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AN INVESTMENT PROBLEM: AN OPTIMAL STOPPING PROBLEM IN WHICH TWO STOPS ARE REQUIRED

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Abstract

An optimal stopping problem without recall in which two stops are required is discussed in this note. The first stop can be interpreted as deciding to buy a commodity at the current price, and the second stop, as the timing to sell it. The explicit solutions are derived for the finite-opportunity case and for the infinite case. The examples are included to illustrate the computations required by the optimal strategy.

Let $X_1, X_2, \cdots$ be independent and identically distributed random variables that can be observed sequentially at a cost of $c(\geqq 0)$ per observation. These random variables can be interpreted as prices of a commodity that a player observes sequentially. The common distribution function $F(x)$ of each of the observation X_i is assumed to be known to him. He must make two stops, buying at the first stop and selling at the second; thus if his stops are at stages m and n $(1 \leqq m < n)$, his gain is

$$Z_{m,n} = X_n - X_m - cn - h(n-m)$$

where h $(\geqq 0)$ is the holding cost which is assumed to be proportional to the time the inventory is held. The problem is to find a strategy that maximizes the expected value $E[Z_{m,n}]$.

This formulation of the problem provides a model for studying some extensions of optimal stopping problems considered in [2] and it corresponds to a particular case of the more general one discussed by Haggstrom [3] in more abstract level. Although the model may be oversimplified for the practical applications, we derive in this short note the explicit solution of the problem because of its importance in wide areas of management science.

First let $c=h=0$. Also we assume that the player knows N, the number of total opportunities permitted to him. If he has not chosen until the final two draws, then he is forced to choose these two observed values. Let V_N be the expected value of the game of length N under the optimal strategy. Let us tentatively assume that the player has stopped for the first time after taking exactly N-k observations, and hence k stages are left for him for his second stop. Let μ_k be the expected gain from his optimal second stop. Then, from the well-known result in optimal stopping theory (see, for example [1; Sec. 13.4]), the sequence $\{\mu_k\}$ satisfies

(1) $$\mu_k = \mu_{k-1} + T_F(\mu_{k-1}) \quad (k=2, 3, \cdots, N-1;\ \mu_1 = \mu \equiv \int x dF(x))$$

where

(2) $$T_F(z) \equiv \int_z^{\infty} (x-z)\, dF(x) \equiv E[(X-z)^+].$$

(We assume that $E|X|<\infty$.)

Moreover μ_k is the optimum decision number for the second stop, that is, he stops at the current x if $x>\mu_k$, continues to the next draw if $x<\mu_k$, and is indifferent if $x=\mu_k$, when k stages are remained after the current observation x.

Now we derive a recurrence relation between V_N and V_{N-1}. Let us denote by $\psi(x)$ the probability of stopping for the first time at the first draw when x is observed. Then clearly

$$V_N = \max_{\psi(\cdot)} \int \{(-x+\mu_{N-1})\,\psi(x)+(1-\psi(x))\,V_{N-1}\}\,dF(x)$$

$$= V_{N-1}+\max_{\psi(\cdot)} \int (\mu_{N-1}-V_{N-1}-x)\,\psi(x)\,dF(x)$$

$$= V_{N-1}+E\,[(\mu_{N-1}-V_{N-1}-X)^+]$$

and hence

(3) $$V_N = V_{N-1}+T_{\tilde{F}}(-(\mu_{N-1}-V_{N-1})) \quad (N=3,4,\cdots;\ V_2=0)$$

where $\tilde{F}(x)\equiv\text{Prob.}\{-X\leqq x\}=1-F(-x-0)$. The function $T_{\tilde{F}}(z)$ is related with $T_F(z)$ by the identity

(4) $$\mu-z = T_F(z)-T_{\tilde{F}}(-z)\,.$$

Let

(5) $$\nu_N = \mu_N-V_N \quad (N=2,3,\cdots;\ \nu_2=\mu_2)\,.$$

Then ν_{N-1} is the optimum decision number for the first stop when the game is of length N, that is, the player stops at the first observation x if $x<\nu_{N-1}$, continues to the next draw if $x>\nu_{N-1}$, and is indifferent if $x=\nu_{N-1}$.

Summarizing the above arguments we obtain the conclusion that *an optimal procedure is to stop at the first m such that $X_m\leqq\nu_{N-m}$ and thereafter at the first n such that $X_n\geqq\mu_{N-n}$. The expected gain using this strategy is $V_N=\mu_N-\nu_N$.*

From (1) and (3), both of μ_N and V_N are increasing in N. From (1) and (3)$\sim$(5) we have

$$\nu_N-\nu_{N+1} = (T_F(\mu_{N-1})-T_F(\mu_N))+(T_F(\nu_N)-T_F(\nu_{N-1}))\,,$$
$$(N\geqq 3;\ \nu_2=\mu_2)\,.$$

Hence it follows, by induction, that ν_N is decreasing in N.

