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QUEUEING SYSTEMS WITH GATES

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1. Introduction

In this paper we study $M/M/r$ type queueing systems with gates. In these models, customers arrive at a system in a Poisson process at a rate λ and have exponentially distributed service times with mean $1/\mu$. A customer who arrives at the system, at first joins in the first queue at the gate, and when the gate opens he goes to the second queue at the counter. Servers serve customers in the second queue only. The gate closes immediately after all the customers in the first queue go to the second queue.

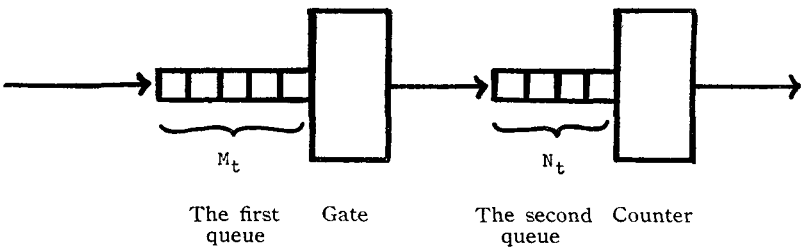

Fig. 1. A queueing system with a gate

Many types of gating rules can be considered. They may depend on the queue sizes, the total service time of the customers in the second queue, etc. But in this paper the gating rule is restricted to two types; (i) the deterministic gating rule, under which the gating intervals are constant and (ii) the exponential gating rule, under which the gating intervals are independent random variables with a common exponential distribution.

Queueing systems with the deterministic gating rule are similar to $D/M/r$ queueing systems with group arrivals and they can be analyzed with the same ideas. However queueing systems with gates generally cannot be analyzed by the ordinary techniques used for the analyses of queueing systems with group arrivals. The reason is that, in a queueing system with a gate, except for the case with the deterministic gating rule, the number of customers arriving at the system in a gating interval depends on the length of the gating interval. By the same reason, we can use queueing systems with gates as more suitable models than bulk queues for some congestion problems in a complex time sharing computer system.

We study single server queues with the exponential gating rule in Section 2, and with the deterministic gating rule in Section 3. The methods used in Section 2 are extended in Section 4 to study many server queues with the exponential gating rule.

2. Single Server Queues with the Exponential Gating Rule

2.1 Limiting Joint Distribution of Queue Sizes

Let λ, μ and η $(0<\lambda, \mu, \eta<\infty)$ be the parameters of the exponential distributions of inter-arrival times, service times and gating intervals, respectively. Denote the number of customers in the first queue at time t by M_t, and the number of customers in the second queue including ones being served at time t by N_t. Then the joint probability $Pr\{M_t=m, N_t=n\}$ $(m, n=0,1,2,\cdots)$ converges to a limit $p(m, n)$ as t tends to infinity, which is independent of the initial value (M_0, N_0) of the process. There are two cases; (i) $p(m, n)=0$ for all m and n, or (ii) $p(m, n)$ is the unique stationary initial distribution satisfying the system of equations

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p(m, n)=1 \tag{2.1}$$

and

$$0=-(\lambda+\mu+\eta)\ p(m, n)+\mu\ \{p(m, n+1)+\delta(n)\ p(m, n)\} + \lambda\ \{1-\delta(m)\}\ p(m-1, n)+\eta\delta(m) \sum_{k=0}^{n} p(k, n-k) \quad (m, n=0, 1, 2, \cdots), \tag{2.2}$$

where $\delta(0)=1$ and $\delta(x)=0$ for $x\neq 0$. As proved later, if $\lambda<\mu$ this system of equations has a unique solution, and if $\lambda\geqq\mu$ the only non-negative solution of (2.2) is $p(m, n)=0$ for all m and n, and it does not satisfy (2.1).

This system of equations can be expressed in equations of the generating function of $p(m, n)$. Denoting the generating function by

$$F(x, y)=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m y^n\ p(m, n) \qquad (0\leqq x,\ y\leqq 1), \tag{2.3}$$

(2.1) becomes as $F(1, 1)=1$, and (2.2) can be written as

$$0=-(\lambda+\mu+\eta)\ F(x, y)+\mu\left\{\frac{1}{y}[F(x, y)-F(x, 0)]+F(x, 0)\right\} +\lambda x F(x, y)+\eta F(y, y). \tag{2.4}$$

Rewriting (2.4) we obtain that

$$\{\lambda xy-(\lambda+\mu+\eta)\ y+\mu\}\ F(x, y)+\eta y F(y, y) =\mu(1-y)\ F(x, 0). \tag{2.5}$$

This equation includes $F(x, y)$ in three different forms; $F(x, y)$, $F(y, y)$ and $F(x, 0)$. To solve the equation, therefore, we have to eliminate two of them.

If we take $y=x$ and divide both sides by $(1-x)$ in (2.5), then we have a relation between $F(x, 0)$ and $F(x, x)$;

$$\mu F(x, 0)=(\mu-\lambda x)\ F(x, x). \tag{2.6}$$

If $\lambda>\mu$, for (2.6) being valid at $x=1$, $F(1, 1)$ must be zero. If $\lambda=\mu$, taking $x=1$ in (2.6), we have $F(1, 0)=0$. The monotonicity of $F(x, y)$ implies that $F(x, 0)=0$ for $0\leqq x<1$, and again from (2.6) it follows that $F(x, x)=0$ for $0\leqq x<1$. Thus if $\lambda\geqq\mu$, the only non-negative solution of (2.2) is $p(m, n)=0$ for all m and n. Hence in later discussions we assume that $\lambda<\mu$.

Combining (2.5) with (2.6) we obtain the equation

$$\begin{aligned}\{\lambda xy-(\lambda+\mu+\eta)\ y+\mu\}\ F(x, y)+\eta yF(y, y)\\ =(1-y)\,(\mu-\lambda x)\ F(x, x).\end{aligned} \tag{2.7}$$

Putting $G(x)=F(x, x)$ and substituting $y=\mu/(\lambda+\mu+\eta-\lambda x)$ in (2.7), it follows that

$$\mu\eta G\left(\frac{\mu}{\lambda+\mu+\eta-\lambda x}\right)=(\lambda+\eta-\lambda x)\,(\mu-\lambda x)\ G(x) \tag{2.8}$$

or

$$G(x)=g(x)\ G(f(x)) \tag{2.9}$$

where

$$f(x)=\frac{\mu}{\lambda+\mu+\eta-\lambda x}, \tag{2.10}$$

and

$$g(x)=\frac{\mu\eta}{(\lambda+\eta-\lambda x)\,(\mu-\lambda x)}. \tag{2.11}$$

For simplicity, we define the functions $f_n(x)$ and $g_n(x)$ recursively as follows:

$$\begin{cases} f_0(x)=x \\ f_n(x)=f(f_{n-1}(x)) & (n=1, 2, 3, \cdots)\end{cases} \tag{2.12}$$

and

$$g_n(x)=g(f_n(x)) \qquad (n=0,\ 1,\ 2,\ \cdots). \tag{2.13}$$

We can show that

$$\lim_{n\to\infty} f_n(x)=\alpha \tag{2.14}$$

and

$$\lim_{n\to\infty} g_n(x)=1 \tag{2.15}$$

for every $0\leqq x\leqq 1$, where α is the smaller one of two positive roots of $x=f(x)$ [see Appendix A].

