J. Operations Research Soc. of Japan
Vol. 14, Nos. 3 & 4, November 1971

WHEN TO STOP: A ZERO-SUM GAME MODEL

MINORU SAKAGUCHI

Faculty of Engineering Science
Osaka University

(Received April 30, 1971)

Abstract

This paper examines a class of optimal stopping problems in which two competitive players are involved. Let X_1, X_2, $\cdots$ be independent and identically distributed random variables that can be observed sequentially at cost c per observation. Two players I and II have the right to stop the sampling process: player I on the interval (a, ∞), player II on $(-\infty, b)$. The real numbers a and b are prescribed. If player I or II stops the process after observing X_n, then II pays $X_n - nc$ to I. The common distribution function of each X_i is assumed to be known to both players. We are required to derive optimal strategies for both players in this zero-sum game. Explicit results are obtained for the case where $c=0$ and N(=the maximum possible number of observations permitted to the players) is finite and the case where $c>0$ and $N=\infty$.

1. Introduction

Let X_1, X_2, $\cdots$ be independent and identically distributed random variables that can be observed sequentially at a cost of $c(\geqq 0)$ per observation. Let two real numbers a and b be given. Two players I and II have the right to stop the sampling process: player I on the interval

(a, ∞), player II on $(-\infty, b)$. If player I or II stops the process after observing X_n, then II pays $X_n - nc$ to I. Each player has his option to stop the process a total of $N-1$ times; if both players did not stop the process during the first $N-1$ times, II is obliged to pay the N-th observation minus Nc to the opponent player I. We assume that the common distribution function $F(x)$ of each observation X_i is known to both players and that $E\{|X_i|\}<\infty$. The problem is to construct optimal strategies for both players. This problem and the related variants when player II is not present, or equivalently, $a=b=-\infty$, have been studied by several authors, among whom Guttman [4], MacQueen and Miller [6], Sakaguchi [8], McCall [7], Gilbert and Mosteller [3], Taylor [9], and Hayes [5] have obtained explicit results which may have applications to management science or operations research. For general summaries of optimal stopping theory, see DeGroot [2], which is based upon the earlier work of Breiman [1]. The purpose of the present paper is to derive some explicit results for the two-person-game version of the problem. In Section 2, the simplest case where $a=b$ and $c=0$ is studied. The result generalizes the earlier obtained one. In Section 3, the case where $a \neq b$ and $c=0$ is discussed. In Section 4, we allow $N=\infty$ when $c>0$. The result obtained here generalizes the earlier known one.

2. The Simplest Case Where $a=b$ and $c=0$

Let $b \leqq a$. Let V_N represent the expected payoff to player I if an optimal procedure is employed by each player when a total of N observations are available. Let $\alpha(x)$ $(\beta(x))$ denote the probability that I(II) stops the process when x is observed at the first trial if $x \geqq a$ $(x \leqq b)$. Then clearly

$$V_N=\max_{\alpha(\cdot)}\min_{\beta(\cdot)}\left[\int_{-\infty}^{b}\{x\beta(x)+(V_{N-1}-c)(1-\beta(x))\}dF(x)\right.$$
$$+(V_{N-1}-c)\int_b^a dF(x)$$
$$\left.+\int_a^\infty\{x\alpha(x)+(V_{N-1}-c)(1-\alpha(x))\}dF(x)\right]$$
$$=V_{N-1}-c+\min_{\beta(\cdot)}\int_{-\infty}^{b}(x-V_{N-1}+c)\beta(x)dF(x)$$
$$+\max_{\alpha(\cdot)}\int_a^\infty(x-V_{N-1}+c)\alpha(x)dF(x).$$

The optimum expected payoff is achieved when

(1)
$$\alpha(x)=\begin{cases}1, & \text{if } x\geqq a\cup(V_{N-1}-c)\\ 0, & \text{if } a\leqq x<a\cup(V_{N-1}-c)\end{cases}$$
$$\beta(x)=\begin{cases}1, & \text{if } x\leqq b\cap(V_{N-1}-c)\\ 0, & \text{if } b\cap(V_{N-1}-c)<x\leqq b,\end{cases}$$

where we denote by $a\cup b$ the larger, and by $a\cap b$ the smaller of the numbers a and b. Thus V_N satisfies the recursion formula

(2)
$$V_N=V_{N-1}-c+\int_{-\infty}^{b\cap(V_{N-1}-c)}(x-V_{N-1}+c)dF(x)$$
$$+\int_{a\cup(V_{N-1}-c)}^{\infty}(x-V_{N-1}+c)dF(x)$$
$$(N=2,3,\cdots;\ V_1=\mu-c,\ \text{where } \mu=E[X]).$$

Consider the special case where $a=b$ and $c=0$. Then equation (2) becomes

(3)
$$V_N-V_{N-1}=\int_{-\infty}^{a\cap V_{N-1}}(x-V_{N-1})dF(x)+\int_{a\cup V_{N-1}}^{\infty}(x-V_{N-1})dF(x).$$

Let

(4)
$$T_F(z)\equiv E[(X-z)\cup 0]=\int_z^\infty(x-z)dF(x).$$

For any distribution function F with finite mean $\mu \equiv \int_{-\infty}^{\infty} x dF(x)$, T_F is non-negative, convex, and strictly decreasing on the set where it is positive. Furthermore $T_F(z) \geqq \mu - z$, $(-\infty < z < \infty)$, and

$$(5) \qquad \lim_{z \to -\infty} \{T_F(z) - (\mu - z)\} = 0, \quad \lim_{z \to \infty} T_F(z) = 0.$$

