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RELIABILITY ANALYSIS FOR SYSTEMS WITH REPAIR

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Abstract

The reliability of standby redundant systems with repair is analyzed by means of semi-Markov processes. The systems are composed of units having general failure time distribution and of repair facilities having exponential repair time distribution. For these systems, the Laplace-Stieltjes transforms of the system failure time distribution function, the mean time to system failure (MTSF), and the steady state availability are derived. The MTSF and the system availability of some examples are calculated numerically, and comparison of different failure distributions is made. Furthermore, the limiting distribution of the system failure time, as repair rate increases indefinitely, is shown to become exponential under certain conditions.

1. Introduction

Nowadays, the reliability problems of complicated systems seem to be of great importance. One of the methods to increase the system

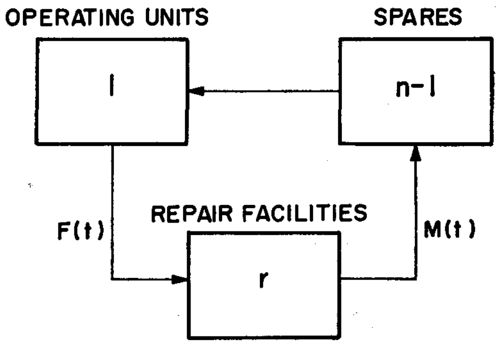

Fig. 1. A Model of Standby Systems with Repair.

reliability is obtained by the introduction of redundancy. Repair maintenance is also well known to improve the reliability of systems. Here, we shall discuss the reliability of repairable redundant systems.

Studies of reliability analysis for such cases have been made by several authors, *e.g.* Barlow [1], Downton [2], Gnedenko [3], Srinivasan [7], Natarajan [4], Osaki [5], and others. In this paper, standby redundant systems with repair maintenance are analyzed by making use of semi-Markov processes.

As shown in Fig. 1, a system is assumed to consist of n identical units, one of which is in operation while the other $n-1$ units are spares. When an operating unit fails, one of the spares is substituted for the failed unit. It is presumed that there are r repairmen, each of whom is capable of dealing with one unit at a time, and that a repaired unit joins spares. If all repairmen are busy, each newly failed unit joins a waiting line and waits until a repairman is freed. The system failure is assumed to occur as soon as all n units are defective. Suppose that the distribution $F(t)$ of the time from the beginning of operation to the failure of the unit is general, and that the repair time distribution $M(t)$ is exponential given as:

$$(1) \qquad M(t)=1-e^{-\mu t} .$$

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It is assumed that a unit whose repair is completed recovers its function perfectly.

Barlow [1] has obtained the mean recurrence time to failed state of the system in the case of $r=n$, and, using this result, steady-state system availability can be calculated. Srinivasan [7] has investigated the system of $r=n-1$ to derive the Laplace-Stieltjes (LS) transform of failure time distribution function by a direct method making use of binomial moments to obtain the MTSF. Natarajan [4] has studied the system having general r repairmen and exponential failure time distribution.

In this paper, a method of treating the case of general r with general failure time distribution by the use of the Pyke's matrix equation [6] is shown, and, further, the LS transform of failure function, the MTSF, and the steady state availability are derived for the sytem in the cases of $r=1, 2$, and $r>n-1$.

2. Semi-Markov Process

As a model for an n-unit system, we consider a semi-Markov process having n states $1, 2, \cdots, n$. The integer valued states denote the number of failed units. The epoch of time to which the process is referred are the instants just after the failure of a unit has occurred. Transition probability from state i to state j within time interval t is denoted by $Q_{ij}(t)$. Furthermore, $G_{ij}(t)$ stands for the distribution function of the first passage time from state i to state j. Here, $Q_{ij}(t)$ and $G_{ij}(t)$ have corresponding LS transforms

$$(2) \qquad q_{ij}(s)=\int_0^\infty e^{-st}dQ_{ij}(t)$$

and

$$(3) \qquad g_{ij}(s)=\int_0^\infty e^{-st}dG_{ij}(t) .$$

Then $\boldsymbol{Q}(s)$ and $\boldsymbol{G}(s)$ are defined as the matrices having these transforms as their elements. For a semi-Markov process with finitely many states, Pyke [6] has shown that