Repeated use of the relation (2.9) shows that

$$G(x)=g_0(x)\ g_1(x)\ \cdots\ g_n(x)\ G(f_{n+1}(x)). \tag{2.16}$$

Since the product $\prod_{n=0}^{N} g_n(x)$ uniformly converges to a function $\prod_{n=0}^{\infty} g_n(x)$ in the interval [0, 1] [see Appendix B] and $G(x)=F(x,\ x)$ is continuous in [0, 1], the right side of (2.16) tends to

$$G(\alpha)\prod_{n=0}^{\infty} g_n(x). \tag{2.17}$$

The unknown constant $G(\alpha)$ is determined by the condition $G(1)=F(1,\ 1)=1$, and it is

$$G(\alpha)=1/\prod_{n=0}^{\infty} g_n(1). \tag{2.18}$$

Hence we have

$$G(x)=\prod_{n=0}^{\infty}\frac{g_n(x)}{g_n(1)}. \tag{2.19}$$

Therefore using this known $G(x)$ we can obtain $F(x,\ y)$ from (2.7);

$$F(x,\ y)=\frac{-\eta yG(y)+(1-y)(\mu-\lambda x)G(x)}{\lambda xy-(\lambda+\mu+\eta)y+\mu} \tag{2.20}$$

for $0 \leqq x,\ y \leqq 1,\ y \neq f(x)$, and

$$F(x, f(x)) = \frac{1-f(x)}{f(x)}[(\mu-\lambda x)\ G'(x) - \lambda G(x)] \tag{2.21}$$

for $y=f(x)$, to be continuous at points satisfying $y=f(x)$.

We have shown that a solution of the system of equations (2.1) and (2.2) must have the generating function $F(x, y)$ given by (2.20) and (2.21). Conversely, if $\lambda<\mu$ we can show that $F(x, y)$ given by (2.20) and (2.21) is a probability generating function and it determines a probability distribution satisfying (2.2). We can also show that the probability distribution has the moments of all orders. Since it is clear that $F(x, y)$ satisfies the equation (2.4) and the condition $F(1, 1)=1$, the only fact to be proved is that $F(x, y)$ has non-negative derivatives of all orders in [0, 1], [see Feller [1], p. 221]. The proof of it is postponed to Appendix C.

Finally we shall calculate the means and the variances of the queue sizes in the limiting distributions. Since the generating function of the total number of the customers in the system is $F(x, x)=G(x)$, and that of the number of the customers in the second queue (including one being served) is $F(1, x)$, the following results can be easily obtained.

The mean and the variance of the total number of customers in the system are as follows.

$$E(M+N) = G'(1), \tag{2.22}$$

$$Var(M+N) = G''(1) + G'(1) - \{G'(1)\}^2. \tag{2.23}$$

The mean and the variance of the number of customers in the second queue including one being served are as follows.

$$E(N) = G'(1) - \frac{\lambda}{\eta}, \tag{2.24}$$

$$Var(N) = G''(1) + G'(1) - \{G'(1)\}^2 + \frac{\lambda}{\eta^2}(2\mu+\eta-\lambda) - \frac{2(\mu-\lambda)}{\eta}G'(1). \tag{2.25}$$

For the numerical values, see Figs. 4 and 5.

2.2 Limiting Sojourn Time Distribution

We shall calculate the limiting sojourn time distribution under the "first come, first served" queue discipline. Let W be the sojourn time in the system (i.e. the sum of the waiting time and the service time) of a customer who arrives at the system in the steady state at time t.

First we shall calculate the conditional distribution of W, assuming that the numbers of customers in the first queue and in the second queue including one being served just before t are m and n, respectively. We denote the first time when the gate opens after t by $t+\tau$. Then the event $\{W \leqq w\}$ can be divided into two events; the first case that the server has served $k(<n)$ customers in the time interval $(t, t+\tau]$ and served more than or equal to $m+n+1-k$ customers in $(t+\tau, t+w]$, and the second case that he has served all n customers in the second queue in $(t, t+\tau]$ and served more than or equal to $m+1$ customers in $(t+\tau, t+w]$. Summing up these events for possible values of τ and k, we obtain that

$$(2.26)\qquad Pr\{W \leqq w | M_t = m,\ N_t = n\}$$
$$= \int_0^w e^{-\eta\tau} \left[\sum_{k=0}^{n-1} \frac{(\mu\tau)^k}{k!} e^{-\mu\tau} E_{m+n+1-k}(w-\tau) + \sum_{k=n}^{\infty} \frac{(\mu\tau)^k}{k!} e^{-\mu\tau} E_{m+1}(w-\tau) \right] d\tau$$

where $E_k(x)$ is the distribution function of Erlang distribution of order k with mean k/μ. Hence the conditional probability density function of W is

$$(2.27)\qquad v(w|m,\ n) = \int_0^w e^{-\eta\tau} \left[\sum_{k=0}^{n-1} \frac{\mu^{m+n+1}\, \tau^k (w-\tau)^{m+n-k}}{k!\ (m+n-k)!} e^{-\mu w} + \sum_{k=n}^{\infty} \frac{\mu^{m+k+1}\, \tau^k (w-\tau)^m}{k!\ m!} e^{-\mu w} \right] d\tau$$

and its Laplace transform is

$$(2.28)\qquad \varphi(s|m, n)=\int_0^\infty e^{-sw}\, v(w|m, n)\, dw$$
$$=\left(\frac{\mu}{s+\mu}\right)^{m+1}\left[\left(\frac{\mu}{s+\mu}\right)^n-\frac{s}{s+\eta}\left(\frac{\mu}{s+\mu+\eta}\right)^n\right].$$

Multiplying the both sides of (2.28) by $p(m, n)$ and summing up for all m and n, we obtain the Laplace transform of the limiting sojourn time distribution;

$$(2.29)\qquad \varphi(s)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\varphi(s|m, n)\; p(m, n)$$
$$=\frac{\mu}{s+\mu}F\left(\frac{\mu}{s+\mu}, \frac{\mu}{s+\mu}\right)-\frac{s}{s+\eta}F\left(\frac{\mu}{s+\mu}, \frac{\mu}{s+\mu+\eta}\right)$$
$$=G\left(1-\frac{s}{\lambda}\right).$$

The mean and the variance of the limiting sojourn time distribution can be obtained as follows.

