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A SMOOTHING METHOD BY ORTHOGONAL POLYNOMIALS AND ITS FREQUENCY RESPONSE

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Abstract

In this paper, the author proposes a smoothing method applying the approximation by orthogonal polynomials, and analyzes the frequency response of it, comparing with the exponential smoothing method as a typical example. The smoothing method by orthogonal polynomials is an expansion of the arithmetic moving average method, which is included in the former as a special case.

1. Introduction

As the smoothing operation of discrete processes, some methods have been utilized to catch the trend of the processes, such as the arithmetic moving average method or the exponential smoothing method.

Now assuming that the process is composed of many frequency components, we may consider the smoothing operation as a low-pass filter, that is, a filter which picks up low frequency components as signals and cuts off high frequency components as noises. The frequency response of the output, or the relation between input and output as a function of frequency of such a filter is of interest.

As for the arithmetic moving average method, a method which has the ideal frequency response of cutting off steeply at arbitrary frequency has been developed [6]. The smoothing method proposed in this paper is an expansion of the arithmetic moving average method. The latter obtains the smoothed value by fitting a straight line to sampled data, while the former obtains it by fitting a linear combination of orthogonal polynomials.

In this paper, the basic theory of the frequency response is first introduced. This theory is based on the fundamental theory of sampled-data control systems in control engineering. Then, our method is illustrated against the exponential smoothing method. In the next section, orthogonal polynomials are explained and the smoothing method by them is evolved. Some figures of the frequency responses of both methods are shown. We can refer to these figures to choose parameters of both methods.

2. Fundamental theory of the frequency response of the smoothing operation

2.1 The frequency response

Let $x(t)$ be a variable representing a continuous process, and $y(t)$ a sampled variable from $x(t)$ with sampling interval T. Fig. 1 illustrates the sampling process. Describing it mathematically, we have

$$
(1) \qquad y(t) = \sum_{n=0}^{\infty} x(t)\,\delta(t-nT) = x(t)\,\delta_T(t)\,,
$$

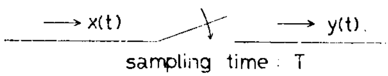

Fig. 1. Sampling process.

where $\delta(t)$ is an impulse at time t, and $\delta_T(t)$ is an impulse series from minus infinity to time t, or

(2) $$\delta_T(t) = \sum_{n=0}^{\infty} \delta(t-nT)\ .$$

The relationship among the variables above mentioned is illustrated in Fig. 2.

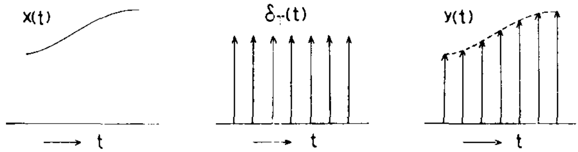

Fig. 2. Relationship among the variables.

Generally, $x(t)$ has frequency components from the zero frequency component, or the average, up to higher frequency components which are usually considered as noises. Then by the sampling theorem, $x(t)$ is perfectly restored from $y(t)$, under the condition that $x(t)$ contains no frequency components above the frequency $1/2T$, that is, half of the sampling frequency [2]. But if this condition is not satisfied, $x(t)$ is not correctly transformed to $y(t)$ in the high frequency range. In this case, $y(t)$ is filtered or smoothed, having high frequency components eliminated (see Fig. 4).

Now, suppose that the smoothed value $z(t)$ is obtained from a linear combination of $y(t)$. Then, we have

(3) $$z(t) = \sum_{n=-N_2}^{N_1} a_n y(t-nT)$$
$$= \sum_{n=-N_2}^{N_1} a_n x(t-nT)\, \delta_T(t)\ .$$

Let $X(s)$, $Y(s)$ and $Z(s)$ be the Laplace transforms of $x(t)$, $y(t)$ and $z(t)$ respectively. Then, we have (see Appendix 1).

