2-p and Problem 2-d. $\tilde{x}\in R$ is a solution of Problem 2-p, but there exists no x^* satisfying (1), (2) and (3), because there exists no $\tilde{x}$ such that the set

$$X[g\leqq z^*]\cap X[f\geqq f(\tilde{x})]$$

is nonempty and bounded.

We shall next derive other relations between Problem 2-p and Problem 2-d under more general assumptions. Let $x^*\in R^l$ be a solution of Problem 2-p. Then there exists no $x\in X[f>f(x^*)]$ such that $g(x)\leqq g(x^*)$ where $X[f>f(x^*)]=\{x\in X; f(x)>f(x^*)\}$. It follows from the continuity of $f(x)$ and $g(x)$ that there exists no $x\in \mathrm{CL}[X[f>f(x^*)]]$ such that $g(x)<g(x^*)$, which implies that

$$g(x^*)\in \mathrm{WL}[g(\mathrm{CL}[X[f>f(x^*)]])]\ .$$

Here we introduce the following assumption:

Assumption 3-d.

$$X[f\geqq f(x^*)]=\mathrm{CL}[X[f>f(x^*)]]\ .$$

If the set $X[f\geqq f(x^*)]$ satisfies Assumption 3-d above then

$$g(x^*)\in \mathrm{WL}[g(X[f\geqq f(x^*)])]\ .$$

Thus we have proved the following theorem:

Theorem 3-2. If x^* is a solution of Problem 2-p and the set $X[f \geqq f(x^*)]$ satisfies Assumption 3-d, then x^* is a solution of Problem 2-d for $y^* = f(x^*)$.

Conversely, if x^* is a solution of Problem 2-d for $y^* = f(x^*)$ and the set $X[g \leqq g(x^*)]$ satisfies the following assumption:

Assumption 3-p.

$$X[g \leqq g(x^*)] = \mathrm{CL}[X[g < g(x^*)]]\,,$$

then

$$f(x^*) \in \mathrm{WU}[f(X[g \leqq g(x^*)])]\,.$$

Thus we obtain the following corollary:

Corollary 3-2. Let $x^* \in X$, let the set $X[f \geqq f(x^*)]$ and the set $X[g \leqq g(X^*)]$ satisfy Assumption 3-d and Assumption 3-p respectively. Then x^* is a solution of Problem 2-p for $z^* = g(x^*)$ if and only if it is a solution of Problem 2-d for $y^* = f(x^*)$.

By Corollary 3-2 we have shown the duality between Problem 2-p and Problem 2-d under Assumption 3-d and Assumption 3-p. This is the duality between objects and constraints.

4. A Saddle-Point Problem

In this section we shall relate Problem 2-p and Problem 2-d to some saddle-point problem under the following assumption:

Assumption 4-1.

(i) X is a convex subset of R^l,

(ii) $f_i(x)$ $(i = 1, 2, \ldots, m)$ are all concave on X, and

(iii) $g_j(x)$ $(j = 1, 2, \ldots, n)$ are all convex on X.

Let x^* be a solution of Problem 2-p; that is $x^* \in X[g \leqq z^*]$ and $f(x^*) \in \mathrm{WU}[f(X[g \leqq z^*])]$. Furthermore, we assume that $X[g \leqq z^*]$ satisfies the Slater's constraint qualification of Problem 2-p, that is,

Assumption 4-p. There exists an $x \in X$ such that $g(x) < z^*$.

Then there exists a $\bar{u}^1 \geq 0$ and a $\bar{v}^1 \geqq 0$ such that

$$L^1(\bar{u}^1, \bar{v}^1, x) \leqq L^1(\bar{u}^1, \bar{v}^1, x^*) \leqq L^1(\bar{u}^1, v, x^*) \qquad (1)$$

for all $x \in X$, $v \geqq 0$

where $L^1(u, v, x) = \langle u, f(x)\rangle - \langle v, g(x) - z^*\rangle$[5]. The first inequality of (1) is equivalent to

(2) $$\langle \hat{u}^1, f(x) - f(x^*)\rangle - \langle \hat{v}^1, g(x) - g(x^*)\rangle \leqq 0 \qquad \text{for all } x \in X.$$

Now we assume that the set $X[f \geqq f(x^*)]$ satisfies Assumption 3-d. It is easily verified that, under Assumption 4-1, Assumption 3-d is equivalent to the Slater's constraint qualification of Problem 2-d for $y^* = f(x^*)$, that is,

Assumption 4-d. There exists an $x \in X$ such that $f(x) > f(x^*)$. Therefore, we may assume Assumption 4-d instead of Assumption 3-d. Under Assumption 4-d, by Theorem 3-2, we have that $g(x^*) \in \mathrm{WL}[g(X[f \geqq f(x^*)])]$. But, since the set $X[f \geqq f(x^*)]$ satisfies Assumption 4-d, there exists a $\hat{u}^2 \geqq 0$ and a $\hat{v}^2 \geqq 0$ such that

(3) $$L^2(\hat{u}^2, \hat{v}^2, x) \leqq L^2(\hat{u}^2, \hat{v}^2, x^*) \leqq L^2(u, \hat{v}^2, x^*) \qquad \text{for all } x \in X,\ u \geqq 0$$

where $L^2(u, v, x) = \langle u, f(x) - f(x^*)\rangle - \langle v, g(x)\rangle$. By observing that

$$L^2(\hat{u}^2, \hat{v}^2, x^*) = L^2(u, \hat{v}^2, x^*) = -\langle \hat{v}^2, g(x^*)\rangle \qquad \text{for all } u \geqq 0,$$

(3) is equivalent to

(4) $$\langle \hat{u}^2, f(x) - f(x^*)\rangle - \langle \hat{v}^2, g(x) - g(x^*)\rangle \leqq 0 \qquad \text{for all } x \in X.$$

Note that (4) has the same form as (2). The difference between (2) and (4) is the difference between $\hat{u}^1 \geq 0$, $\hat{v}^1 \geqq 0$ and $\hat{u}^2 \geqq 0$, $\hat{v}^2 \geq 0$. By letting $u^* = \hat{u}^1 + \hat{u}^2$ and $v^* = \hat{v}^1 + \hat{v}^2$, it follows from (2) and (4) that

(5) $$\langle u^*, f(x) - f(x^*)\rangle - \langle v^*, g(x) - g(x^*)\rangle \leqq 0 \qquad \text{for all } x \in X.$$

Conversely, it is obvious that if x^* satisfies (5) for some $u^* \geq 0$ and some $v^* \geq 0$ then x^* is a solution of Problem 2-p and Problem 2-d. Consequently, we obtain the following theorem:

5) For any two vectors $a = (a_1, a_2, \ldots, a_k)$, $b = (b_1, b_2, \ldots, b_k)$ in R^k, $\langle a, b\rangle$ represents the inner product of a and b; that is $\langle a, b\rangle = \sum_{i=1}^{k} a_i b_i$.

Theorem 4–1. Let $X, f(x)$ and $g(x)$ satisfy Assumption 4-2, Assumption 4-p for $z^*=g(x^*)$ and Assumption 4-d. Then the following conditions (i), (ii) and (iii) are all equivalent:

(i) x^* is a solution of Problem 2-p for $z^*=g(x^*)$,

(ii) x^* is a solution of Problem 2-d for $y^*=f(x^*)$, and

(iii) there exist a $u^*\geq 0$ and a $v^*\geq 0$ satisfying (5).

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