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AN ALGORITHM FOR THE ASYMPTOTIC PROBLEM OF AN INTEGER PROGRAM

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1. Introduction

Since Gomory has pointed out the significance of the asymptotic problem associated with a pure integer problem [1], a considerable effort has been made to establish its solution algorithms.

An asymptotic problem is

$$P: \quad \text{minimize} \quad \sum_{j=1}^{n} c_j x_j \tag{1}$$

$$\text{subject to} \quad \sum_{j=1}^{n} \alpha_j x_j = \beta$$

$$x_j: \quad \text{nonnegative integers,}$$

where $c_j \geq 0$, $j=1, 2, \cdots, n$. $\beta \in G$, and G is a finite module of order D, which is generated by $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ (see [1] [7] for details).

Basically two approaches are known to solve Problem P. One is the algorithm proposed by Gomory [1] and improved by White [8] and Mine-Narihisa [6], which is based on the dynamic programming recursion on j of x_j. The other (Shapiro [7], Hu [4], Greenberg [3] and Ibaraki [5]) is based on the shortest path in a directed graph associated with G.

In this note, we will consider Problem Q which is an asymptotic

problem P augmented with a set of constraints:

$$(2) \qquad 0 \leq x_j \leq b_j \quad \text{for} \quad j=1, 2, \cdots, n,$$

where b_j is a nonnegative integer and can take on ∞ if no restriction is imposed on x_j.

Constraints (2) are frequently encountered in practice, *e.g.*, integer programs with 0−1 variables x_j (*i.e.*, $b_j=1$). Another important example is found in the case where the asymptotic problems are considered as partial problems generated during the computation based on the branch-and-bound method (*e.g.*, [2]). Let the best value among the solutions thus far known be z^* and the value of the optimal solution of the present partial problem be z', then any solution x satisfying

$$(3) \qquad \sum_{j=1}^{n} c_j x_j > z^* - z'$$

is excluded from the further consideration. For x not to satisfy (3), we obtain

$$(4) \qquad x_j \leq (z^* - z')/c_j \quad \text{for all } j \text{ such that } c_j > 0.$$

These may be taken as constraints (2).

Problem Q was treated by Greenberg [3]. His algorithm is an extension of the algorithm for P based on the shortest path formulation. This note will point out that the Gomory's approach for P can also be extended to handle Q.

2. Recursion Formula for Problem Q

For $\alpha \in G$, let

$$F_s(\alpha) = \{(x_1, x_2, \cdots, x_s) \mid \sum_{j=1}^{s} a_j x_j = \alpha,\ 0 \leq x_j \leq b_j,\ x_j : \text{integer}\}$$

$$(5) \qquad \phi_s(\alpha) = \min \{\sum_{j=1}^{s} c_j x_j \mid (x_1, x_2, \cdots, x_s) \in F_s(\alpha)\}$$

$$O_s(\alpha) = \{x = (x_1, x_2, \cdots, x_s) \mid x \in F_s(\alpha),\ \sum_{j=1}^{s} c_j x_j = \phi_s(\alpha)\}\ .$$

First note that

$$(6)\qquad \phi_s(\alpha)=\min\{\phi_{s-1}(\alpha-k\,\alpha_s)+k\,c_s \mid k=0, 1, \cdots b_s\}$$

$$(7)\qquad (x_1^*, x_2^*, \cdots, x_{s-1}^*, x_s^*)\in O_s(\alpha)$$

hold, where

$$\phi_s(\alpha)=\phi_{s-1}(\alpha-x_s^*\,\alpha_s)+x_s^*\,c_s$$

$$(x_1^*, x_2^*, \cdots, x_{s-1}^*)\in O_{s-1}(\alpha-x_s^*\,\alpha_s).$$

(6) is proved as follows. Let $(x_1', x_2', \cdots, x_s')\in O_s(\alpha)$.
Then from the definition of $O_s(\alpha)$

$$\sum_{j=1}^{s-1} c_j\,x_j'+c_s\,x_s'\leq\sum_{j=1}^{s-1} c_j\,x_j+c_s\,x_s$$

for all $(x_1, x_2, \cdots, x_s)\in F_s(\alpha)$.