(4) $$Y(s)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X(s+jk\omega_s)\,,$$

(5) $$Z(s)=\frac{1}{T}\sum_{k=-\infty}^{\infty}\left\{X(s+jk\omega_s)\sum_{n=-N_2}^{N_1}a_n\,e^{-nT(s+jk\omega_s)}\right\},$$

where $\omega_s=2\pi/T$ is called the sampling angular frequency. That is, (4) and (5) represent the sampling and smoothing operations respectively, shown in Fig. 3.

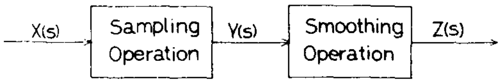

Fig. 3. Sampling and smoothing operations.

Substituting $j\omega$ for s in the Laplace transform, we obtain the frequency response of the variable. Now putting $F(j\omega)=Z(j\omega)/Y(j\omega)$, we define the frequency response of the smoothing operation. Then, we have

(6) $$F(j\omega)=\sum_{n=-N_2}^{N_1}a_n\,e^{-jnT(\omega+k\omega_s)}\,,\qquad (k=-\infty,\cdots,0,\cdots,\infty).$$

As the response $F(j\omega)$ is repeated with ω_s, it is sufficient to analyze the response only in the case of $k=0$. Therefore, instead of (6), we may consider the following frequency response

(7) $$F(j\omega)=\sum_{n=-N_2}^{N_1}a_n\,e^{-jn\omega T}\,.$$

Fig. 4 shows the relationship among the responses.

2.2 The examples of the frequency responses of the exponential smoothing method [1]

At first, we consider the first degree of this method. The smoothed value is obtained from the following:

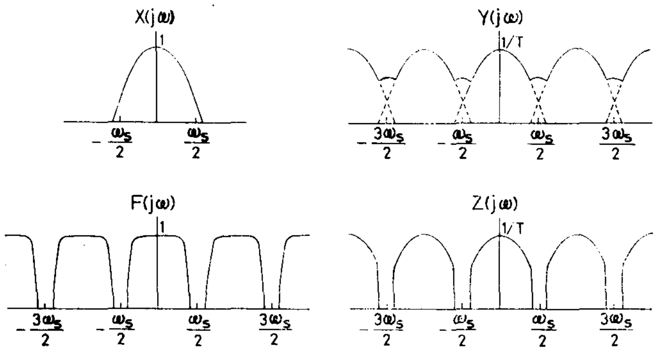

Fig. 4. Relationship among the responses.

(8) $$z(t)=S(t)\,,$$

where $S(t)$ is the smoothing term of first order,

(9) $$S(t)=a\,y(t)+b\,S(t-T)\,,$$

where a is a smoothing constant satisfying the condition $0<a<1$, and $b=1-a$.

From (8) and (9), we have

(10) $$\begin{aligned} z(t) &= a\,y(t)+b\,\{a\,y(t-T)+b\,S(t-2T)\} \\ &= a\,\{y(t)+b\,y(t-T)\}+b^2\,S(t-2T) \\ &\quad \cdots\cdots\cdots\cdots \\ &= a\sum_{n=0}^{N} b^n\,y(t-nT)+b^{N+1}\,S(t-\overline{N+1}\,T)\,. \end{aligned}$$

The last term of the right-hand side of (10) approaches zero, if N approaches infinity. Then from (7) and (10), we obtain the frequency response

(11) $$F(j\omega)=a\sum_{n=0}^{\infty} b^n\,e^{-jn\omega T}$$

$$= \frac{a}{1-b\,e^{-j\omega T}},$$

for $\quad |b\,e^{-j\omega T}| < 1\,.$

As equation (11) is represented in the complex form, it is decomposed into its absolute value and its argument

(12) $$|F(j\omega)| = \frac{a}{\sqrt{1-2b\cos\omega T+b^2}},$$

(13) $$\arg F(j\omega) = -\tan^{-1}\frac{b\sin\omega T}{1-b\cos\omega T},$$

where the former is called the gain and the latter the phase shift. They are shown in Fig. 5.