The mean sojourn time in the system is

$$(2.30)\qquad E(W)=\frac{1}{\lambda}G'(1)\qquad\qquad \left(=\frac{1}{\lambda}E(M+N)\right).$$

The variance of the sojourn time is

$$(2.31)\qquad Var(W)=\frac{1}{\lambda^2}[G''(1)-\{G'(1)\}^2]$$
$$\left(=\frac{1}{\lambda^2}[Var(M+N)-E(M+N)]\right).$$

For the numerical values, see Figs. 2 and 3.

3. Single Server Queues with the Deterministic Gating Rule

3.1 Limiting Distribution of the Number of Customers in the System

Since the lengths of the gating intervals in this model are constant, we may assume without loss of generality that the gate opens at times $t=0, 1, 2, \cdots$. We call the time interval from $t-1$ to t, period t, and denote the number of customers in the first queue at the end of period t by M_t, and that in the second queue including one being served at that time by N_t. As M_t denotes the number of arrivals in period t, M_t $(t=1, 2, 3, \cdots)$ are mutually independent random variables and have the common Poisson distribution with mean λ. Let X_t denote the possible number of customers whom the server can serve in period t assuming there are sufficiently many customers in the second queue. Then M_t's and X_t's are independent of each other, and the mutually independent random variables X_t $(t=1, 2, 3, \cdots)$ have the common Poisson distribution with mean μ. If the initial condition is that $(M_0, N_0)=(m_0, n_0)$, then N_t's are determined by the relation

$$(3.1) \qquad N_{t+1}=[N_t+M_t-X_{t+1}]^+ \qquad (t=0, 1, 2, \cdots),$$

where $[x]^+=x$ if $x\geqq 0$ and $=0$ if $x<0$.

The relation (3.1) has the same form as the well known equation for the waiting times in a $GI/G/1$ type queue. Hence we can easily obtain the following results [see Kingman [2]].

If $\lambda\geqq\mu$, $\lim_{t\to\infty} Pr\{N_t=n\}=0$ for all n, and if $\lambda<\mu$, the limiting distribution $p(n)$ of N_t exists and its probability generating function is given by

$$(3.2) \qquad F(y)=\sum_{n=0}^{\infty} y^n\, p(n) = \exp\left\{-\sum_{n=1}^{\infty}\frac{1}{n}E(1-y^{[U_n]^+})\right\},$$

where

$$(3.3) \qquad U_n=\sum_{t=1}^{n}(M_t-X_{t+1}) \qquad (n=1, 2, 3, \cdots).$$

Therefore the mean and the variance of the limiting distribution of the number of customers in the second queue including ones being served at the end of a period are given by

$$(3.4)\qquad E(N)=\sum_{n=1}^{\infty}\frac{1}{n}E([U_n]^+)$$

and

$$(3.5)\qquad Var(N)=\sum_{n=1}^{\infty}\frac{1}{n}E(\{[U_n]^+\}^2).$$

For the numerical values, see Figs. 4 and 5.

3.2 Limiting Sojourn Time Distribution

Here we shall again assume the "first come, first served" queue discipline. Let W be the sojourn time of a customer in the system. The distribution of W under the condition that the customer arrives at the system at time $t-\tau$, where t is an integer and $0\leqq\tau<1$, and that $N_t=n$, is given by

$$(3.6)\qquad Pr\{W\leqq w|\tau,\ N_t=n\}$$
$$=\sum_{m=0}^{\infty}E_{m+n+1}(w-\tau)\frac{\{\lambda(1-\tau)\}^m}{m!}e^{-\lambda(1-\tau)}\qquad (w\geqq\tau)$$

where $E_k(x)$ is the distribution function of the Erlang distribution of order k with mean k/μ. Therefore the conditional probability density function of W is

$$(3.7)\qquad v(w|\tau,\ n)=\begin{cases}\displaystyle\sum_{m=0}^{\infty}\frac{\mu^{m+n+1}\{\lambda(1-\tau)\}^m(w-\tau)^{m+n}}{m!\ (m+n)!}e^{-\mu(w-\tau)-\lambda(1-\tau)} & (w\geqq\tau)\\ 0 & (w<\tau)\end{cases}$$

and its Laplace transform is

$$(3.8)\qquad \varphi(s|\tau,\ n)=\int_0^{\infty}e^{-sw}v(w|\tau,n)\,dw$$
$$=\left(\frac{\mu}{s+\mu}\right)^{n+1}e^{-s\lambda(1-\tau)/(s+\mu)-s\tau}.$$

Hence the Laplace transform of the limiting sojourn time distribution is

(3.9)
$$\varphi(s)=\sum_{n=0}^{\infty}p(n)\int_0^1\varphi(s|\tau,n)\,d\tau$$
$$=\frac{\mu}{s+\mu}F\left(\frac{\mu}{s+\mu}\right)\cdot\frac{s+\mu}{s(s+\mu-\lambda)}\left(e^{-s/(s+\mu)}-e^{-s}\right).$$

The mean and the variance of it are given by

(3.10)
$$E(W)=\frac{1}{\mu}\left\{E(N)+\frac{\lambda}{2}+1\right\}+\frac{1}{2}$$

and

(3.11)
$$Var\ (W)=\frac{1}{\mu^2}\{Var(N)+E(N)+\lambda+1\}+\frac{1}{12}\left(1-\frac{\lambda}{\mu}\right)^2.$$

For the numerical values, see Figs. 2 and 3.

4. Many Server Queues with the Exponential Gating Rule

4.1 Limiting Joint Distribution of the Queue Sizes

In the case of exponential gating rule, the technique used in Section 2 can be applied to many server queues. In this section we omit the discussion concerning the existence of the limiting distribution, and restrict ourselves to discuss how to find the limiting distribution.