Next we consider the case where c, $h>0$ and $N=\infty$. This is of course the limiting case of one where c, $h>0$ and $N<\infty$, which itself has obviously modified equations corresponding to (1) and (3). Although the existence of optimal strategies in the limiting case is guaranteed by

Haggstrom [3] (or more generally by Degroot [1; Sec. 13.8~9]), we are interested in actually deriving an optimal strategy.

Let us tentatively assume that the player has stopped for the first time after taking several observations and hence he is required to choose his second stop. Let a be the expected gain from his optimal second stop. Then it is well-known (see for example [1; Sec. 13.5]) that a satisfies the equation

(6) $$T_F(a) = c+h\,,$$

and moreover a is the optimum decision number. The meaning of the latter statement is as follows: an optimal strategy is to stop sampling at the first n such that $X_n \geqq a$. The expected gain using this strategy equals a.

Now let V be the expected value of the two-stop game under the optimal strategy. Also let us denote by $\psi(x)$ the probability of choosing x as his first stop when it is observed at the first draw. Then clearly

$$V = \max_{\psi(\cdot)} \int \{(-x+a)\,\psi(x)+(1-\psi(x))\,V-c\}dF(x)$$

$$\therefore \quad c = \max_{\psi(\cdot)} \int (-x+a-V)\,\psi(x)\,dF(x) = E\,[(a-V-X)^+]$$

Therefore

(7) $$c = T_F(-(a-V))\,.$$

Let

(8) $$\beta = a-V\,.$$

Then β is the optimum decision number for the first stop. Thus we obtain from (7) and (4),

(9) $$c+\mu = \beta+T_F(\beta)\,.$$

$T_F(z)$, defined by (2), is a familiar function in many areas. It is non-negative, convex and strictly decreasing on the set where it is positive. Furthermore $T_F(z) \geqq \mu - z$, and

$$\lim_{Z\to-\infty}\{T_F(z)-(\mu-z)\}=0,$$

$$\lim_{Z\to\infty}T_F(z)=0\,.$$

The inverse function $T_F^{-1}(c)$ exists for $c>0$. The function, defined by the following equation,

$$S_F(z)\equiv z+T_F(z) \tag{10}$$

is convex, non-decreasing and $S_F(z)\geqq\max(z,\mu)$, and

$$\lim_{Z\to-\infty}S_F(z)=\mu\,,\quad \lim_{Z\to\infty}(S_F(z)-z)=0\,.$$

The inverse function $S_F^{-1}(c)$ exists for $c>\mu$. Hence (6), (9) and (8) give

$$\begin{aligned} &\alpha=T_F^{-1}(c+h)\,,\\ (11)\qquad &\beta=S_F^{-1}(c+\mu)\,,\\ &V=\alpha-\beta=T_F^{-1}(c+h)-S_F^{-1}(c+\mu)\,.\end{aligned}$$

Summarizing the above arguments we can state as follows: *If $\alpha>\beta$ an optimal procedure is to stop at the first m such that $X_m\leqq\beta$ and thereafter at the first n such that $X_n\geqq\alpha$. The expected gain using this strategy is $V=\alpha-\beta$. If $\alpha\leqq\beta$, then it is best for the player to take no observation at all and have a total gain of* 0.

The behaviors of V_N and V depend on the form of the distribution of the observations and especially on the shapes of both lower and upper tails. Consequently, to give variety for tails of several shapes, the following examples are worked out by three kinds of distributions: the uniform, the normal and the exponential.

Example 1. *Uniform distribution*: $F(x)=x$ $(0\leqq x\leqq 1)$. Then

$$T_F(z)=\begin{cases}\dfrac{1}{2}-z\,, & z\leqq 0\\[2ex] \dfrac{1}{2}(1-z)^2\,, & 0\leqq z\leqq 1\\[2ex] 0\,, & z\geqq 1\,,\end{cases}$$

and hence we get from (1) and (3)~(5)

$$\mu_k = \frac{1}{2}(1+\mu_{k-1}^2) \qquad \left(k=2, 3, \cdots; \mu_1=\frac{1}{2}\right)$$

$$V_N = V_{N-1}+\frac{1}{2}(\mu_{N-1}-V_{N-1})^2 \quad (N=3, 4, \cdots; V_2=0).$$

A short table of the μ's, V's and ν's, together with those in Examples 2 and 3, is in Table 1.