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$$G(s)=Q(s)[I+G_o(s)] \tag{4}$$

where $G_o(s)$ denotes a matrix composed of the off-diagonal elements of $G(s)$ and with diagonal elements of zeros, and I is the $n\times n$ identity matrix. When some elements of $G(s)$ are obtained from $Q(s)$ by using (4), the MTSF and the steady-state system availability can be derived.

3. Distribution of System Failure Time

The LS transform of failure time distribution of the n-unit system with r repairmen, $\phi_{nr}(s)$, is given by

$$\phi_{nr}(s)=f(s)_r g_{1n}(s)\ , \tag{5}$$

where $f(s)$ is defined by the equation

$$f(s)=\int_0^\infty e^{-st}dF(t)\ . \tag{6}$$

Here ${}_r g_{ij}$ and ${}_r q_{ij}$ denote g_{ij} and q_{ij} for the system with r repairmen respectively. Therefore, by solving (4) to find ${}_r g_{1n}(s)$ for given $Q(s)$, $\phi_{nr}(s)$ can be obtained as follows:

$$\phi_{nr}(s)=\frac{f(s)_r q_{12}(s)_r q_{23}(s)\cdots {}_r q_{n-1,n}(s)}{{}_r S_{n-1}} \tag{7}$$

because ${}_r q_{ij}(s)=0$ for $j>i+1$. Here, ${}_r S_i$ is the determinant consisting of the first i rows and of the first i columns of the matrix $I-Q(s)$, namely,

$$ {}_r S_i=\begin{vmatrix} 1-{}_r q_{11} & -{}_r q_{12} & 0 & \cdots & \cdots & 0 \\ -{}_r q_{21} & 1-{}_r q_{22} & -{}_r q_{23} & 0 & \cdots & 0 \\ \vdots & -{}_r q_{32} & 1-{}_r q_{33} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & -{}_r q_{i-1,i} \\ -{}_r q_{i1} & -{}_r q_{i2} & \cdots & \cdots & -{}_r q_{i,i-1} & 1-{}_r q_{ii} \end{vmatrix}\ . \tag{8}$$

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For ${}_rS_i$, the following recurrence relation holds:

$$(9)\qquad {}_rS_i={}_rS_{i-1}(1-{}_rq_{ii})-{}_rS_{i-2}\cdot{}_rq_{i-1,\,i}\cdot{}_rq_{i,\,i-1}$$
$$-{}_rS_{i-3}\cdot{}_rq_{i-2,\,i-1}\cdot{}_rq_{i-1,\,i}\cdot{}_rq_{i,\,i-2}-\cdots\cdots\cdots$$
$$-{}_rS_1\cdot{}_rq_{23}\cdot{}_rq_{34}\cdots{}_rq_{i-1,\,i}\cdot{}_rq_{i2}$$
$$-{}_rq_{12}\cdot{}_rq_{23}\cdots{}_rq_{i-1,\,i}\cdot{}_rq_{i1}\ .$$