Let λ, μ and η be the parameters of the exponential distributions of inter-arrival times, service times and gating intervals, respectively. There are r servers, and for the existence of the proper limiting distribution we assume that $\lambda<r\mu$. We denote the number of customers in the first queue at time t by M_t, and the number of customers in the second queue including ones being served at time t by N_t. Then the limiting distribution

(4.1)
$$p(m,\ n)=\lim_{t\to\infty}Pr\ \{M_t=m,\ N_t=n\}\qquad(m,\ n=0,\ 1,\ 2,\ \cdots)$$

satisfies the system of equations

$$(4.2)\qquad 0=-(\lambda+\eta)\,p(m,n)+\lambda\{1-\delta(m)\}\,p(m-1,n)+\eta\delta(m)\sum_{k=0}^{n}p(k,n-k)-\mu\left\{r-\sum_{k=0}^{r-1}(r-k)\,\delta(n-k)\right\}p(m,n)+\mu\left\{r-\sum_{k=1}^{r-1}(r-k)\,\delta(n-k-1)\right\}p(m,n+1)$$

where $\delta(0)=1$ and $\delta(x)=0$ for $x\neq 0$.

(If $\lambda\geqq r\mu$, the only non-negative solution of (4.2) is $p(m,n)=0$ for all m and n, and if $\lambda<r\mu$, it has a unique solution satisfying the condition that $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}p(m,n)=1$. These results can be proved by analogous methods as in Section 2. Hence the proofs are omitted here.)

Let the probability generating function of $p(m,n)$ be denoted by

$$(4.3)\qquad F(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}x^m y^n p(m,n)\qquad (0\leqq x,\ y\leqq 1).$$

Using the relation

$$(4.4)\qquad \sum_{m=0}^{\infty}x^m p(m,n)=\frac{1}{n!}\left[\frac{\partial^n}{\partial y^n}F(x,y)\right]_{y=0}$$

(4.2) can be rewritten in terms of $F(x,y)$ as

$$(4.5)\qquad 0=-(\lambda+\eta)F(x,y)+\lambda xF(x,y)+\eta F(y,y)-r\mu F(x,y)+\mu\sum_{k=0}^{r-1}\frac{r-k}{k!}y^k\left[\frac{\partial^k}{\partial y^k}F(x,y)\right]_{y=0}+r\mu\frac{1}{y}[F(x,y)-F(x,0)]-\mu\sum_{k=1}^{r-1}\frac{r-k}{k!}y^{k-1}\left[\frac{\partial^k}{\partial y^k}F(x,y)\right]_{y=0}$$

To simplify the notations, we define

$$(4.6)\qquad \begin{cases}F^{(0)}(x,0)=F(x,0)\\ F^{(k)}(x,0)=\left[\dfrac{\partial^k}{\partial y^k}F(x,y)\right]_{y=0}\qquad (k=1,2,3,\cdots).\end{cases}$$

Then (4.5) can be written as

$$\{\lambda xy-(\lambda+\eta+r\mu)\,y+r\mu\}\;F(x,y)+\eta yF(y,y)=(1-y)\,\mu\sum_{k=0}^{r-1}y^k\,F^{(k)}(x,0)\,. \tag{4.7}$$

This is a generalized form of (2.5). In Section 2, three unknown functions are $F(x,y)$, $F(y,y)$ and $F(x,0)$. At first, taking $y=x$, we obtain the relation (2.6) between $F(y,y)$ and $F(x,0)$, and then substituting $y=\mu/(\lambda+\mu+\eta-\lambda x)$ we get the main functional equation (2.9). However in this case we have $r+2$ unknown functions, $F(x,y)$, $F(y,y)$, $F^{(0)}(x,0)$, $\cdots$, $F^{(r-1)}(x,0)$. In order to determine the form of $F(x,y)$, we shall first get the relations among $F^{(k)}(x,0)$'s, and write them in terms of $F^{(0)}(x,0)=F(x,0)$. Then we shall use the same technique as in Section 2 to get the functional equation of $G(x)=F(x,x)$.

Differentiate (4.7) k times ($k=1,2,\cdots,r-1$), with respect to y and put $y=0$, then

$$F^{(k)}(x,0)=\frac{1}{\mu}\,(\lambda+\eta+(k-1)\,\mu-\lambda x)\,F^{(k-1)}(x,0)-\frac{\eta}{\mu}\,a_k \tag{4.8}$$

where

$$\begin{cases}a_1=F(0,0)\\ a_{k+1}=\left[\dfrac{d^k}{dy^k}\;F(y,y)\right]_{y=0} & (k=2,3,\cdots,r-1)\,.\end{cases} \tag{4.9}$$

Hence $F^{(k)}(x,0)$ is written in terms of $F(x,0)=F^{(0)}(x,0)$ such as

$$F^{(k)}(x,0)=F(x,0)\prod_{j=1}^{k-1}\frac{1}{\mu}\,(\lambda+\eta+j\mu-\lambda x)-\frac{\eta}{\mu}\sum_{i=1}^{k-1}a_i\prod_{j=i}^{k-1}\frac{1}{\mu}\,(\lambda+\eta+j\mu-\lambda x)-\frac{\eta}{\mu}\,a_k\,. \tag{4.10}$$

Substituting the right hand side of (4.10) for $F^{(k)}(x,0)$ in (4.7), we obtain an equation including only three unknown functions, $F(x,y)$, $F(y,y)$ and $F(x,0)$ as

$$(4.11)\quad \{\lambda xy-(\lambda+\eta+r\mu)y+r\mu\}F(x,y)+\eta yF(y,y)$$
$$=(1-y)\mu F(x,0)\left[r+\sum_{k=1}^{r-1}\frac{r-k}{k!}y^k\prod_{j=0}^{k-1}\frac{1}{\mu}(\lambda+\eta+j\mu-\lambda x)\right]$$
$$-(1-y)\eta\left[(r-1)a_1y+\sum_{k=2}^{r-1}\frac{r-k}{k!}y^k\sum_{i=1}^{k-1}a_i\prod_{j=i}^{k-1}\frac{1}{\mu}(\lambda+\eta+j\mu-\lambda x)\right.$$
$$\left.+\sum_{k=2}^{r-1}\frac{r-k}{k!}y^ka_k\right]$$
$$=(1-y)\mu F(x,0)B(x,y)-(1-y)\eta\sum_{i=1}^{r-1}a_iA_i(x,y),$$

where

$$(4.12)\quad B(x,y)=r+\sum_{k=1}^{r-1}\frac{r-k}{k!}y^k\sum_{j=0}^{k-1}\frac{1}{\mu}(\lambda+\eta+j\mu-\lambda x)$$

and

$$(4.13)\quad A_i(x,y)=\frac{r-i}{i!}y^i+\sum_{k=i+1}^{r-1}\frac{r-k}{k!}y^k\prod_{j=i}^{k-1}\frac{1}{\mu}(\lambda+\eta+j\mu-\lambda x)$$
$$(i=1,2,\cdots,(r-2).$$

