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A MATHEMATICAL MODEL FOR A DECENTRALIZED DECISION-INFORMATION SYSTEM WITH AUTONOMOUS DIVISIONS

HAJIME ETO

Hitachi Central Research Laboratory, Kokubunji, Tokyo

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Abstract

This paper presents a mathematical model to design and construct a decentralized decision-information system incorporatable with taxational control. Divisions can purchase the resources as much as they demand at the prices decided by the center. Hence a time-consuming negotiation between the center and divisions for resource allocation can be avoided. The coordination process on the central approval of divisional plans terminates when all divisions spontaneously retire from the process. In this manner the decentralized decision is maintained thoroughly up to the final stage of the process, and thereby the information flow and the file are significantly saved in comparison with the decentralized systems thus far proposed. For the enhancement of the quick controllability the price information can be embodied in "tax" and "interest". This model is extensible to include the resource procurement and the corporate-constitutional improvement. It is also applicable to administration of a group of cities or countries in which each should enjoy the fullest autonomy, and to a governmental control of private industries.

Introduction

The availability of informations sometimes affects the decision behavior. The maximization of sales amounts is at present more preferably adopted as the objectives in many firms than the maximization of profit, probably because the information of profit is not readily available. Hence a designer of management information system (MIS) is required to single out the essential information so that it be readily available. An indirect control through informations allows each division autonomous decisions for its manager in order to utilize his talent to the fullest extent in harmony with the global optimum of the whole organization. The indirect control of this kind seems to dissolve the conflict between the global and local efficiency and to make the technological efficiency at, or probably above the expected level while the contrary sometimes occurs in a centralized organization.

The decomposition algorithm for the mathematical programming has attracted a mathematical attention [1], [3], [4], [5], [6], [7], [8], [9] and the decentralized managerial interpretation has been given to it [1], [2], [4], [8], [11], [12]. In contrast to these computational and interpretative works, little has been attempted to make the algorithm a formal tool for design, construction and control of a system. Mathematical devices below aid us (1) to design a total decision-information system which is step-by-step constructive in the sense that a subsystem works without the completion of the total system, (2) to control the system in terms of a quick-response policy consistent to the growth target, (3) to exclude an abrupt interruption of autonomy which was the case in the foregoing decentralized models (see [2]), and (4) to administrate the organization in terms of policy evaluation.

Assumptions

(A1) A large firm consists of mutually independent divisions details of whose activities the center does not go into.

(A2) Every division sells all its products to the markets at market prices. This is deduced from (A1) which implies that a division (say, computer division) can buy an intermediate goods (say, integrated circuits) at will from the market or from another division (say, integrated circuits division) at the market price. Thus all the goods can safely be regarded to be sold to and be purchased from the market.

(A3) The competence of the divisions is to make its own plan for the given prices of resources, while the competence of the center is to control the prices of the resources in response to their demand.

Problem Formulation in Terms of Reform Evaluation

The global problem (G) may be expressed as follows;

$$(G)\begin{cases} (1) \text{ maximize } & w=\sum_{k\in K} c_k x_k \\ (2) \text{ subject to } & \sum_{k\in K} A_k x_k \leqq b, \\ (3) & F_k x_k \leqq b_k \quad \text{for each } k\in K \\ \quad\text{and} & \\ (4) & x_k \geqq 0 \quad \text{for each } k\in K \end{cases}$$

and a divisional objective for division k at stage l may be as follows;

$$(5) \quad \text{maximize} \quad w_k{}^l=(c_k-p^l A_k)x_k \quad \text{for each } k\in K$$

where c_k, x_k, A_k, b, p^l, F_k and b_k are $1\times n_k$, $n_k\times 1$, $m\times n_k$, $m\times 1$, $1\times m$, $m_k\times n_k$ and $m_k\times 1$ matrices or vectors respectively, w and $w_k{}^l$ are real numbers, k and l are natural numbers, and K is the set of the divisions indices ranging from 1 to n. (5), (3) and (4) for a k and a l form a divisional problem $(D_k{}^l)$, and $x_k{}^l$ denotes the solution to $(D_k{}^l)$ for p^l given.