Moreover, by utilizing the relation

$$(10)\qquad \sum_{j=1}^{n}{}_rq_{ij}=f(s)\ ,\quad (i\leq n-1)\ ,$$

${}_rS_i$ is given by

$$(11)\qquad {}_rS_i=$$

$$\begin{vmatrix}
1-f(s) & -{}_rq_{12} & 0 & 0 & \cdots\cdots & 0\\
1-f(s) & 1-{}_rq_{22} & -{}_rq_{23} & 0 & \cdots\cdots & 0\\
1-f(s) & -{}_rq_{32} & 1-{}_rq_{33} & -{}_rq_{34} & \ddots & \vdots\\
\vdots & \vdots & -{}_rq_{43} & 1-{}_rq_{44} & \ddots & \vdots\\
\vdots & \vdots & \vdots & -{}_rq_{54} & \ddots & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
1-f(s) & -{}_rq_{i-1,2} & -{}_rq_{i-1,3} & \vdots & \ddots & -{}_rq_{i-1,i}\\
1-f(s)+{}_rq_{i,i+1} & -{}_rq_{i2} & -{}_rq_{i3} & -{}_rq_{i4} & \cdots\cdots -{}_rq_{i,i-1} & 1-{}_rq_{ii}
\end{vmatrix}$$

$$=(1-f(s))\sum_{j=1}^{1}{}_r\Delta_{j1}+{}_rq_{12}\cdot{}_rq_{23}\cdot{}_rq_{34}\cdots\ {}_rq_{i-1,\,i}\cdot{}_rq_{i,\,i+1}\ ,$$

where ${}_r\Delta_{km}$ denotes the cofactor for the element of the kth row, mth column of this matrix. Therefore, letting ${}_rW_i(s)$ and $U_{ir}(s)$ denote $\sum_{j=1}^{i}{}_r\Delta_{j1}$, $\prod_{j=1}^{i}{}_rq_{j,\,j+1}$, respectively, we have

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$$(12)\qquad {}_rS_i=(1-f(s))_rW_i(s)+U_{ir}(s)\ .$$

Thus, from (7) and (12), $\psi_{nr}(s)$ is given by

$$(13)\qquad \psi_{nr}(s)=\frac{f(s)U_{n-1,\,r}(s)}{(1-f(s))_rW_{n-1}(s)+U_{n-1,\,r}(s)}\ .$$

Considering the elements ${}_rq_{j,\,j+1}$, we can write

$$(14)\qquad U_{ir}(s)=\begin{cases}[\prod_{j=1}^{r} f(s+j\mu)]\cdot[f(s+r\mu)]^{i-r}\ , & (i>r)\ ,\\ \prod_{j=1}^{i} f(s+j\mu) & , \quad (i\leq r)\ .\end{cases}$$

For example, in case $r=n-1$, terms ${}_{n-1}q_{ij}(s)$ are as follows:

$$(15)\qquad {}_{n-1}q_{ij}(s)=\int_0^\infty e^{-st}\cdot \binom{i}{i-j+1}e^{-(j-1)\mu t}(1-e^{-\mu t})^{i-j+1}dF(t)\ ,$$

$$(1\leq j\leq i<n)$$

$$(16)\qquad {}_{n-1}q_{i,\,i+1}(s)=\int_0^\infty e^{-st}\cdot e^{-i\mu t}dF(t)\ ,\qquad (1\leq i<n)\ .$$

Then from (7) and (15),

$$(17)\qquad \psi_{n,\,n-1}(s)=\frac{f(s)\prod_{j=1}^{n-1}f(s+j\mu)}{{}_{n-1}S_{n-1}}\ .$$

By using (9), (15) and (16), ${}_{n-1}S_i$ follows the recurrence relation

$$(18)\qquad {}_{n-1}S_i={}_{n-1}S_{i-1}\cdot\left(1-\binom{i}{1}(f(s+\overline{i-1}\mu)-f(s+i\mu))\right)$$

$$-{}_{n-1}S_{i-2}\cdot f(s+\overline{i-1}\mu)\binom{i}{2}(f(s+\overline{i-2}\mu)-2f(s+\overline{i-1}\mu)+f(s+i\mu))$$

$$-{}_{n-1}S_{i-3}\cdot f(s+\overline{i-2}\mu)f(s+\overline{i-1}\mu)\ \begin{matrix}i\\3\end{matrix}\ (f(s+\overline{i-3}\mu)-3f(s+\overline{i-2}\mu)$$