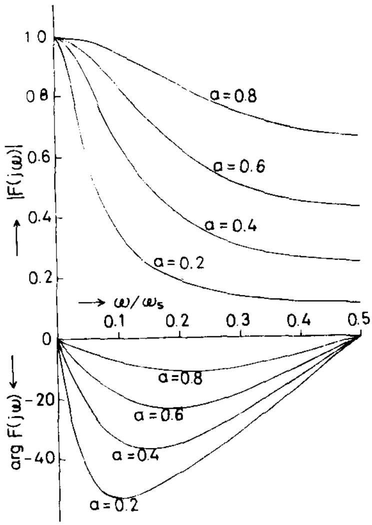

Fig. 5. The frequency response of exponential smoothing method of first degree.

As for the second degree, the smoothed value $z(t)$ is obtained as follows:

$$z(t) = 2S(t) - R(t) , \tag{14}$$

where $S(t)$ is given by (9), and

$$R(t) = aS(t) + bR(t-T) . \tag{15}$$

Then the frequency response in the second degree is, as shown in Appendix 2,

$$F(j\omega) = \frac{2a}{1-b\,e^{-j\omega T}} - \left(\frac{a}{1-b\,e^{-j\omega T}}\right)^2 . \tag{16}$$

Divided into the gain and the phase shift, it is shown in Fig. 6.

As for the third degree, we similarly have the smoothed value

$$z(t) = 3S(t) - 3R(t) + Q(t) , \tag{17}$$

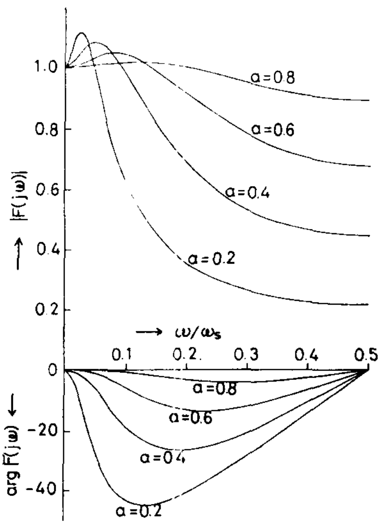

Fig. 6. The frequency response of exponential smoothing method of second degree.

where $S(t)$ and $R(t)$ are given by (9) and (15) respectively, and

(18) $$Q(t)=aR(t)+bQ(t-T)\,.$$

Then, we have the frequency response (see Appendix 3)

(19) $$F(j\omega)=\frac{3a}{1-b\,e^{-j\omega T}}-3\left(\frac{a}{1-b\,e^{-j\omega T}}\right)^2+\left(\frac{a}{1-b\,e^{-j\omega T}}\right)^3.$$

Divided into the gain and the phase shift, it is shown in Fig. 7.

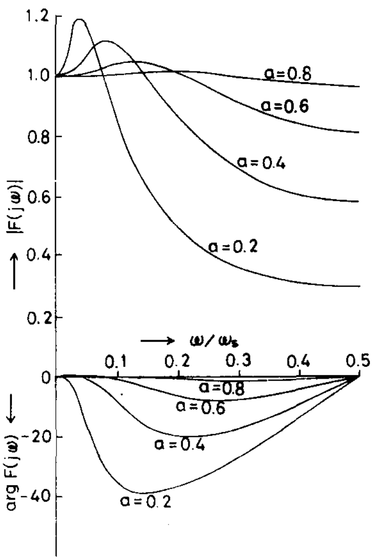

Fig. 7. The frequency response of exponential smoothing method of third degree.

A low-pass filter is ideal when the gain is unity below a certain limiting frequency, which is called the cut-off frequency, and is zero above it, while the phase shift is zero in the entire frequency range. Now from Figs. 5, 6 and 7, it is seen that the exponential smoothing method operates as a kind of low-pass filter, but that its characteristics deteriorates as the value of the smoothing constant a increases. Furthermore, we must take notice of a large phase shift in the case of small value of constant a.