Now we use the same technique as in Section 2. In (4.11), we take $y=x$ and divide it by $(1-x)$, then we have

$$(4.14)\quad (r\mu-\lambda x)F(x,x)=\mu F(x,0)B(x,x)-\eta\sum_{i=1}^{r-1}a_iA_i(x,x).$$

Solving $F(x,0)$ from the above equation and substituting it in (4.11), we have

$$(4.15)\quad \{\lambda xy-(\lambda+\eta+r\mu)y+r\mu\}F(x,y)+\eta yF(y,y)$$
$$=(1-y)(r\mu-\lambda x)\frac{B(x,y)}{B(x,x)}F(x,x)$$
$$-\eta(1-y)\sum_{i=1}^{r-1}a_i\left\{A_i(x,y)-\frac{B(x,y)}{B(x,x)}A_i(x,x)\right\}.$$

As in Section 2, we define

$$(4.16)\quad f(x)=\frac{r\mu}{\lambda+\eta+r\mu-\lambda x}$$

and

(4.17) $$G(x)=F(x,x).$$

Taking $y=f(x)$, (4.15) will be reduced to

(4.18) $$\eta f(x)G(f(x))=(1-f(x))(r\mu-\lambda x)\frac{B(x,f(x))}{B(x,x)}G(x)$$
$$-\eta(1-f(x))\sum_{i=1}^{r-1}a_i\left\{A_i(x,f(x))-\frac{B(x,f(x))}{B(x,x)}A_i(x,x)\right\}.$$

It follows that

(4.19) $$G(x)=\frac{\eta f(x)}{(1-f(x))(r\mu-\lambda x)}\cdot\frac{B(x,x)}{B(x,f(x))}G(f(x))$$
$$+\frac{\eta}{r\mu-\lambda x}\sum_{i=1}^{r-1}a_i\left\{\frac{B(x,x)}{B(x,f(x))}A_i(x,f(x))-A_i(x,x)\right\}$$
$$=g(x)G(f(x))+\sum_{i=1}^{r-1}a_i h^i(x),$$

where

(4.20) $$g(x)=\frac{\eta f(x)}{(1-f(x))(r\mu-\lambda x)}\cdot\frac{B(x,x)}{B(x,f(x))}$$

and

(4.21) $$h^i(x)=\frac{\eta}{r\mu-\lambda x}\left\{\frac{B(x,x)}{B(x,f(x))}A_i(x,f(x))-A_i(x,x)\right\}$$
$$(i=1,2,\cdots,r-1).$$

Now we define $r+1$ sequences of functions recursively as follow:

(4.22) $$\begin{cases} f_0(x)=x & \\ f_n(x)=f(f_{n-1}(x)) & (n=1,2,3,\cdots)\end{cases}$$

(4.23) $$g_n(x)=g(f_n(x)) \qquad (n=0,1,2,\cdots)$$

(4.24) $$h_n^i(x)=h^i(f_n(x)) \qquad (i=1,2,\cdots,r-1)$$
$$(n=0,1,2,\cdots).$$

Then clearly, we have that for every $0\leqq x\leqq 1$,

$$(4.25)\qquad f_n(x)\rightarrow\alpha=\frac{1}{2\lambda}\left[\lambda+\eta+r\mu-\sqrt{(\lambda+\eta+r\mu)^2-4r\lambda\mu}\right]$$

$$(4.26)\qquad g_n(x)\rightarrow g(\alpha)=1$$

$$(4.27)\qquad h_n^i(x)\rightarrow h^i(\alpha)=0\qquad (i=1, 2, \cdots, r-1)$$

as n increases to infinity.

From (4.19), using these functions, $G(x)$ can be written as

$$(4.28)\qquad G(x)=G(\alpha)\prod_{n=0}^{\infty} g_n(x)+\sum_{i=1}^{r-1} a_i\left[h^i(x)+\sum_{n=1}^{\infty} h_n^i(x)\prod_{k=0}^{n-1} g_k(x)\right].$$

Though this expression has r unknown parameters, a_1, a_2, $\cdots$, a_{r-1} and $G(\alpha)$, they can be determined by the simultaneous linear equations which consist of (4.9) and $G(1)=1$. Thus we have obtained $F(x, x)=G(x)$, and also $F(x, y)$ from (4.15).

The above solution is rather complicated, but for small r, we would be able to calculate the means and the variances with the help of computer.

5. Numerical Examples

Some numerical examples of single server queues with gates under the exponential or the deterministic gating rules are represented in Figs. 2~5.

Figs. 2 and 3 show the mean and the variance of the sojourn time in the system in the steady state for various values of the relative traffic intensity $\rho=\lambda/\mu$ and of the mean length $1/\eta$ of gating intervals. The graphs in solid lines represent the exponential case and those in broken lines represent the deterministic case.

Figs. 4 and 5 show the mean and the variance of the number of customers in the second queue in the steady state immediately after the gate opens. The same notations as in Figs. 2 and 3 are used here. The number of customers in the second queue immediately after the gate opens, can also be considered as the total number of customers in

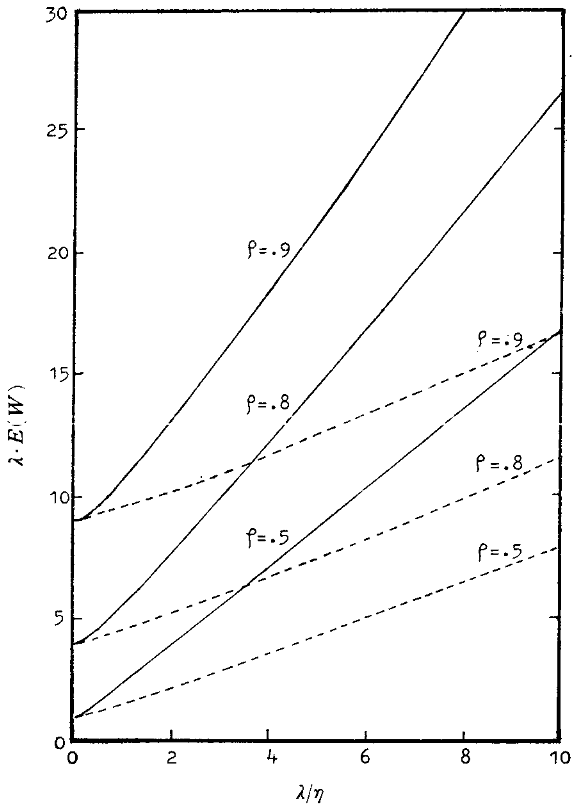

Fig. 2. The mean sojourn time in the system.