$$+3f(s+\overline{i-1}\mu)-f(s+i\mu)\Big)$$

$$-\ \cdots\cdots\cdots\cdots$$

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$$-{}_{n-1}S_1\cdot\left(\prod_{j=2}^{i-1} f(s+j\mu)\right)\binom{i}{i-1}\sum_{k=0}^{i-1}\binom{i-1}{k}(-1)^k f(s+\overline{k+1}\mu)$$

$$-\prod_{j=1}^{i-1} f(s+j\mu)\Big)\binom{i}{i}\sum_{k=0}^{i}\binom{i}{k}(-1)^k f(s+k\mu)\ , \quad (1<i<n)\ .$$

From (12), (15) and (16), ${}_{n-1}S_i$ can be written as

$$(19)\qquad {}_{n-1}S_i=(1-f(s))_{n-1}W_i(s)+\prod_{j=1}^{i} f(s+j\mu)\ .$$

Using (18) and (19), we obtain the recrrence relation for ${}_{n-1}W_i$, namely

$$(20)\qquad {}_{n-1}W_i=\sum_{j=1}^{i-1}{}_{n-1}W_jH_{i,\,i-j}+\prod_{j=1}^{i-1} f(s+j\mu)\ , \quad (1<i<n)\ ,$$

where ${}_{n-1}W_1=1$ while H_{ij} is defined by

$$(21)\quad H_{ij}=-\binom{i}{j}\left[\prod_{k=i-j+1}^{i-1} f(s+k\mu)\right]\cdot\sum_{k=1}^{j+1}\binom{j}{k-1}(-1)^{k-1}f(s+\overline{i+k-j-1}\mu)\ ,$$
$$(i>j\geq 2)\ ,$$

and

$$(22)\qquad H_{i1}=1-\binom{i}{1}(f(s+\overline{i-1}\mu)-f(s+i\mu))\ .$$

From (18), (19) and (20), ${}_{n-1}W_i$ is given by

$$(23)\qquad {}_{n-1}W_i(s)=\sum_{j=1}^{i-1}\left(H_{ij}\prod_{k=2}^{i-1}H_{k1}\right)$$
$$+\prod_{j=1}^{i-1} f(s+j\mu)+\sum_{j=2}^{i-1}\left(H_{i,\,i-j}\prod_{m=1}^{j-1} f(s+m\mu)\right)$$
$$+\sum_{j=3}^{i-1}\sum_{k=2}^{j-1}\left(H_{i,\,i-j}\,H_{j,\,j-k}\prod_{m=1}^{k-1} f(s+m\mu)\right)+\cdots\cdots\cdots$$
$$+\sum_{m_1=i-2}^{i-1}\ \sum_{m_2=i-3}^{m_1-1}\cdots\sum_{m_{i-3}=2}^{m_{i-4}-1}\Big(H_{i,i-m_1}\cdot H_{m_1,m_1-m_2}\cdot H_{m_2,m_2-m_3}\cdots$$
$$\cdot H_{m_{i-4},m_{i-4}-m_{i-3}}\prod_{k=1}^{m_{i-3}-1} f(s+k\mu)\Big)$$
$$+\left(\sum_{j=3}^{i} H_{j1}\right)\cdot f(s+\mu)\ , \quad (i>1)\ .$$

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In this paper, it is to be understood that for subscripted variables X_i the product $\prod_{i=j}^{k} X_i$ is taken as unity for $j>k$ while the summation $\sum_{i=j}^{k} X_i$ is taken as zero for the same case. From (19) and (20), the following equation

$$(24)\qquad {}_{n-1}S_{n-1}=(1-f(s))\Big[\sum_{j=1}^{n}\binom{n}{j}\Big[\prod_{k=1}^{j-1}(1-f(s+k\mu))\Big]\prod_{k=j}^{n-1}f(s+k\mu) - \prod_{k=1}^{n-1}f(s+k\mu)\Big]+\prod_{k=1}^{n-1}f(s+k\mu)$$

can be proved by induction. Thus the equation

$$(25)\qquad \psi_{n,\,n-1}(s)=\frac{1}{1+\sum_{j=1}^{n}\binom{n}{j}\prod_{k=0}^{j-1}\Big(\frac{1-f(s+k\mu)}{f(s+k\mu)}\Big)}$$

is obtained by using (17) and (24). This result agrees with Srinivasan's formula [7].