This function has been known to play an important role in optimal stopping problems (DeGroot [2]). Since

$$\int_{-\infty}^{z} (z-x)\, dF(x) = z - \mu - \int_{z}^{\infty} (z-x)\, dF(x) = z - \mu + T_F(z)$$

we have, from (3),

$$\begin{aligned} V_N = V_{N-1} - \int_{-\infty}^{V_{N-1}} (V_{N-1} - x)\, dF(x) + \int_{a}^{\infty} (x - a + a - V_{N-1})\, dF(x) \\ = \mu + T_F(a) - T_F(V_{N-1}) + (a - V_{N-1})(1 - F(a)) \end{aligned}$$

if $V_{N-1} \leqq a$. Hence

$$(6) \qquad V_N - \mu = \begin{cases} T_F(a) - T_F(V_{N-1}) + (a - V_{N-1})(1 - F(a)), & \text{if } V_{N-1} < a \\ 0, & \text{if } V_{N-1} = a \\ T_F(V_{N-1}) - T_F(a) + (V_{N-1} - a)(1 - F(a)), & \text{if } V_{N-1} > a. \end{cases}$$

In the righthand side of this equation, we find that the upper expression is negative and the lower, positive, and both are increasing with V_{N-1}, since $T_F(z)$ is convex and $\frac{d}{dz} T_F(z) = -(1 - F(z))$, $(a.e.)$. Therefore induction arguments give

$$(7) \qquad V_N - \mu = \begin{cases} T_F(V_{N-1}) - T_F(a) + (V_{N-1} - a)(1 - F(a)), & \text{if } a < \mu \\ 0, & \text{if } a = \mu \\ T_F(a) - T_F(V_{N-1}) + (a - V_{N-1})(1 - F(a)), & \text{if } a > \mu \end{cases}$$

and that:

$$\text{(8)}\qquad \begin{array}{l} \text{if } a<\mu \text{ then } a<\mu=V_1<V_2<\cdots \\ \text{if } a=\mu \text{ then } a=\mu=V_1=V_2=\cdots \\ \text{if } a>\mu \text{ then } a>\mu=V_1>V_2>\cdots \end{array}$$

Combining equations (1) with $a=b$ and $c=0$ with monotonicity in (8), the following is shown:

$$\text{(9)}\qquad \begin{array}{l} \text{if } a\leqq\mu, \text{ then} \quad \alpha_N{}^*(x)=\begin{cases} 0, & (a\leqq x<V_{N-1}) \\ 1, & (x\geqq V_{N-1}) \end{cases} ;\ \beta_N{}^*(x)\equiv 1,\ (x\leqq a) \\ \text{if } a\geqq\mu, \text{ then} \quad \alpha_N{}^*(x)\equiv 1,\ (x\geqq a);\ \beta_N{}^*(x)=\begin{cases} 1, & (x<V_{N-1}) \\ 0, & (V_{N-1}<x\leqq a) \end{cases} \end{array}$$

where the subscript N in the optimal strategies $\alpha^*(\cdot)$ and $\beta^*(\cdot)$ represents that there are N stages remaining before the first observation is sampled. We shall summarize the above results in the following theorem:

[**Theorem 1**] *In the case where* $a=b$ *and* $c=0$, *the value of the game is given by the recurrence formula* (7), *and an optimal strategy of each player is given by* (9).

A check of the following three special cases may be instructive. If $a=-\infty$, the game is a one-person game of player I. By use of (5), equations (7) and (9) give

$$\text{(10)}\qquad V_N=V_{N-1}+T_F(V_{N-1}),\qquad \alpha_N{}^*(x)=\begin{cases} 1, & \text{if } x\geqq V_{N-1} \\ 0, & \text{if otherwise.} \end{cases}$$

The sequence of the decision numbers $\{V_N\}_1{}^\infty$, is strictly increasing. If $a=\infty$, the game is a one-person game of player II. In this case we have

$$\text{(11)}\qquad (-V_N)=(-V_{N-1})+T_{\tilde F}(-V_{N-1}),\quad \beta_N{}^*(x)=\begin{cases} 1, & \text{if } x\leqq V_{N-1} \\ 0, & \text{if otherwise} \end{cases}$$

where $\tilde{F}$ is the distribution function of $-X$, *i.e.*, $\tilde{F}(x)=1-F(-x)$, and we have used the identity $\mu-z=T_F(z)-T_{\tilde{F}}(-z)$. $\{V_N\}$ is strictly decreasing with N. Finally if $a=\mu$, we have $V_N=\mu\,(N=1, 2, \cdots)$. The optimal play is terminated after one observation is sampled yielding the expected payoff μ even when $N\geqq 2$.

Although simple in principle, the calculation of V_N is generally complicated because of the fact that $T_F(z)$ is seldom representable in a closed form. We give three examples where closed-form expressions do exist, however, and these serve to illustrate the optimal play.

Example 1. Normal distribution: $\dfrac{dF(x)}{dx}=(2\pi)^{-1/2}\exp(-x^2/2)$.