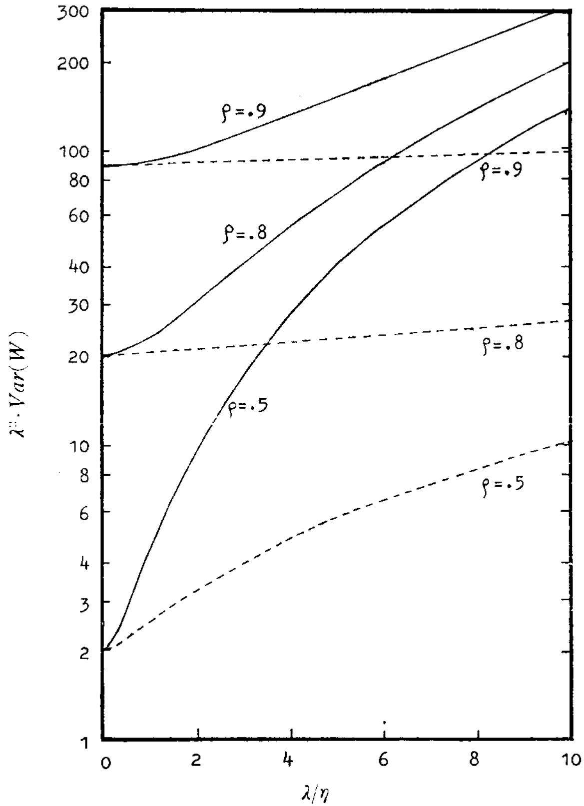

Fig. 3. The variance of the sojourn time in the system.

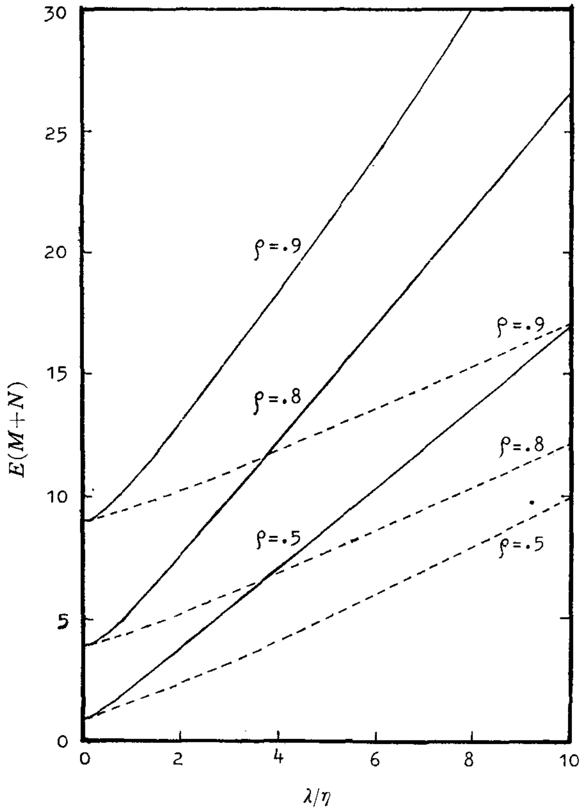

Fig. 4. The mean number of customers in the second queue immediately after the gate opens.

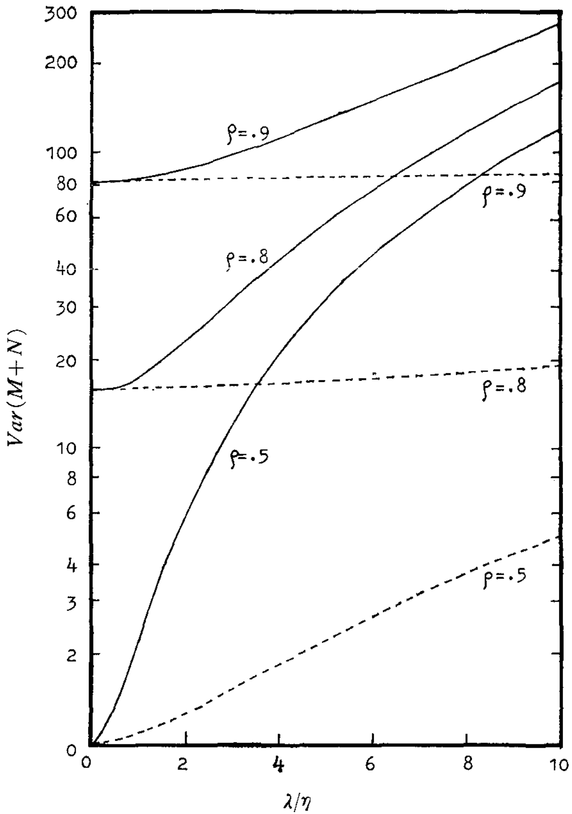

Fig. 5. The variance of the number of customers in the second queue immediately after the gate opens.

the system just before the gate opens, and in the case of the exponential gating rule, its distribution coincides with that for arbitrary time if the system is in the steady state. Hence the graphs in solid lines in Figs. 4 and 5 also represent the mean and the variance of the total number of customers in the system with exponential gating rule at arbitrary time in the steady state. (This is not true for the deterministic case.)

Appendices

Appendix A. Convergences of $f_n(x)$ and $g_n(x)$

Proposition A. (i) The sequence $f_n(x)$ uniformly converges to α in the interval [0, 1].

(ii) The sequence $g_n(x)$ uniformly converges to 1 in [0, 1].