In particular, by letting $x_s=x_s'$ and $(x_1, x_2, \cdots, x_{s-1})\in O_{s-1}(\alpha-x_s'\,\alpha_s)$,

$$\sum_{j=1}^{s-1} c_j\,x_j'+c_s\,x_s'\leq\phi_{s-1}(\alpha-x_s'\,\alpha_s)+c_s\,x_s'$$

$$i.e.,\quad \sum_{j=1}^{s-1} c_j\,x_j'\leq\phi_{s-1}(\alpha-x_s'\,\alpha_s)$$

follows. But this implies $\sum_{j=1}^{s-1} c_j\,x_j'=\phi_{s-1}(\alpha-x_s'\,\alpha_s)$ from the definition of $\phi_{s-1}(\alpha-x_s'\,\alpha_s)$. This proves (6). (7) is also an immediate consequence of this argument.

Relation (6) yields

$$(8)\qquad \phi_s(\alpha-\alpha_s)+c_s\leq\phi_{s-1}(\alpha-k\alpha_s)+kc_s,\quad k=1, 2, \cdots, b_j.$$

By using this property, (6) and (7) can be simplified further.

$$(9)\qquad \phi_s(\alpha)=\min\{\phi_{s-1}(\alpha),\ \phi_s(\alpha-\alpha_s)+c_s\}$$

$$(10)\qquad (x_1^*, x_2^*, \cdots, x_{s-1}^*, x_s^*)\in O_s(\alpha)$$

where

$$\begin{cases} (x_1^*, x_2^*, \cdots, x_{s-1}^*) \in O_{s-1}(\alpha) \text{ and } x_s^* = 0 \\ \qquad\qquad \text{if } \psi_s(\alpha) = \psi_{s-1}(\alpha), \\ (x_1^*, x_2^*, \cdots, x_s^* - 1) \in O_s(\alpha - \alpha_s) \\ \qquad\qquad \text{otherwise,} \end{cases}$$

if $\psi_s(\alpha-\alpha_s)$ has been already calculated and if

$$x_s^* - 1 < b_s$$

holds.

Consequently, by applying the recursion formula (6) (or (9) if possible), we can calculate $\psi_s(\alpha)$ for all $\alpha \in G$ and for $s=1, 2, \cdots, n$ successively. Evidently $x \in O_n(\beta)$ and $\psi_n(\beta)$ are an optimal solution of Q and its value respectively.

The computation of $\psi_s(\alpha)$, $\alpha \in G$, for a fixed s is separately carried out for each coset defined by H_s, where H_s is the submodule of G generated by α_s. Let $K=\gamma+H_s$, $\gamma \in G$, be a coset of G and let $\alpha^* \in K$ satisfy

$$\psi_{s-1}(\alpha^*) = \min \{\psi_{s-1}(\alpha) \mid \alpha \in K\}. \tag{11}$$

Then from the fact that $\alpha^* - k\alpha_s \in K$ for $k=0, 1, \cdots, b_s$ and $c_j \geq 0$ in (6)

$$\psi_s(\alpha^*) = \psi_{s-1}(\alpha^*) \tag{12}$$

follows. This property is useful to increase the computational efficiency and was used by White [8] and Mine-Narihisa [6] for solving P.

3. Algorithm

The following algorithm will provide us with $x \in O_n(\beta)$ and $\psi_n(\beta)$ for a given Problem Q. In the algorithm, A is used to keep $\alpha \in G$ for which $\psi_s(\alpha)$ is not calculated yet. $\lambda_s(\alpha)$ (s-dimensional vector) represents a solution in $O_s(\alpha)$.

Step 1: Let

$$\psi_0(\alpha) = (D-1) \max c_j, \qquad \alpha \in G, \ \alpha \neq e$$

$$\psi_0(e) = 0,$$

where e is the zero element of module G. Let $s=1$, $A=G$, and go to Step 2.

Step 2:

(i) If $A=\phi$, go to Step 3; otherwise go to (ii).