Table 1. Optimum Decision Numbers and the Maximum Expected Gain.

k	Uniform			Normal			Exponential		
	μ_k	V_k	ν_k	μ_k	V_k	ν_k	μ_k	V_k	ν_k
1	.5000			.00			1.00		
2	.6250	.0000	.6250	.40	.00	0.40	1.37	.00	1.37
3	.6953	.1953	.5000	.63	.63	0.00	1.62	.62	1.00
4	.7414	.3203	.4211	.79	1.03	−0.24	1.82	.99	.83
5	.7751	.4091	.3660	.91	1.32	−0.41	1.98	1.26	.72
6	.8004	.4761	.3243	1.01	1.55	−0.54	2.12	1.47	.65
7	.8203	.5287	.2916	1.09	1.74	−0.65	2.24	1.64	.60
8	.8364	.5712	.2652	1.16	1.90	−0.74	2.35	1.79	.56
9	.8498	.6064	.2434	1.22	2.03	−0.81	2.44	1.92	.52
10	.8611	.6360	.2251	1.28	2.15	−0.87	2.53	2.03	.50
11	.8707	.6613	.2094	1.33	2.26	−0.93	2.61	2.14	.47
12	.8791	.6833	.1958	1.37	2.36	−0.99	2.68	2.24	.44

(The values for the Uniform Distribution are those reproduced from Haggstrom [3])

Furthermore we obtain, from (11),

$$a = T_F^{-1}(c+h) = \begin{cases} 1-\sqrt{2(c+h)}, & 0<c+h<\frac{1}{2} \\ \frac{1}{2}-(c+h), & c+h\geqq\frac{1}{2} \end{cases}$$

$$\beta = S_F^{-1}\left(c+\frac{1}{2}\right) = \begin{cases} \sqrt{2c} & 0<c<\frac{1}{2} \\ c+\frac{1}{2}, & c\geqq\frac{1}{2}. \end{cases}$$

Note that

$$V\begin{Bmatrix} > \\ = \\ < \end{Bmatrix}0, \text{ according as } \sqrt{2c}+\sqrt{2(c+h)}\begin{Bmatrix} < \\ = \\ > \end{Bmatrix}1.$$

Example 2. *Normal distribution*: $\dfrac{dF(x)}{dx}=(2\pi)^{-1/2}\exp(-x^2/2)$.

Then

$$T_F(z)=\phi(z)-z\Phi(z) \qquad (\equiv \Psi(z), \text{say}),$$

where we have set

$$\phi(z)\equiv(2\pi)^{-1/2}\exp(-z^2/2)\,, \qquad \Phi(z)\equiv\int_z^\infty \phi(t)\,dt\,.$$

By using the facts $\Psi(-z)=z+\Psi(z)$ and $F=\tilde{F}$ we obtain, from (1) and (3).

$$\mu_k=\mu_{k-1}+\Psi(\mu_{k-1})\,, \qquad (k=2,3,\cdots;\ \mu_1=0)$$

$$V_N=\mu_{N-1}+\Psi(\mu_{N-1}-V_{N-1})=V_{N-1}+\Psi(-(\mu_{N-1}-V_{N-1}))$$

$$(N=3,4,\cdots;\ V_2=0)\,.$$

Furthermore we get, from (9) and (11),

$$\alpha=\Psi^{-1}(c+h)\,, \quad \beta=-\Psi^{-1}(c)\,, \quad V=\Psi^{-1}(c+h)+\Psi^{-1}(c)\,.$$

Note that if $0<c\leqq(2\pi)^{-1/2}$

$$V\begin{Bmatrix} > \\ = \\ < \end{Bmatrix}0, \text{ according as } \Psi^{-1}(c)\begin{Bmatrix} > \\ = \\ < \end{Bmatrix}h\,.$$

Example 3. *Exponential distribution*: $\dfrac{dF(x)}{dx}=e^{-x}(x\geqq 0)$. Then

$$T_F(z)=\begin{cases}1-z\,, & z<0\\ e^{-z}\,, & z\geqq 0\,.\end{cases}$$

We obtain, from (1) and (3)~(5), that

$$\mu_k=\mu_{k-1}+e^{-\mu_{k-1}} \qquad (k=2,3,\cdots;\ \mu_1=1)$$

$$V_N=\mu_{N-1}-1+e^{-(\mu_{N-1}-V_{N-1})} \quad (N=3,4,\cdots;\ V_2=0)\,.$$

Furthermore we get

$$a = T_F^{-1}(c+h) = \begin{cases} -\log(c+h), & \text{if } 0<c+h<1 \\ 1-(c+h), & \text{if } c+h \geqq 1 \end{cases}$$

and β is the unique positive root of the equation

$$\beta + e^{-\beta} = c+1 \, .$$

We can find that for any $0<h<1$

$$V \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0, \quad \text{according as} \begin{Bmatrix} 0<c\leqq\bar{c}(h) \\ c>\bar{c}(h) \end{Bmatrix},$$

where $\bar{c}(h)$ is the unique and positive root of the equation (in c)

$$-\log(c+h) = H(c+1) \, ,$$

and $H(k)$ is also the unique and positive root of the equation (in z)

$$z + e^{-z} = k \qquad (\text{if } k>1) \, .$$

References

[1] DeGroot, M.H., *Optimal Statistical Decisions*, McGraw-Hill Book Co , New York, 1970, Chapter 13.

[2] Gilbert, J.P. and F. Mosteller, "Recognizing the maximum of a sequence," *J. Amer. Stat. Assoc.* **61** (1966), 35–73.

[3] Haggstrom, G.W., "Optimal sequential procedures when more than one stop is required," *Ann. Math. Stat.*, **38** (1967), 1618–1626.