In what follows a feasible solution is assumed to be already available for all $k\in K$ and all $l\geqq 1$ without loss of generality because the phase one device in the simplex method for linear programming provides a formally feasible solution and because a feasible solution is known from an experience in a realistic case.

$$
(C^l)\begin{cases}
(6) & \text{maximize} \quad w = \sum_{k\in K} \sum_{i=1}^{2} c_{ki}^{l}\, y_{ki} \\
(7) & \text{subject to} \quad \sum_{k\in K} \sum_{i=1}^{2} q_{ki}^{l}\, y_{ki} \leqq b\,, \\
(8) & \qquad \sum_{i=1}^{2} y_{ki} = 1 \quad \text{for all } k\in K \\
 & \text{and} \\
(9) & \qquad y_{ki} \geqq 0 \quad \text{for all } k\in K \quad \text{and} \quad i=1,2
\end{cases}
$$

The transformed central problem (C^l) given as above at stage l in scalars y_{ki}'s is recursively constructed as follows: for x_k^l's and y_{k1}^{l-1}'s already obtained as solutions to (D_k^l) and (C^{l-1}) respectively, c_{ki}^l's are scalars such that

$$(6.1) \qquad c_{k1}^1 = 0 \quad \text{for all } k\in K$$

$$(6.2) \qquad c_{k2}^l = c_k\, x_k^l \quad \text{for all } k\in K \text{ and all } l \geqq 1$$

$$(6.3) \qquad c_{k1}^l = \sum_{i=1}^{2} c_{ki}^{l-1} y_{ki}^{l-1} \quad \text{for all } k\in K \text{ and all } l \geqq 2$$

and q_{ki}^l's are column vectors of order m such that

$$(7.1) \qquad q_{k1}^1: \text{ arbitrary large for all } k\in K$$

$$(7.2) \qquad q_{k2}^l = A_k\, x_k^l \quad \text{for all } k\in K \text{ and all } l \geqq 1$$

$$(7.3) \qquad q_{k1}^l = \sum_{i=1}^{2} q_{ki}^{l-1} y_{ki}^{l-1} \quad \text{for all } k\in K \text{ and all } l \geqq 2.$$

The dual optimal solution to (C^{l-1}) is denoted by $(p^l;\, p_k^l)$ partitioned in accordance with (7) and (8). p^l, a row vector of order m appearing in (5), is interpreted as the price or the marginal value of the common resources b. When b consists of the capital, the labor and the land, p^l means the marginal values of the capital-profitability, the labor-productivity and the land-productivity.

The solution $\bar{x}_k^{l+1}$ to (G) for each $k\in K$ in correspondence to $(p^{l+1};\, p_k^{l+1})$ is recovered at stage l from x_k^j and y_{ki}^j for each $k\in K$, $i=1$, 2 and all $j \leqq l$ by the equation (10) below.

$$(10) \qquad \bar{x}_k^{l+1} = \sum_{h=1}^{l-1} x_k^h\, y_{k2}^h \prod_{g=h+1}^{l} y_{k1}^g + x_k^l\, y_{k2}^l$$

where $\sum_{h=1}^{0} a^h$ is supposed to vanish.

The optimality of (C^l) is attained when the benefit $\bar{w}_k^l$ of the k-

th divisional reform plan no longer exceeds the lower bound $p_k{}^l$ given each time by the center for all $k \in K$.

$$(11) \qquad \bar{w}_k{}^l \leqq p_k{}^l$$

where $p_k{}^l$ is interpreted as the marginal contribution of the division k. By (5), (6.2) and (7.2), (11) is equivalent to (11′) below which is the optimality criterion in the Dantzig and Wolfe's procedure [4].

$$(11') \qquad c_{k2} - p^l \cdot q_{k2} - p_k{}^l \leqq 0$$

The problem (C^l) and ($D_k{}^l$) for all k and l are assumed to be feasible and bounded in the subsequent discussions.

Reform Target-Reform Approval Decision Behavior and Procedure

A realistic decision behavior of a firm meeting ($A1$) to ($A3$) may be as follows;

step 0. Set $l=1$. The initial prices p^1 of the common resources and the initial lower bound $p_k{}^1$ of the benefit of the k-th divisional reform plan for each $k \in K$ are known from the past experience. (Otherwise, set $p^1=0$ and $p_k{}^1=-\infty$ for all $k \in K$).

step 1. The center informs division k (for each $k \in K$) of the provisional prices p^l of the common resources and the lower bound $p_k{}^l$ of the net profit of the k-th divisional reform plan.

step 2. Division k (for each $k \in K$) makes the k-th divisional detailed reform plan $x_k{}^l$ by solving ($D_k{}^l$) to maximize its "net profit" $w_k{}^l$ of (5) that means the profit after the prices of the common resources are deducted in proportion to their consumption.

step 3. Division k (each $k \in K$) reviews the inequality (11) and reports the center whether or not the optimal net profit $\bar{w}_k{}^l$ of the k-th divisional reform plan exceeds the lower bound $p_k{}^l$ given by the center.

step 4. If no division reports the excess, the center judges that the global optimality is attained, and announces that no further reform is needed. Go to step 8. Otherwise go to step 5.

step 5. The division k (each $k \in K$) which attained the excess calculates the cost $q_{k2}{}^l$ and the benefit $c_{k2}{}^l$ of the k-th divisional reform plan and informs the center of these result.

step 6. The center coordinates the submitted k-th divisional reform plan in terms of the extent $y_{k2}{}^l$ of its approval, and decides the prices p^{l+1} of the common resources and the lower bound $p_k{}^{l+1}$ of the k-th divisional reform plan for each $k \in K$.

step 7. The center informs a division k (each $k \in K$) of the extent $y_{k2}{}^l$ to which the reform plan is approved, and a division k, based on these informations, renews its detailed plan $\bar{x}_k{}^{l+1}$. Reset $l=l+1$ and return to step 1.

step 8. If no reform is needed any longer, the center authorizes the latest decision on each divisional plan at step 6 as the final one. The procedure terminates.