Then

$$T_F(z)=\phi(z)-z\Phi(z)\ (\equiv\Psi(x),\ \text{say}), \tag{12}$$

where we have set

$$\phi(z)=(2\pi)^{-1/2}\exp(-z^2/2),\quad \Phi(z)=\int_z^\infty \phi(t)\,dt\,.$$

By using $\Psi(-x)=x+\Psi(x)$, we obtain from (6) the recursion formula

$$V_N=\begin{cases}-\phi(a)+V_{N-1}\Phi(a)+\Psi(V_{N-1}), & \text{if } a\leqq 0\\ \phi(a)-V_{N-1}\Phi(a)-\Psi(V_{N-1}), & \text{if } a\geqq 0.\end{cases}$$

Sometimes we denote V_N by $V_N(a)$ in order to emphasize that it is a function of a. We find that $V_N(0)=0$, and by the induction argument, that $V_N(a)+V_N(-a)=0$ for all N and a. In fact we get $V_1(a)\equiv 0$,

$$V_2(a)=\begin{cases}-\phi(a)+\phi(0), & \text{if } a\leqq 0\\ \phi(a)-\phi(0), & \text{if } a\geqq 0\end{cases}$$

$$V_3(a)=\begin{cases}-\phi(a)+(\phi(0)-\phi(a))\Phi(a)+\Psi(\phi(0)-\phi(a)), & \text{if } a\leqq 0\\ \phi(a)+(\phi(0)-\phi(a))\Phi(a)-\Psi(-(\phi(0)-\phi(a))), & \text{if } a\geqq 0.\end{cases}$$

$V_2(a)$ and $V_3(a)$ are both strictly decreasing (*see* Fig. 1.).

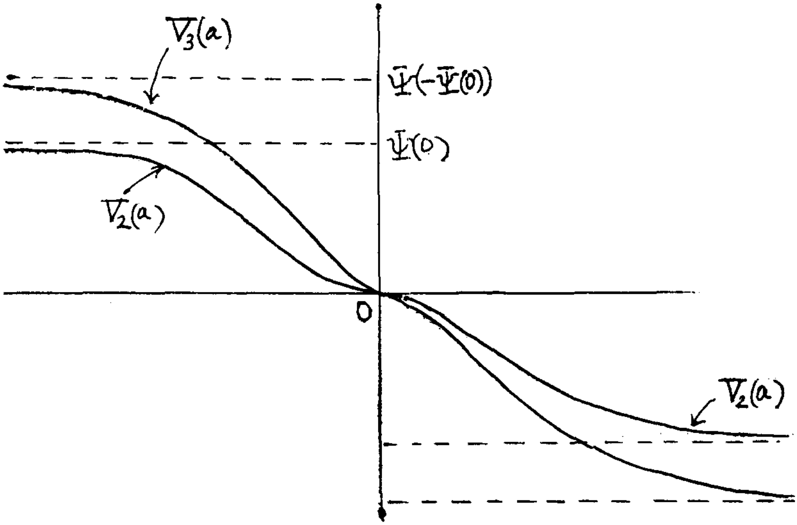

Fig. 1. Normal Distribution.

For $N=3$, $a>0$, the optimal play procedes as follows:

1st move — 2nd move — 3rd move

- If $a \leqq X_1$, I stops.
- If $X_1 \leqq -(\phi(0)-\phi(a))$, II stops.
- If $-(\phi(0)-\phi(a))<X_1<a$, observe X_2 ⟶
  - if $a \leqq X_2$, I stops.
  - if $X_2 \leqq 0$, II stops.
  - if $0<X_2<a$, observe X_3 ⟶
    - ⟶ stop with X_3.

Expected payoff $V_3(a)$.

Example 2. Uniform distribution: $F(x)=x$, $(0\leqq x\leqq 1)$. Then

$$(13)\qquad T_F(z)=\begin{cases}\dfrac{1}{2}-z, & z\leqq 0\\[2mm] \dfrac{1}{2}(1-z)^2, & 0\leqq z\leqq 1\\[2mm] 0, & z\geqq 1\end{cases}$$

and for $0\leqq a\leqq 1$

$$V_N-\frac{1}{2}=\begin{cases}\dfrac{1}{2}(a-V_{N-1})^2, & \text{if } a\leqq\mu\\[2mm] -\dfrac{1}{2}(a-V_{N-1})^2, & \text{if } a\geqq\mu.\end{cases}$$

Therefore

$$V_1(a)\equiv\frac{1}{2},\quad V_2(a)=\begin{cases}\dfrac{1}{2}+\dfrac{1}{2}\left(a-\dfrac{1}{2}\right)^2, & 0\leqq a\leqq\dfrac{1}{2}\\[2mm] \dfrac{1}{2}-\dfrac{1}{2}\left(a-\dfrac{1}{2}\right)^2, & \dfrac{1}{2}\leqq a\leqq 1.\end{cases}$$

$$V_3(a)=\begin{cases}\dfrac{1}{2}+\dfrac{1}{2}\left(a-\dfrac{1}{2}\right)^2\left\{1-\dfrac{1}{2}\left(a-\dfrac{1}{2}\right)\right\}^2, & 0\leqq a\leqq\dfrac{1}{2}\\[2mm] \dfrac{1}{2}-\dfrac{1}{2}\left(a-\dfrac{1}{2}\right)^2\left\{1+\dfrac{1}{2}\left(a-\dfrac{1}{2}\right)\right\}^2, & \dfrac{1}{2}\leqq a\leqq 1.\end{cases}$$

$V_2(a)$ and $V_3(a)$ are both strictly decreasing and symmetric about the center point $\left(\frac{1}{2},\ \frac{1}{2}\right)$ (*see* Fig. 2).