3. The smoothing method by orthogonal polynomials

We may consider that a linear combination of orthogonal polynomials is fitted to sampled data $y(t-N_1T),\ldots,y(t),\ldots,y(t+N_2T)$, as shown in Fig. 8. Then, we have

$$(20)\qquad y(t+\overline{n-N_1-1}T)=\sum_{i=1}^{m}h_i g_i(n)+\varepsilon(n),\quad n=1,2,\cdots,N,$$

where each $g_i(n)$ is an orthogonal polynomial, h_i is the coefficient of $g_i(n)$, $\varepsilon(n)$ is the approximation error, and $N=N_1+N_2+1$ [4].

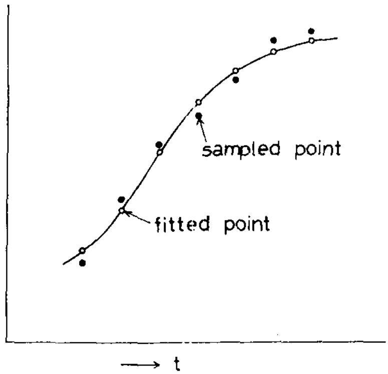

Fig. 8. Fitting of sampled data on orthogonal polynomials.

According to the orthogonal condition, we have the following relation:

$$(21)\qquad \sum_{n=1}^{N}g_i(n)g_k(n)=1,\qquad (i=k),$$

$$=0,\qquad (i\neq k).$$

Now, $g_i(n)$ are represented as follows:

$$(22)\qquad g_0(n)=\lambda_0\,,$$

$$g_1(n)=\lambda_1\left(n-\frac{N+1}{2}\right),$$

$$g_2(n)=\lambda_2\left\{\left(n-\frac{N+1}{2}\right)^2-\frac{N^2-1}{12}\right\},$$

$$g_3(n)=\lambda_3\left\{\left(n-\frac{N+1}{2}\right)^3-\frac{3N^2-7}{20}\left(n-\frac{N+1}{2}\right)\right\},$$

$$\cdots\cdots\cdots\cdots\cdots,$$

where $\lambda_i(i=0,\ 1,\ldots.)$ are coefficients to normalize their respective $g_i(n)$ [5]. Usually, $g_i(n)$ is multiplied by S_i to be turned into integer. Then instead of (21), we have

$$(23)\qquad \sum_{n=1}^{N} g_i'(n)\,g_k'(n)=S_i^2\,,\qquad (i=k)\,,$$

$$=0\,,\qquad (i\neq k)\,.$$

The $g_i'(n)$ for $i=1,\ldots.,\ 5$ are shown in Table 1 [3].

In (20), if the order m of orthogonal polynomials is equal to N, the approximation error $\varepsilon(n)$ is zero. If $m<N$, we obtain

$$(24)\qquad \varepsilon(n)=\sum_{i=m+1}^{N} h_i\,g_i(n)\,.$$

Multiplying (20) by $g_i(n)$ and considering (21), we obtain

$$(25)\qquad h_i=\sum_{n=1}^{N} y(t+\overline{n-N_1-1}\,T)\,g_i(n)\,,\qquad i=1,\cdots,m\,.$$

Now, we may consider

$$(26)\qquad z(t)=\sum_{i=0}^{m} h_i\,g_i(N_1+1)$$

as the smoothed value at time t, and neglect $\varepsilon(n)$ in (24). Then we have the frequency response (see Appendix 4).

$$F(j\omega)=\sum_{i=0}^{m} g_i(N_1+1)\sum_{n=1}^{N} g_i(n)\, e^{j(n-N_1-1)\omega T}\,. \tag{27}$$

From some cases shown in Figs. 9 and 10, it may be considered that the smoothing operation by orthogonal polynomials performs a kind of low-pass filter. Furthermore, the smoothed value may be obtained at any point in the fitted section. But as seen from these figures, there remain some problems as follows:

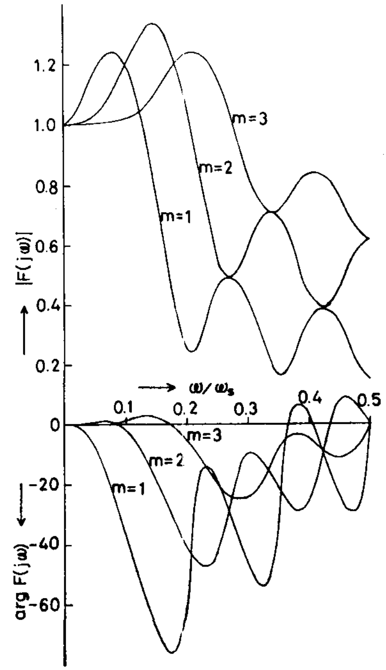

Fig. 9. The frequency response of the smoothing method by orthogonal polynomials approximation in the case of $N_1=6$ and $N_2=0$.

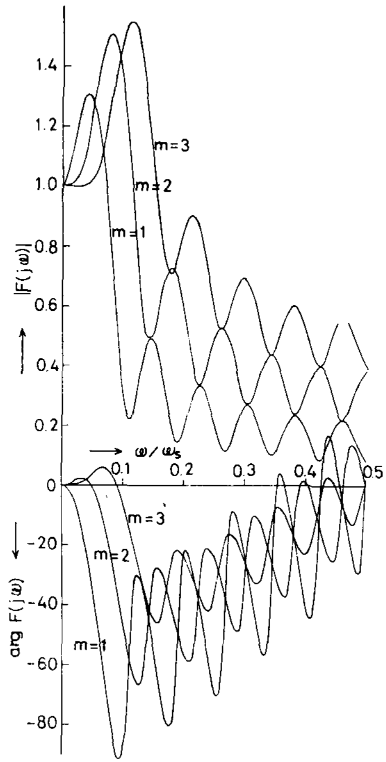

Fig. 10. The frequency response of the smoothing method by orthogonal polynomial approximation in-the case of $N_1=12$ and $N_2=0$.

(1) a high gain in the low frequency range,
(2) a considerable amount of gain in the high frequency range,
(3) a large phase shift.

We must note that here the smoothed value is obtained from the right end of the fitted section, which may be considered the most severe case to apply this method to the smoothing operation. Therefore, if the smoothed value is obtained inside of the fitted section, the smoothing characteristic may be more improved.

Table 1. The table of orthogonal polynomials in the case of $N=13$ and $m=5$.

$(+)n(-)$		g_0	$g_1(\pm)^*$	g_2	$g_3(\pm)^*$	g_4	$g_5(\pm)^*$
7		1	0	−14	0	84	0
8	6	1	1	−13	−4	64	20
9	5	1	2	−10	−7	11	26
10	4	1	3	−5	−8	−54	11
11	3	1	4	2	−6	−96	−18
12	2	1	5	11	0	−66	−33
13	1	1	6	22	11	99	22
S_i^2		13	182	2002	572	68068	6188

* The sign ($\pm$) means that the sign (+) or (−) in the marked column is changed according to the sign in the column n.

As a special case, we may point out that the smoothing method by orthogonal polynomials in the case of $m=1$ and $N_1=N_2$ is equivalent to the arithmetic moving mean method.

Acknowledgement

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Appendix 1.

Laplace transform of $\delta_T(t)$ is

$$\mathfrak{L}[\delta_T(t)] = 1 + e^{-Ts} + e^{-2Ts} + \cdots\cdots. \tag{28}$$

The right hand side of (28) converges to

$$\mathfrak{L}[\delta_T(t)] = \frac{1}{1-e^{-Ts}} \tag{29}$$

in the range of $|e^{-Ts}|<1$.

Then, Laplace transform of $y(t)$ is, by the theorem of convolution in the complex plane,

$$Y(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{X(p)}{1-e^{-T(s-p)}}\, dp\,, \tag{30}$$

where $j=\sqrt{-1}$.