The calculational procedure is easily derived in stepwise correspondence to the behavior described earlier. Step 6 should read this time:

step 6. Construct and solve the central problem (C^l) to obtain $(p^{l+1}; p_k{}^{l+1})$ and $y_{ki}{}^l$ for all $k \in K$ and $i=1, 2$.

To keep the uniformity for all $k \in K$, the columns with $i=2$ all appeared in (C^l). But in stating the behavior and the procedure above the $i=2$ columns with (11) held were not generated.

Fullest-Autonomous Decision

The geometric interpretation of the construction of (C^l) and $(D_k{}^l)$ and that of $\bar{x}_k{}^l$ are as follows. The point $t_1{}^l$ given by setting $y_{k1}=1$ and $y_{k2}=0$ for each $k \in K$ in (C^l) represents the solution to (C^{l-1}), and the point $t_2{}^l$ given by setting $y_{k1}=0$ and $y_{k2}=1$ for each $k \in K$ in (C^l) represents the submitted reform plan which is possibly infeasible in (C^l). The point $t_0{}^l$ given by assigning y_{k1} and y_{k2} appropriate values meeting (8) and (9) represents a convex linear combination of the previous solution $t_1{}^l$ and the reform plan $t_2{}^l$.

Our procedure forms a convex linear combination of the preceeding solutions at every iteration while the Dantzig and Wolfe's procedure [4] conducts the same thing just once at the conclusion of the whole process. This minor change overcomes the defect of the Dantzig and Wolfe's decentralization that the decentralization breaks down suddenly at the final stage as Baumol and Fabian pointed out [2]. On the contrary our system allows all divisions more spontaneous motivation of their own volition throughout the whole process until the final stage at which all divisions spontaneously retire from the reform process.

Theorem 1. The calculational procedure above yields the optimal feasible solution to the original problem (G).

Proof. Noting the equivalence between (10) and the recursive definition (10′) below, the equivalence between the procedure above and the procedure in [4] is clear by the foregoing comparison of the both procedures.

$$(10') \qquad \begin{cases} \bar{x}_k{}^2 = x_k{}^1 \\ \bar{x}_k{}^{l+1} = \bar{x}_k{}^{l-1} y_{k1}{}^l + x_k{}^l y_{k2}{}^l \qquad \text{for } l \geqq 2 \end{cases} \qquad \text{Q.E.D.}$$

The decision in terms of policy evaluation raises autonomy and thereby significantly reduces informations to be filed at the center in comparison with the decentralized systems thus far proposed (e.g. [4]).

Quick Controllability of the System

Since $p_k{}^l$ denotes the weighted sum of the prices of the k-th divisional resources (Theorem 2 below), the division k is urged to increase (some of) its resources if $p_k{}^l$ is high. The k-th divisional decision as to which k-th divisional resource should be increased is to be based, by Theorem 4, on its price $\pi_k{}^l$ obtained at the conclusion of solving $(D_k{}^l)$. $\pi_k{}^l$, a m-vector, need not be sent to the center which knows $p_k{}^l$, a scalar. This saves the information file at the center. If $p_k{}^l$ is low, the division k is urged a technological innovation.

A seemingly desirable way to induce divisions to make their optimal plans for p^l given without falsehood is to allow divisions to earn their "net profit" after deduction of $p^l A_k \bar{x}_k^l$ from the gross profit $c_k \bar{x}_k^l$ so that the center and divisions share the whole profit at the conclusion of the whole process. The central dividend $p^l b$ (note $p^l \sum_{k \in K} A_k \bar{x}_k^l = p^l b$ because the slacks associated with the resources of non-zero price vanish) vanishes only when $p^l = 0$ which means all the common resources are beyond demand. By Theorem 3 the k-th divisional dividend vanishes only when $p_k^l = 0$ which means all the k-th divisional resources exceed the demand (Theorem 2). Both cases are unlikely in a realistic problem. Hence this taxational control seems acceptable both to the center and the divisions. A tax and interest policy is empirically known to be sensitive to the environmental changes and to have a quick effect on planning behaviors.