Example 3. Exponential distribution: $\dfrac{dF(x)}{dx}=e^{-x}\,(x>0)$. Then

$$(14)\qquad T_F(z)=\begin{cases}1-z, & z<0\\ e^{-z}, & z\geqq 0\end{cases}$$

and for $0\leqq a<\infty$

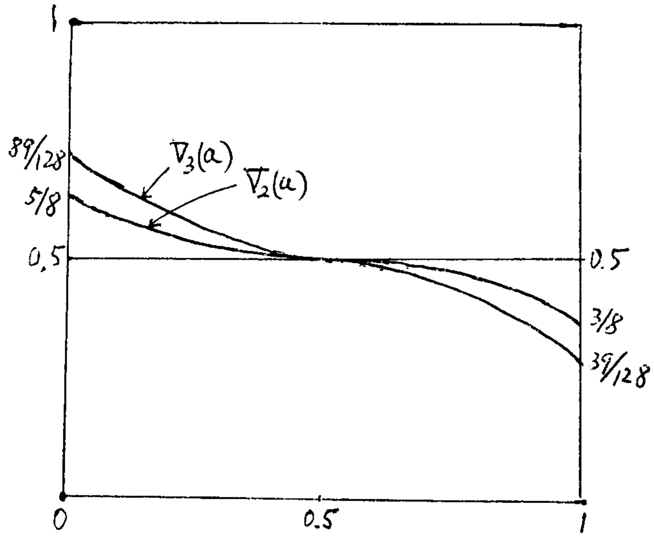

Fig. 2. Uniform Distribution.

$$V_N - 1 = \begin{cases} e^{-V_{N-1}} - e^{-a} + (V_{N-1} - a)\, e^{-a}, & \text{if } 0 \leqq a \leqq 1 \\ e^{-a} - e^{-V_{N-1}} + (a - V_{N-1})\, e^{-a}, & \text{if } a \geqq 1. \end{cases}$$

Hence we have

$$V_1(a) \equiv 1,\ \ V_2(a) = \begin{cases} 1 + e^{-1} - ae^{-a}, & 0 \leqq a \leqq 1 \\ 1 - e^{-1} + ae^{-a}, & a \geqq 1, \end{cases}$$

$$V_3(a) = \begin{cases} 1 + (e^{-1} - ae^{-a} - a)\, e^{-a} + e^{-(1 + e^{-1} - ae^{-a})}, & 0 \leqq a \leqq 1 \\ 1 + (e^{-1} - ae^{-a} + a)\, e^{-a} - e^{-(1 - e^{-1} + ae^{-a})}, & a \geqq 1. \end{cases}$$

$V_2(a)$ and $V_3(a)$ are both strictly decreasing (*see* Fig. 3).

3. The Case Where $a \neq b$ and $c = 0$

Let us denote the game analysed in the previous section by Γ_a. The subscript a represents that player I has the right to stop the process on the set (a, ∞), player II on the set $(-\infty, a)$. Suppose that, given two

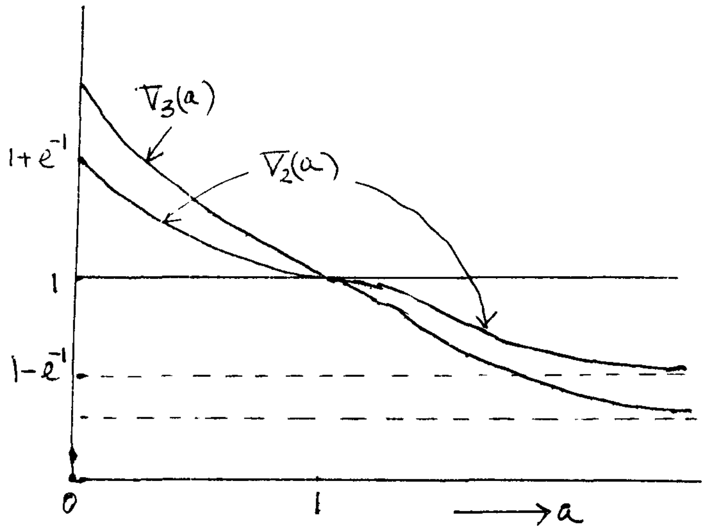

Fig. 3. Exponential Distribution.

numbers a and b with $a \neq b$, player I has the right to stop the process on the set $[a, \infty)$, player II on the set $(-\infty, b]$. Denote this game by $\Gamma_{a,b}$. The first subscript a is the lower limit of player I's 'stopping region', and the second subscript b is the upper limit of player II's 'stopping region'. If $b<a$, then the interval (b, a) constitutes the 'continuation region'. If $a<b$, then both players have the right to stop the process on the interval (a, b), and the next observation is sampled only when both of them did not stop the process. We discuss in this section the solution of the game $\Gamma_{a,b}$. Since we can follow the same line as in the previous section, we shall omit the details.