Proof. Since α is the smaller one of two positive roots of the equation $x=f(x)$, it is given by

$$(\mathrm{A}\cdot 1)\qquad \alpha=\frac{1}{2\lambda}[\lambda+\mu+\eta-\sqrt{(\lambda+\mu+\eta)^2-4\lambda\mu}]\,.$$

Clearly $0<\alpha<1$. If $x=\alpha$ then $f_n(\alpha)=\alpha$ for all n. If $x\neq\alpha$,

$$(\mathrm{A}\cdot 2)\qquad \begin{aligned}\frac{f(x)-\alpha}{x-\alpha}&=\frac{\mu-(\lambda+\mu+\eta-\lambda x)}{(x-\alpha)\,(\lambda+\mu+\eta-\lambda x)}\\&=\frac{\alpha\lambda}{\lambda+\mu+\eta-\lambda x}\\&=\frac{\alpha\lambda}{\mu}f(x)\,.\end{aligned}$$

Since $0<f(x)\leqq\frac{\mu}{\mu+\eta}$ in [0, 1], we have

$$(\mathrm{A}\cdot 3)\qquad 0<\frac{f(x)-\alpha}{x-\alpha}\leqq\frac{\alpha\lambda}{\mu+\eta}\qquad(<1)$$

for every $0\leqq x\leqq 1$, $x\neq\alpha$. Hence from the definition (2.12) it follows that

$$(\mathrm{A}\cdot 4)\qquad |f_n(x)-\alpha|=|f(f_{n-1}(x))-\alpha| \leqq \frac{\alpha\lambda}{\mu+\eta}|f_{n-1}(x)-\alpha| \leqq\cdots\leqq\left(\frac{\alpha\lambda}{\mu+\eta}\right)^n(x-\alpha)<\left(\frac{\alpha\lambda}{\mu+\eta}\right)^n.$$

Letting n tend to infinity, it proves (i). (ii) follows immediately from (i) and from the continuity of $g(x)$ in [0, 1].

Appendix B. Convergence of $\prod_{n=0}^{N} g_n(x)$ and Differentiability of the Limit Function $\prod_{n=0}^{\infty} g_n(x)$

Proposition B. (i) $\prod_{n=0}^{N} g_n(x)$ uniformly converges to a function $\prod_{n=0}^{\infty} g_n(x)$ in [0, 1].

(ii) The limit function $\prod_{n=0}^{\infty} g_n(x)$ has non-negative derivatives of all orders in [0. 1].

Proof. From the Taylor's formula,

$$(\mathrm{A}\cdot 5)\qquad g(x)=1+\frac{\sqrt{(\lambda+\mu+\eta)^2-4\lambda\mu}}{\mu\eta}(x-\alpha)+o(x-\alpha)$$

where $o(t)$ represents a term such that $o(t)/t\to 0$ as $t\to 0$. From the definition (2.13) it follows that

$$(\mathrm{A}\cdot 6)\qquad \frac{\log g_{n+1}(x)}{\log g_n(x)}=\frac{\frac{\sqrt{(\lambda+\mu+\eta)^2-4\lambda\mu}}{\mu\eta}(f_{n+1}(x)-\alpha)+o(f_{n+1}(x)-\alpha)}{\frac{\sqrt{(\lambda+\mu+\eta)^2-4\lambda\mu}}{\mu\eta}(f_n(x)-\alpha)+o(f_n(x)-\alpha)} =\frac{f(f_n(x))-\alpha}{f_n(x)-\alpha}+o(f_n(x)-\alpha).$$

Since $f_n(x)$ uniformly converges to α by Proposition A, (A·3) implies that

$$(\mathrm{A}\cdot 7)\qquad 0<\frac{\log g_{n+1}(x)}{\log g_n(x)}<\frac{\alpha\lambda}{\mu+\eta}<1$$

for sufficiently large n and every $0\leqq x\leqq 1$. This proves (i).

In order to prove (ii), it is sufficient to prove

(a) for every $k=1,2,3,\cdots$, the k-th derivative of $\prod_{n=0}^{N} g_n(x)$ uniformly converges in $[0,1]$.

Note that $f(x)$ and $g(x)$ have positive derivatives of all orders in $[0,1]$, and hence $f_n(x)$ and $g_n(x)$ have also positive derivatives of all orders in $[0,1]$. Since

$$(\mathrm{A}\cdot 8)\qquad \frac{d}{dx}\left\{\prod_{n=0}^{N} g_n(x)\right\}=\left\{\prod_{n=0}^{N} g_n(x)\right\}\cdot\frac{d}{dx}\left\{\sum_{n=0}^{N}\log g_n(x)\right\},$$

differentiating both sides of (A·8) k times, we have

$$(\mathrm{A}\cdot 9)\qquad \left\{\prod_{n=0}^{N} g_n(x)\right\}^{(k+1)}=\sum_{\nu=0}^{k}\binom{k}{\nu}\left\{\prod_{n=0}^{N} g_n(x)\right\}^{(\nu)}\cdot\left[\sum_{n=0}^{N}\{\log g_n(x)\}^{(k-\nu+1)}\right]$$

where $\{h(x)\}^{(k)}$ denotes the k-th derivative of $h(x)$. Hence using the mathematical induction, (a) follows from

(b) for each $k=1,2,3,\cdots$, the series $\sum_{n=0}^{N}\{\log g_n(x)\}^{(k)}$ converges uniformly in $[0,1]$.

By the definition of $g_n(x)$, we can write as

$$(\mathrm{A}\cdot 10)\qquad \{\log g_n(x)\}^{(k)}=-\{\log(\lambda+\eta-\lambda f_n(x))\}^{(k)}-\{\log(\mu-\lambda f_n(x))\}^{(k)}$$
$$=\sum_i c_i\left[\left\{\frac{1}{\lambda+\eta-\lambda f_n(x)}\right\}^{\nu_{i0}}+\left\{\frac{1}{\mu-\lambda f_n(x)}\right\}^{\nu_{i0}}\right]\cdot\prod_{j=1}^{k}\{f_n^{(j)}(x)\}^{\nu_{ij}}$$

with positive constants c_i and $(k+1)$-tuples of integers $(\nu_{i0},\cdots,\nu_{ik})$. Since c_i and $(\nu_{i0},\cdots,\nu_{ik})$ are independent of n, and $f_n(x)$ uniformly converges to α, (b) follows from

(c) for each $k=1,2,3,\cdots$, $\sum_{n=0}^{N}\{f_n(x)\}^{(k)}$ converges uniformly in $[0,1]$.