If the denominator of $X(s)$ is at least two orders higher than its numerator, the closed integral can be substituted for the complex integral. Then, taking the closed integral path outside the poles of $X(s)$, the complex integral of the right hand side of (30) is obtained from the sum of residues at the poles of $1/(1-e^{-T(s-p)})$, and

$$\begin{aligned} Y(s) &= \sum_{k=-\infty}^{\infty}\left[-\frac{X(p)}{\frac{d}{dp}\left(1-e^{-T(s-p)}\right)}\right]_{p=s+j\frac{2\pi k}{T}} \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(s+j\,\frac{2\pi k}{T}\right) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(s+jk\omega_s\right). \end{aligned} \tag{31}$$

Similarly, we have

$$Z(s) = \sum_{n=-N_2}^{N_1} a_n \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{X(p)e^{-nTp}}{1-e^{-T(s-p)}}\, dp \tag{32}$$

$$= \sum_{n=-N_2}^{N_1} a_n \sum_{k=-\infty}^{\infty} \left[-\frac{X(p)\, e^{-nTp}}{\frac{d}{dp}\left(1-e^{-T(s-p)}\right)} \right]_{p=s+j\frac{2\pi k}{T}}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(s+j\frac{2\pi k}{T}\right) \sum_{n=-N_2}^{N_1} a_n\, e^{-nT\left(s+j\frac{2\pi k}{T}\right)}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(s+jk\omega_s) \sum_{n=-N_2}^{N_1} a_n\, e^{-nT(s+jk\omega_s)} .$$

Appendix 2.

In the same manner as the first degree,

(33) $$R(t) = a\, S(t) + b\, R(t-T)$$

$$= a \sum_{n=0}^{\infty} b^n S(t-nT)$$

Substituting (8) and (10) into (33),

(34) $$R(t) = a^2 \sum_{n=0}^{\infty} b^n \sum_{l=0}^{\infty} b^l y(t-\overline{n+l}\,T)$$

$$= a^2 \sum_{n=0}^{\infty} (n+1)\, b^n y(t-nT) .$$

Substituting (8), (10) and (34) into (14),

(35) $$z(t) = 2a \sum_{n=0}^{\infty} b^n y(t-nT) - a^2 \sum_{n=0}^{\infty} (n+1)\, b^n y(t-nT) .$$

Thus we obtain

(36) $$F(j\omega) = 2a \sum_{n=0}^{\infty} b^n e^{-jn\omega T} - a^2 \sum_{n=0}^{\infty} (n+1)\, b^n e^{-jn\omega T}$$

$$= \frac{2a}{1-b\,e^{-j\omega T}} - \left(\frac{a}{1-b\,e^{-j\omega T}}\right)^2 ,$$

$$(37)\qquad |F(j\omega)|=\frac{\sqrt{\{2a(1-b\cos\omega T)-a^2\}^2+(2ab\sin\omega T)^2}}{1-2b\cos\omega T+b^2},$$

$$(38)\qquad \arg F(j\omega)=\tan^{-1}\frac{2ab\sin\omega T}{2a(1-b\cos\omega T)-a^2}$$

$$-2\tan^{-1}\frac{b\sin\omega T}{1-b\cos\omega T}.$$

Appendix 3.

Similarly, we have

$$(39)\qquad Q(t)=a\,R(t)+b\,Q(t-T)$$

$$=a\sum_{n=0}^{\infty}b^n R(t-n\,T).$$

Substituting (34) into (39),

$$(40)\qquad Q(t)=a^3\sum_{n=0}^{\infty}b^n\sum_{l=0}^{\infty}(l+1)\,b^l\,y(t-\overline{n+l}\,T)$$

$$=a^3\sum_{n=0}^{\infty}\frac{(n+1)\,(n+2)}{2}\,b^n\,y(t-n\,T).$$

Substituting (8), (10), (34) and (40) into (17),

$$(41)\qquad z(t)=3a\sum_{n=0}^{\infty}b^n\,y(t-n\,T)-3a^2\sum_{n=0}^{\infty}(n+1)\,b^n\,y(t-n\,T)$$