First let $b<a$. Replacing a in the first term of the right-hand side of equation (3) by b, we obtain

$$(15)\qquad V_N-\mu=\begin{cases} T_F(a)-T_F(V_{N-1})+(a-V_{N-1})\,(1-F(a))\,, & \text{if } V_{N-1}\leqq b \\ T_F(a)-T_F(b)+(V_{N-1}-b)(1-F(b))+(a-V_{N-1})(1-F(a)), & \text{if } b\leqq V_{N-1}\leqq a \\ T_F(V_{N-1})-T_F(b)+(V_{N-1}-b)\,(1-F(b)), & \text{if } V_{N-1}\geqq a \end{cases}$$

corresponding to (6). Hence we find that:

if $\mu\leqq b$, then (i) $V_N-\mu=T_F(a)-T_F(V_{N-1})+(a-V_{N-1})(1-F(a))$, (ii) $\{V_N\}$ is decreasing, and (iiii) $\alpha_N{}^*(x)\equiv 1\,(x\geqq a)$, $\beta_N{}^*(x)=\begin{cases}1, & (x<V_{N-1})\\ 0, & (V_{N-1}<x\leqq b)\end{cases}$;

if $\mu\geqq a$, then (i′) $V_N-\mu=T_F(V_{N-1})-T_F(b)+(V_{N-1}-b)\,(1-F(b))$, (ii′) $\{V_N\}$ is increasing, and (iii′) $\alpha_N{}^*(x)\equiv\begin{cases}0 & (a\leqq x<V_{N-1})\\ 1 & (x\geqq V_{N-1})\end{cases}$; $\beta_N{}^*(z)\equiv 1\ \ (x\leqq b)$.

This result is rewritten as the first statement in the following theorem:

[Theorem 2] *If* $\mu\leqq b<a$ $(b<a\leqq\mu)$ *then the game* $\Gamma_{a,b}$ *has the same solution as* $\Gamma_a(\Gamma_b)$. *If* $b<\mu<a$, *then the value of the game* $\Gamma_{a,b}$ *is given by the recurrence formula* (15), *and an optimal strategy of each player is given by*

$$(16)\qquad \alpha_N{}^*(x)=\begin{cases}0, & (a\leqq x<a\cup V_{N-1})\\ 1, & (x\geqq a\cup V_{N-1})\end{cases},\qquad \beta_N{}^*(x)=\begin{cases}1, & (x\leqq b\cap V_{N-1})\\ 0, & (b\cap V_{N-1}<x\leqq b)\,.\end{cases}$$

Next we shall consider the case where $a<b$. We show in the following that the results are not changed if we allow that $a<b$. We have in this case that

$$(17)\qquad V_N-V_{N-1}=\left\{\int_{\infty}^{a\cap V_{N-1}}+\int_{b\cup V_{N-1}}^{\infty}\right\}(x-V_{N-1})\,dF(x)$$
$$+\max_{\alpha(\cdot)}\ \min_{\beta(\cdot)}\int_a^b(x-V_{N-1})\,(\alpha(x)+\beta(x)-\alpha(x)\beta(x))dF(x)$$

analogously to (3). The optimal decisions made by each player are easily determined, except on the interval (a, b), as follows:

$$\text{(18)} \quad \alpha_N^*(x)=\begin{cases} 0, & (b\leqq x<b\cup V_{N-1}) \\ 1, & (x\geqq b\cup V_{N-1}) \end{cases}, \qquad \beta_N^*(x)=\begin{cases} 1, & (x\leqq a\cap V_{N-1}) \\ 0, & (a\cap V_{N-1}<x\leqq a). \end{cases}$$

This is (16) with a and b interchanged.

(**Lemma** 3) *Let a, b and g be given numbers, where $a<b$. A solution of the saddle value problem*

$$\max_{\phi(\cdot)}\ \min_{\psi(\cdot)} \int_a^b (x-g)\phi(x)\psi(x)dF(x),$$

subject to $0\leqq\phi(\cdot)\leqq1$ and $0\leqq\psi(\cdot)\leqq1$, *is given by*

$$\phi^*(x)=1-\psi^*(x)=\begin{cases} 0, & a\leqq x<b\cap g \\ 1, & b\cap g<x\leqq b \end{cases}, \quad \textit{Max-min value}=0.$$

Proof. A direct application of Neyman-Pearson lemma.

[**Theorem** 4] *If $\mu\leqq a<b\,(a<b\leqq\mu)$, then the game $\Gamma_{a,b}$ has the same solution as $\Gamma_a(\Gamma_b)$. If $a<\mu<b$, then the game $\Gamma_{a,b}$ has the same solution as Γ_μ.*

(**Proof**) From (17) and Lemma 3 we have

$$V_N - V_{N-1} = \left\{\int_a^b + \int_{-\infty}^{a\cap V_{N-1}} + \int_{b\cup V_{N-1}}^{\infty}\right\} (x-V_{N-1})\, dF(x)$$

and hence

$$\text{(19)} \qquad V_N-\mu=\begin{cases} T_F(a)-T_F(V_{N-1})+(a-V_{N-1})(1-F(a)), & \text{if } V_{N-1}<a \\ 0, & \text{if } a\leqq V_{N-1}\leqq b \\ T_F(V_{N-1})-T_F(b)+(V_{N-1}-b)(1-F(b)), & \text{if } V_{N-1}>b, \end{cases}$$

analogously to (6) and (15). From (17), (18) and Lemma 3 we find that

$$(20)\qquad \alpha_N{}^*(x)=\begin{cases}0, & (a\leqq x<b\cap V_{N-1})\\ 1, & (b\cap V_{N-1}<x<b)\\ 0, & (b<x<b\cup V_{N-1})\\ 1, & (x>b\cup V_{N-1})\end{cases}\ ;\quad \beta_N{}^*(x)=\begin{cases}1, & (x<a\cap V_{N-1})\\ 0, & (a\cap V_{N-1}<x<a)\\ 1, & (a<x<b\cap V_{N-1})\\ 0, & (b\cap V_{N-1}<x<b)\end{cases}$$

The rest of the proof is easily carried out using (19), (20) and Theorem 1, and will hence be omitted. Theorems 2 and 4 indicate that if $a\neq b$ then the only one important case is $b<\mu<a$. We show an example of this case.