Now we shall prove (c) by induction. First we prove that $\sum_{n=0}^{N} f_n^{(1)}(x)$ uniformly converges in [0, 1]. By the definition (2.12)

$$f_{n+1}^{(1)}(x)=\frac{d}{dx}\{f(f_n(x))\}=f^{(1)}(f_n(x))\,f_n^{(1)}(x)\,. \tag{A·11}$$

Since $f^{(1)}(x)=\frac{\lambda}{\mu}\{f(x)\}^2$ and $0<f_n(x)\leqq\frac{\mu}{\mu+\eta}$ in [0, 1], it follows that

$$0<\frac{f_{n+1}^{(1)}(x)}{f_n^{(1)}(x)}=\frac{\lambda}{\mu}\{f_{n+1}(x)\}^2<\frac{\lambda\mu}{(\mu+\eta)^2}\qquad(<1)\,. \tag{A·12}$$

This proves the uniform convergence of $\sum_{n=0}^{N} f_n^{(1)}(x)$. Next we assume that for every $j=1, 2, \cdots, k-1$, the series $\sum_{n=0}^{N} f_n^{(j)}(x)$ uniformly converges in [0, 1]. Since we can write as

$$\begin{aligned} f_{n+1}^{(k)}(x)&=\{f(f_n(x))\}^{(k)}=\{f^{(1)}(f_n(x))f_n^{(1)}(x)\}^{(k-1)} \\ &=f^{(1)}(f_n(x))\cdot f_n^{(k)}(x)+\sum_i c_i'\,f^{(\nu_{i0})}(f_n(x))\prod_{j=1}^{k-1}\{f_n^{(j)}(x)\}^{\nu_{ij}} \end{aligned} \tag{A·13}$$

with positive constants c_i' and k-tuples of integers $(\nu_{i0}, \cdots, \nu_{i\,k-1})$, we have

$$f_{n+1}^{(k)}(x)-\frac{\lambda}{\mu}\{f_{n+1}(x)\}^2 f_n^{(k)}(x)=\sum_i c_i'\,f^{(\nu_{i0})}(f_n(x))\prod_{j=1}^{k-1}\{f_n^{(j)}(x)\}^{\nu_{ij}}. \tag{A·14}$$

Summing up with n, we obtain that

$$\begin{aligned} f_{N+1}^{(k)}(x)+\sum_{n=0}^{N}\left[1-\frac{\lambda}{\mu}\{f_{n+1}(x)\}^2\right]f_n^{(k)}(x) \\ =\sum_{n=0}^{N}\sum_i c_i\,f^{(\nu_{i0})}(f_n(x))\prod_{j=1}^{k-1}\{f_n^{(j)}(x)\}^{\nu_{ij}}. \end{aligned} \tag{A·15}$$

Since $f^{(j)}(f_n(x))$ uniformly converges to $(\lambda/\mu)^j\,\alpha^{j+1}$, the right hand side of (A·15) converges uniformly by the assumption. Since $f_{N+1}(x)>0$ and $(\lambda/\mu)\{f_{n+1}(x)\}^2<1$ in [0, 1], the series of positive functions

$\sum_{n=0}^{N}\left[1-\frac{\lambda}{\mu}\{f_{n+1}(x)\}^2\right]f_n^{(k)}(x)$ must converge monotonically pointwise in [0, 1] by (A·15). From Proposition A, for an arbitrary positive number ε there exists an integer n_0 such that for any $n>n_0$, $|f_n(x)-\alpha|<\varepsilon$. Hence the inequalities

$$(A\cdot 16)\quad \frac{1}{1-(\lambda/\mu)(\alpha+\varepsilon)^2}\sum_{n=n_0}^{N}\left[1-\frac{\lambda}{\mu}\{f_{n+1}(x)\}^2\right]f_n^{(k)}(x) < \sum_{n=n_0}^{N} f_n^{(k)}(x) < \frac{1}{1-(\lambda/\mu)(\alpha-\varepsilon)^2}\sum_{n=n_0}^{N}\left[1-\frac{\lambda}{\mu}\{f_{n+1}(x)\}^2\right]f_n^{(k)}(x)$$

hold, and we obtain that the series $\sum_{n=0}^{N} f_n^{(k)}(x)$ converges pointwise in in [0, 1] and that $f_n^{(k)}(x)$ converges to zero. Since $f_n^{(k)}(x)$ is a positive and increasing function in [0, 1], the convergence of the sequence is uniform in [0, 1]. Hence from (A·15) the convergence of $\sum_{n=0}^{N}\left[1-\frac{\lambda}{\mu}\{f_{n+1}(x)\}^2\right]f_n^{(k)}(x)$ is uniform and from (A·16) the convergence of $\sum_{n=0}^{N} f_n^{(k)}(x)$ is also uniform in [0, 1]. Thus (c) is proved and we complete the proof of (ii).

Appendix C. Absolute Monotonicity of $F(x, y)$

Proposition C. $F(x, y)$ defied by (2.20) and (2.21), has non-negative derivatives of all orders in [0, 1].

Proof. From Proposition B, $\prod_{n=0}^{\infty} g_n(x)$ has non-negative derivatives of all orders. $G(x)$ defined by (2.19) has also non-negative derivatives of all orders and by the Taylor's formula we can write as

$$(A\cdot 17)\qquad G(x)=\sum_{k=0}^{\infty} a_k x^k \qquad (0\leqq x\leqq 1)$$

where a_k are non-negative coefficients. Furthermore we can write as

(A·18) $$\frac{x}{1-x}G(x)=\sum_{k=1}^{\infty}b_k x^k \qquad (0\leqq x<1)$$

where

(A·19) $$b_k=\sum_{i=0}^{k-1}a_i \qquad (k=1,2,3,\cdots).$$

Using these constants, we can rewrite $F(x, y)$ as follows:

(A·20)
$$\begin{aligned}F(x, y) &= \frac{f(x)}{(f(x)-y)}[-\eta yG(y)+(1-y)(\mu-\lambda x)G(x)]\\ &=\frac{\eta}{\mu}\frac{f(x)(1-y)}{f(x)-y}\left[\frac{f(x)}{1-f(x)}G(f(x))-\frac{y}{1-y}G(y)\right]\\ &=\frac{\eta}{\mu}f(x)(1-y)\sum_{k=1}^{\infty}b_k[(f(x))^{k-1}+(f(x))^{k-2}y+\cdots+y^{k-1}]\\ &=\frac{\eta}{\mu}\sum_{k=1}^{\infty}b_k(f(x))^k\\ &\quad+\frac{\eta}{\mu}f(x)y\sum_{k=1}^{\infty}a_k[(f(x))^{k-1}+(f(x))^{k-2}y+\cdots+y^{k-1}].\end{aligned}$$

Since the constants a_k and b_k are non-negative and $f(x)$ has positive derivatives of all orders in [0, 1], $F(x, y)$ has also non-negative derivatives of all orders in [0, 1].

References

[1] Feller, W.; *An introduction to probability theory and its applications*, Vol. 2, John Wiley & Sons, Inc., New York, 1966.

[2] Kingman, J.F.C.; "On the algebra of queues" *J. Appl. Prob.*, **3**, 285-326, 1966.