(ii) Obtain $\alpha^* \in A$ such that

$$\phi_{s-1}(\alpha^*)=\min\{\phi_{s-1}(\alpha) \mid \alpha \in A\}. \tag{13}$$

Let $\phi_s(\alpha^*)=\phi_{s-1}(\alpha^*)$, $\lambda_s(\alpha^*)=(\lambda_{s-1}(\alpha^*), 0)$, $A \leftarrow A-\{\alpha^*\}$, and $\bar{\alpha}=\alpha^*$. Go to (iii).

(iii) If $\bar{\alpha}+\alpha_s=\alpha^*$, return to (i); otherwise go to (iv) if $[\lambda_s(\bar{\alpha})]_s+1>b_s$ or go to (v) if $[\lambda_s(\bar{\alpha})]_s+1 \leq b_s$, where $[\lambda_s(\bar{\alpha})]_s$ is the s-th element of $\lambda_s(\bar{\alpha})$.

(iv) For $\alpha'=\bar{\alpha}+\alpha_s$, calculate

$$\phi_s(\alpha')=\min\{\phi_{s-1}(\alpha'-k\alpha_s)+kc_s \mid k=0, 1, \cdots, b_s\} \tag{14}$$

$$\lambda_s(\alpha')=(\lambda_{s-1}(\alpha'-x_s^*\alpha_s), x_s^*)$$

where $\phi_s(\alpha')=\phi_{s-1}(\alpha'-x_s^*\alpha_s)+x_s^* c_s$.

Let $\bar{\alpha}=\alpha'$, $A \leftarrow A-\{\alpha'\}$ and return to (iii).

(v) For $\alpha'=\bar{\alpha}+\alpha_s$, calculate

$$\phi_s(\alpha')=\min\{\phi_{s-1}(\alpha'), \phi_s(\alpha'-\alpha_s)+c_s\}$$

$$\lambda_s(\alpha')=(x_1^*, x_2^*, \cdots, x_s^*) \tag{15}$$

where
$$\begin{cases} (x_1^*, x_2^*, \cdots, x_{s-1}^*)=\lambda_{s-1}(\alpha') \text{ and } x_s^*=0 \\ \qquad\qquad \text{if } \phi_s(\alpha')=\phi_{s-1}(\alpha') \\ (x_1^*, x_2^*, \cdots, x_{s-1}^*, x_s^*-1)=\lambda_s(\alpha'-\alpha_s) \\ \qquad\qquad \text{otherwise.} \end{cases}$$

Let $\bar{\alpha}=\alpha'$, $A \leftarrow A-\{\alpha'\}$ and return to (iii).

Step 3: If $s=n$, terminate the computation. Otherwise, increase s by

1, let $A=G$ and return to Step 2.

Let $R(\alpha_s)$ be the order of α_s. Then, for each s, the above algorithm requires $D(1-1/R(\alpha_s))$ operations of calculating (14) or (15), and $D/R(\alpha_s)$ operations of calculating (13). When $b_j=\infty$ for $j=1, 2, \cdots, n$, this algorithm reduces to that proposed by Gomory [1] and improved by White [8] and Mine-Narihisa [6].

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CORRIGENDA

Corrigenda to "Some extensions of dynamic economic lot size model with backlogging," Vol. 14, No. 1 (1971). In Figure-3 in page 11, the vertical axis c_i represents "marginal production cost".

Corrigenda to "General scheduling algorithms with applications to parallel scheduling and multiprogramming scheduling," Vol. 14, No. 2 (1971).

page	line	for	read
77	5	Determined	Determine
78	7, 20, 21	LB(f), UB(f)	LB(S), UB(S)
78	22, 24, 26, 28		
79	15, 14		
86	Fig. 12	d	d_2
88	Fig. 14	⑤	$⑤^{15}$
90	Fig. 15	⑤	⑤ (2, 5)
91	4	Algorithm 1,	Algorithm 1 or 2,
91	9	Min. f of	Min. f_o of
91	10	of f found	of f_o found
91	14, 15	f	f_o
92	1	f	f_o
92	27	1 or 2	1, or 2 except the first node
93	14, 29		