Example 4. Normal distribution in Example 1: Let $b\leqq 0\leqq a$. Then from (15)

$$V_N=\begin{cases}\phi(a)-V_{N-1}\Phi(a)-\Psi(V_{N-1}), & \text{if } V_{N-1}\leqq b\\ \phi(a)-\phi(b)+V_{N-1}(\Phi(b)-\Phi(a)), & \text{if } b<V_{N-1}<a\\ -\phi(b)+V_{N-1}\Phi(b)+\Psi(V_{N-1}), & \text{if } V_{N-1}\geqq a\end{cases}$$

and hence $V_1(a,b)\equiv 0$, $V_2(a,b)=\phi(a)-\phi(|b|)$. With b fixed negative, we consider V_3 as a function of $a\geqq 0$. Denote by $\underline{a}(b)$ the unique positive root of the equation (in a)

$$\phi(a)-\phi(|b|)=a,$$

and define

$$\bar{a}(b)=\begin{cases}\text{the unique positive root of the equation } \phi(a)=\phi(|b|)-|b|, & -\gamma\leqq b<0\\ +\infty, \text{ if } b<-\gamma\end{cases}$$

where γ ($\fallingdotseq 0.37$) is the unique positive root of the equation $\phi(z)=z$, *i.e.* $\gamma=\underline{a}(-\infty)$. Clearly we have $\underline{a}(b)<|b|<\bar{a}(b)$. Then

$$V_3(a,\ b)=\begin{cases}-\phi(b)+(\phi(a)-\phi(b))\Phi(b)+\Psi(\phi(a)-\phi(b)), & \text{if } 0\leqq a\leqq\underline{a}(b)\\ (\phi(a)-\phi(b))(1+\Phi(b)-\Phi(a)), & \text{if } \underline{a}(b)<a<\bar{a}(b)\\ \phi(a)-(\phi(a)-\phi(b))\Phi(a)-\Psi(\phi(a)-\phi(b)), & \text{if } a\geqq\bar{a}(b).\end{cases}$$

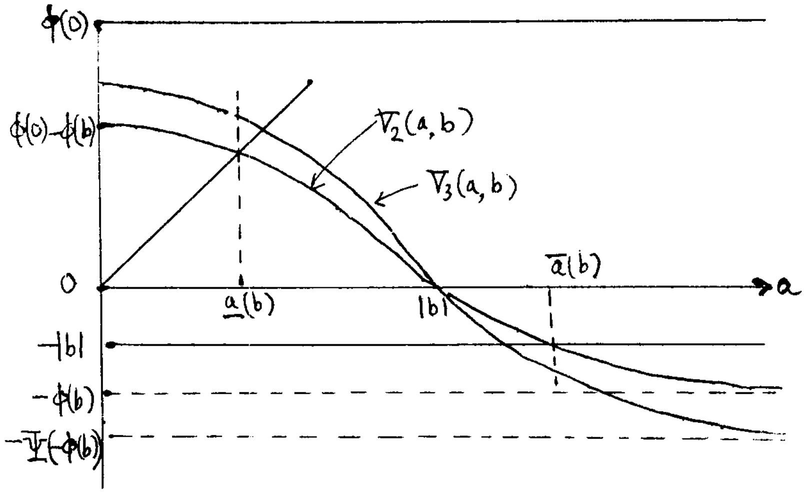

Fig. 4. $V_N(a, b)$ for Normal Distribution.

This function is continuous, strictly decreasing for $a \geqq 0$, and $V_2(a, b)\{\gtreqless\}V_3(a, b)$ if $(0<)a\{\gtreqless\}|b|$ (*see* Fig. 4). Note that $V_3(0, b)$ is equal to $V_3(a)$ in Example 1 with a replaced by b.

4. The Case Where $c>0$ and $N=\infty$

In this section we consider the case where cost c per observation is positive and the maximum possible number N of observations permitted to the players is infinite. For the case $a=b=-\infty$, that is, if player II is not involved in the problem, it is well known that an optimal rule is to stop sampling as soon as one gets an observation at least as large as v^*, where v^* is the unique solution of the equation

(21) $$T_F(v^*)=c.$$

If $v^* \geqq 0$, then the player's expected gain under the optimal procedure is also v^*. If $v^*<0$ then, of course, it is best for the player to do no sampling at all and have a gain of 0.

Now we set $b \leqq a$. Let us denote by $V(x)$ the expected payoff to player I under the condition that the value x is observed and the optimal procedures are employed by both players thereafter during the process. Then the function $V(x)$ satisfies the equation

$$V(x) = \begin{cases} \min\left(x, \int V(y)\,dF(y) - c\right), & x < b \\ \int V(y)\,dF(y) - c, & b \leqq x \leqq a \\ \max\left(x, \int V(y)\,dF(y) - c\right), & x > a, \end{cases}$$

where the integral signs without limits denote $\int_{-\infty}^{\infty}$.