$$+\,a^3\sum_{n=0}^{\infty}\frac{(n+1)\,(n+2)}{2}\,b^n\,y(t-n\,T)$$

Thus we obtain

$$(42)\qquad F(j\omega)=3a\sum_{n=0}^{\infty}b^n\,e^{-jn\omega T}-3\,a^2\sum_{n=0}^{\infty}(n+1)\,b^n\,e^{-jn\omega T}$$

$$+ a^3 \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} b^n e^{-jn\omega T}$$

$$= \frac{3a}{1-b\,e^{-j\omega T}} - 3\left(\frac{a}{1-b\,e^{-j\omega T}}\right)^2$$

$$+ \left(\frac{a}{1-b\,e^{-j\omega T}}\right)^3,$$

(43) $$|F(j\omega)| = \frac{\sqrt{\begin{array}{l}[3a\{1-2b\cos\omega T+b^2(\cos^2\omega T-\sin^2\omega T)\} \\ \quad -3a^2(1-b\cos\omega T)+a^3]^2 \\ \quad +\{6ab\sin\omega T(1-b\cos\omega T)-3a^2b\sin\omega T\}^2\end{array}}}{(1-2b\cos\omega T+b^2)^{3/2}},$$

(44) $$\arg F(j\omega) = \tan^{-1}\frac{6ab\sin\omega T(1-b\cos\omega T)-3a^2b\sin\omega T}{\begin{array}{l}3a\{1-2b\cos\omega T+b^2(\cos^2\omega T-\sin^2\omega T)\} \\ \quad -3a^2(1-b\cos\omega T)+a^3\end{array}}$$

$$-3\tan^{-1}\frac{b\sin\omega T}{1-b\cos\omega T}.$$

Appendix 4.

From (25) and (26),

(45) $$z(t) = \sum_{i=0}^{m}\left\{\sum_{n=1}^{N} y(t+\overline{n-N_1-1}\,T)\,g_i(n)\right\} g_i(N_1+1).$$

The Laplace transform of $z(t)$ is

(46) $$Z(s) = \sum_{i=0}^{m} g_i(N_1+1) \sum_{n=1}^{N} g_i(n) \frac{1}{2\pi j}$$

$$\times \int_{c-j\infty}^{c+j\infty} \frac{X(p)\,e^{(n-N_1-1)Tp}}{1-e^{-(s-p)T}}\,dp$$

$$= \sum_{i=0}^{m} g_i(N_1+1) \sum_{n=1}^{N} g_i(n)$$

$$\times \left[- \sum_{k=-\infty}^{\infty} \frac{X(p)\, e^{(n-N_1-1)Tp}}{\frac{d}{dp}\left(1-e^{-(s-p)T}\right)} \right]_{p=s+j\frac{2\pi k}{T}}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(s+j\,\frac{2\pi k}{T}\right) \sum_{i=0}^{m} g_i(N_1+1)$$

$$\times \sum_{n=1}^{N} g_i(n)\, e^{(n-N_1-1)\left(s+j\frac{2\pi k}{T}\right)T}$$

Considering the case of $k=0$, we obtain

(47) $$F(j\omega) = \sum_{i=0}^{m} g_i(N_1+1) \sum_{n=1}^{N} g_i(n)\, e^{j(n-N_1-1)\omega T}$$

$$= \sum_{i=0}^{m} g_i(N_1+1) \sum_{n=1}^{N} g_i(n) \cos\,(n-N_1-1)\,\omega T$$

$$-j \sum_{i=0}^{m} g_i(N_1+1) \sum_{n=1}^{N} g_i(n) \sin\,(n-N_1-1)\,\omega T$$

$$= u - jv\,,$$

(48) $$F(j\omega) = \sqrt{u^2+v^2}\,,$$

(49) $$\arg F(j\omega) = -\tan^{-1}\frac{v}{u}\,.$$