Theorem 2. Let $m_k \times 1$ vector π_k^l be the optimal and dual solution to (D_k^l) for p^l given by (C^{l-1}). Then

$$p_k^l = \pi_k^l b_k$$

Proof. At the optimality of (C^l), by the complementarity theorem of Linear Programming the equality holds

$$c_{ki}^l - p^{l+1} q_{ki}^l = p_k^{l+1} \tag{11''}$$

for all k and i such that $y_{ki}^l > 0$ in (C^l) (note $(p^{l+1}; p_k^{l+1})$ is the optimal and dual solution to (C^l)) meanwhile the duality theorem of Linear Programming on (D_k^l) yields

$$\bar{w}_k^l = \pi_k^l b_k . \tag{12}$$

Thus the theorem follows from (11″) and (12).

Q.E.D.

Theorem 3. At the optimality of (C^l) for all $l \geqq 1$

$$c_k \bar{x}_k^{l+1} - p^{l+1} A_k \bar{x}_k^{l+1} = p_k^{l+1} \quad \text{for all } k \in K \tag{13}$$

(Note the global optimality is found after p^{l+1}, p_k^{l+1} and $\bar{x}_k^{l+1}$ are obtained.)

Proof. Since the equality (11″) holds only if $y_{ki}>0$,

$$c_{ki}^l y_{ki}^l - p^{l+1} q_{ki}^l y_{ki}^l = p_k^{l+1} y_{ki}^l \quad \text{for all } k \in K \text{ and } i=1,2$$

which implies by (8)

$$\text{(14)} \qquad \sum_{i=1}^2 c_{ki}^l y_{ki}^l - p^{l+1} \sum_{i=1}^2 q_{ki}^l y_{ki}^l = p_k^{l+1} \quad \text{for all } k \in K$$

It shall be inductively shown that the left side of (14) is equivalent to that of (13). Clearly for $l=1$

$$\sum_{i=1}^2 c_{ki}^1 y_{ki}^1 = c_k x_k^1 = c_k x_k^1 y_{k2}^1 = c_k \bar{x}_k^2$$

If $\sum_{i=1}^2 c_{ki}^{l-1} y_{ki}^{l-1} = c_k \bar{x}_k^l$, then by (6.2) and (6.3)

$$\sum_{i=1}^2 c_{ki}^l y_{ki}^l = \sum_{i=1}^2 c_{ki}^{l-1} y_{ki}^{l-1} y_{k1}^l + c_k x_k^l y_{k2}^l$$

$$= c_k(\bar{x}_k^l y_{k1}^l + x_k^l y_{k2}^l) = c_k \bar{x}_k^{l+1} \quad \text{by (10}'\text{)}.$$

The analogous argument leads to $\sum_{i=1}^2 q_{ki}^l y_{ki}^l = A_k \bar{x}_k^{l+1}$. Q.E.D.

Theorem 4. Let π_k^l be the same as in theorem 2 and suppose the global optimality criterion holds for all $k \in K$ at l. Then π_k^l is also the optimal and dual solution associated with (3) in (G). ($l \geqq 2$ since the optimality test is done at the beginning of iteration l before solving (C^l)).

Proof. By theorem 2 and 3

$$c_k \bar{x}_k^l - p^l A_k \bar{x}_k^l = \pi_k^l b_k \quad \text{for all } l \geqq 2 \text{ and } k \in K$$

and $\bar{x}_k^l$ is, by theorem 1, the optimal k-th divisional solution to (G). By the duality theorem of Linear Programming on (G), π_k^l is the optimal and dual solution to (G). Q.E.D.

Theorem 4, combined with theorem 1, gives an alternative proof of the theorem by Walker [10] for obtaining the optimal and dual solution to Linear Programming of the Dantzig-Wolfe's decomposition type.

Extensibility and Step-by-Step Constructivity

The personnel and financial departments which consume the resources at the negative level can be included among divisions. In this extension constraints must be added to keep or improve constitutional ratios. For example the equity-liability ratio must be constrained for α and β given

$$\alpha \leqq \frac{b^d + d + x_{1d}}{b^e + e + x_{1e}} \leqq \beta$$

where b^d, b^e, d, e, x_{1d} and x_{1e} respectively denote the levels of debt to be allocated, own fund to be allocated, debt fixed, own fund fixed, debt to be procured and own fund to be procured. This constraint is easily restated by two linear inequalities. Therefore the model above is subject to no change in this extension. Thus it is feasible to build the system for resources fixed first and to incorporate the resources procurement and constitutional improvement system later because the former system works by itself consistently to the latter before the latter's completion. The incorporated system may help to solve a conflict between a corporate expansion and constitutional unsoundness (e.g., decline in equity-liability ratio) in the Japanese firms.

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