Let $v = \int V(y)\,dF(y) - c$. Then

$$V(x) = \begin{cases} x \cap v, & \text{if } x < b \\ v, & \text{if } b \leqq x \leqq a \\ x \cup v, & \text{if } x > a. \end{cases} \tag{23}$$

Hence it follows by straightforward calculation of $\int V(y)\,dF(y)$, that

$$v = \begin{cases} \mu - c + T_F(a) - T_F(v) + (a-v)(1-F(a)), & \text{if } v < b \\ \mu - c + T_F(a) - T_F(b) + a(1-F(a)) - b(1-F(b)) + v(F(a)-F(b)), & \text{if } b \leqq v \leqq a \\ \mu - c + T_F(v) - T_F(b) + (v-b)(1-F(b)), & \text{if } v > a. \end{cases} \tag{24}$$

We denote by $h(v)$ the righthand side of the above equation which is the function of v, and moreover, we denote the first to third expressions there by $h_1(v)$, $h_2(v)$ and $h_3(v)$, respectively. We can easily find that $h_1(v)$ is increasing and convex, that $h_3(v)$ is increasing and convex and that $h(b) < \mu - c < h(a)$. Moreover $h(v)$ is continuously differentiable and $h_2(v)$

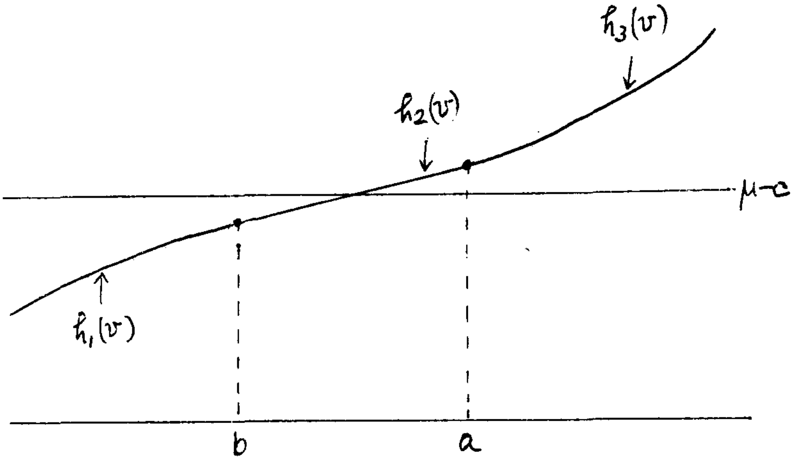

Fig. 5. Function $h(v)$.

is linearly increasing (*see* Fig. 5). Since $0<h_i'(v)<1$ $(i=1, 2, 3)$, it follows that equation (24) has a unique root v^*. This fact can be obtained by solving the equation

$$
(25) \qquad \begin{aligned} v&=h_1(v), && \text{if } h(b)<b \\ v&=h_2(v), && \text{if } h(b)>b \text{ and } h(a)<a \\ v&=h_3(v), && \text{if } h(a)>a \end{aligned}
$$

where $h(a)=\mu-c+T_F(a)-T_F(b)+(a-b)(1-F(b))$, and $h(b)=\mu-c+T_F(a)-T_F(b)+(a-b)(1-F(a))$.

We shall summarize the above result in the following

[Theorem 5] *Equation* (24) *has a unique root* v. *Denote this root by* v^*. *An optimal strategy for player I is to stop sampling as soon as he gets an observation at least as large as* $a\cup v^*$. *An optimal strategy for player II is to stop sampling as soon as he gets an observation at most as large as* $b\cap v^*$. *The expected payoff when the optimal strategy is employed* *by each player is also* v^*.

In the special case where $a=b$, the linear part $h_2(v)$ vanishes, and equations (25) may be rewritten as

$$(26)\qquad \begin{aligned} &v=\mu-c+T_F(a)-T_F(v)+(a-v)(1-F(a)), &&\text{if } \mu-c<a \\ &v=\mu-c+T_F(v)-T_F(a)+(v-a)(1-F(a)), &&\text{if } \mu-c>a. \end{aligned}$$

A check of the limiting case $a=-\infty$ will be instructive. The second equation of (26), along with (5), leads to equation (21) in this limiting case, which is well-known in the one-person-game version of our optimal stopping problem.

Example 5. Uniform distribution in Example 1: Let $0\leqq a<1$. Then, by (13) and (24), computation gives

$$h(v)=\begin{cases} \frac{1}{2}-c-\frac{1}{2}a^2+av, & \text{if } v<0 \\ \frac{1}{2}-c-\frac{1}{2}(a-v)^2, & \text{if } 0<v<a \\ \frac{1}{2}-c+\frac{1}{2}(a-v)^2, & \text{if } a<v<1 \\ \frac{1}{2}-c-\frac{1}{2}(1-a^2)+(1-a)v, & \text{if } v>1 \end{cases}$$

and hence (26) becomes:

if $\frac{1}{2}-c<a$, then

$$v=\begin{cases} \frac{1}{2}-c-\frac{1}{2}a^2+av, & \text{if } v<0 \\ \frac{1}{2}-c-\frac{1}{2}(a-v)^2, & \text{if } 0<v<a \end{cases}$$

if $\frac{1}{2}-c>a$, then

$$v=\frac{1}{2}-c+\frac{1}{2}(a-v)^2, \qquad \text{if } a<v<1.$$

Thus it follows that:

if $0\leqq a\leqq\frac{1}{2}$, then

$$v^*=\begin{cases}1+a-\sqrt{2(a+c)}, & \text{if } 0<c\leqq\frac{1}{2}-a\\ -(1-a)+\sqrt{2(1-a-c)}, & \text{if } \frac{1}{2}-a<c\leqq\frac{1}{2}(1-a^2)\\ \left\{\frac{1}{2}(1-a^2)-c\right\}/(1-a), & \text{if } c>\frac{1}{2}(1-a^2)\end{cases}$$

if $\frac{1}{2}\leqq a\leqq 1$, then

$$v^*=\begin{cases}-(1-a)+\sqrt{2(1-a-c)}, & \text{if } 0<c\leqq\frac{1}{2}(1-a^2)\\ \left\{\frac{1}{2}(1-a^2)-c\right\}/(1-a), & \text{if } c>\frac{1}{2}(1-a^2)\end{cases}$$

(*see* Fig. 6).

Note that if $c=\frac{1}{2}(1-a^2)$ the game is fair, that is, $v^*=0$.

Example 6. Exponential distribution in Example 3: Let $0\leqq a<\infty$. By (14) and (24), we obtain

$$h(v)=\begin{cases}1-c+(1+a)e^{-a}-1+v(1-e^{-a}), & \text{if } v<0\\ 1-c+(1+a)e^{-a}-ve^{-a}-e^{-v}, & \text{if } 0<v<a\\ 1-c-(1+a)e^{-a}+ve^{-a}+e^{-v}, & \text{if } v>a\end{cases}$$

and hence (26) becomes:

if $1-c<a$, then

$$v=\begin{cases}1-c+(1+a)e^{-a}-1+v(1-e^{-a}), & \text{if } v<0\\ 1-c+(1+a)e^{-a}-ve^{-a}-e^{-v}, & \text{if } 0<v<a\end{cases}$$

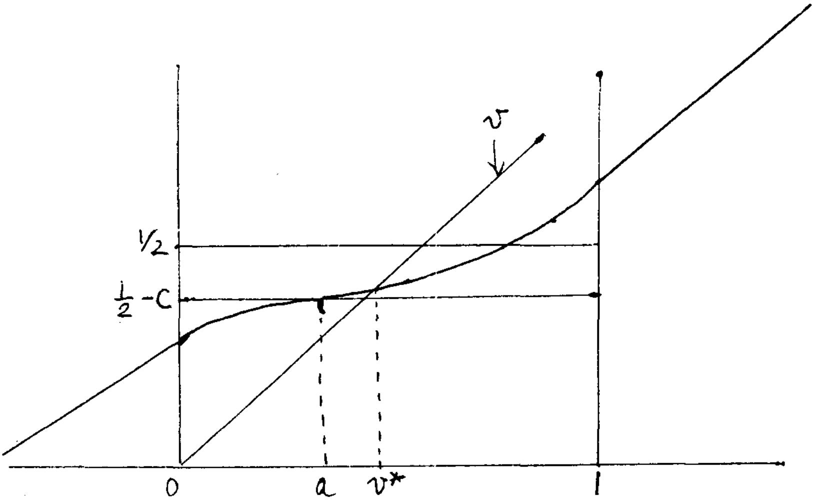

Fig. 6. Function $h(v)$ and the Value v^* for Uniform Distribution when $0 \leqq a \leqq \frac{1}{2}$ and $0 < c \leqq \frac{1}{2} - a$.

if $1-c>a$, then

$$v=1-c-(1+a)e^{-a}+ve^{-a}+e^{-v}, \qquad \text{if } v>a.$$

From these equations we can derive the value of v^* and optimal strategies for both players as functions of a and c. We omit the detail, only remarking that the game is fair, *i.e.* $v^*=0$, if $c=(1+a)e^{-a}$.

References

[1] Breiman, L. (1964), Stopping rule problems. Chap. 10 of *Applied Combinatorial Mathematics*, E.F. Beckenbach (ed.), Wiley, New York.

[2] DeGroot, M.H. (1970), *Optimal Statistical Decisions*, McGraw-Hill Book Company, New York, Chapter 13.

[3] Gilbert, J.P. and Mosteller, F. (1966), "Recognizing the maximum of a sequence," *J. Amer. Stat. Assoc.* **61**, 35-73.

[4] Guttman, I. (1960), "On a problem of L. Moser," *Canadian Math. Bull.* **3**, 35-39.

[5] Hayes, R.H. (1969), "Optimal strategies for divestiture," *Operations Res.* **17**, 292–310.

[6] MacQueen, J. and Miller, J.G., Jr. (1960), "Optimal persistence policies," *Operations Res.* 8, 362–380.

[7] McCall, J.J. (1965), "The economics of information and optimal stopping rules," *J. Business*, **38**, 300–317.

[8] Sakaguchi, M. (1961), "Dynamic programming of some sequential sampling design," *J. Math. Anal. Appl.* **2**, 446–466.

[9] Taylor, H. (1967), "Evaluating a call option and optimal timing in the stock market," *Manag. Sci.* (A) **14**, 111–120.