In case $r<n-1$, where waiting for repair occurs, $\psi_{nr}(s)$ becomes more complex than in case $r=n-1$. For the case where $r=1$ and $n\geq 3$, the terms ${}_1q_{ij}(s)$ are as follows:

$$(26)\qquad {}_1q_{ij}(s)=\frac{(-\mu)^{i-j+1}}{(i-j+1)!}\cdot f^{(i-j+1)}(s+\mu)\,,\quad (1<j\leq i)\,,$$

$$(27)\qquad {}_1q_{i1}(s)=f(s)-\sum_{k=0}^{i-1}f^{(k)}(s+\mu)\cdot\frac{(-\mu)^k}{k!}\,,$$

$$(28)\qquad {}_1q_{i,\,i+1}(s)=f(s+\mu)\,,$$

where $f^{(k)}(s)$ denotes the kth derivative of $f(s)$. Therefore, from (7) and (28), the equation

$$(29)\qquad \psi_{n1}(s)=\frac{f(s)[f(s+\mu)]^{n-1}}{{}_1S_{n-1}}$$

holds, where ${}_1S_i$ is given by

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(30) $${}_1S_i=(1-f(s))_1W_i(s)+[f(s+\mu)]^i$$

by using (12) and (28). Thus from (9), (12), (26), (27) and (28), ${}_1W_i$ satisfies the recurrence relation

(31) $${}_1W_i(s)=\sum_{j=1}^{i-1}{}_1W_j(s)B_{i-j}[f(s+\mu)]^{i-j+1}+[f(s+\mu)]^{i-1}\ , \quad (i>1)$$

where ${}_1W_1=1$ and B_k is given by

(32) $$B_k=(-1)^{k+1}\cdot\frac{\mu^k f^{(k)}(s+\mu)}{k!}\ , \quad (k>1)$$

and

(23) $$B_1=1+\mu f'(s+\mu)\ .$$

Now, (31) can be solved for ${}_1W_i$ as

(34) $$\begin{aligned}{}_1W_i(s)&=[f(s+\mu)]^{i-1}+[f(s+\mu)]^{i-2}\cdot\sum_{j=1}^{i-1}B_{i-j}\\&+[f(s+\mu)]^{i-3}\sum_{j=2}^{i-1}\sum_{k=1}^{j-1}B_{i-j}\cdot B_{j-k}+\cdots\cdots\\&+f(s+\mu)\sum_{m_1=i-2}^{i-1}\sum_{m_2=i-3}^{m_1-1}\cdots\sum_{m_{i-2}=1}^{m_{i-3}-1}B_{i-m_1}B_{m_1-m_2}\cdots B_{m_{i-3}-m_{i-2}}\\&+(B_1)^{i-1}\ , \quad (i>1)\ .\end{aligned}$$

In the case of $r=2$ and $n\geq 4$, the transition matrix elements ${}_2q_{ij}(s)$ are given by the following equations:

(35) $${}_2q_{1j}(s)={}_{n-1}q_{1j}(s)\ ,$$

(36) $${}_2q_{ij}(s)=\frac{(-2\mu)^{i-j+1}}{(i-j+1)!}f^{(i-j+1)}(s+2\mu)\ , \quad (i\geq j>2)\ ,$$

(37) $${}_2q_{i,\,i+1}(s)=f(s+2\mu)\ , \quad (i\geq 2)\ ,$$

(38) $${}_2q_{i2}(s)=2^{i-1}\left[f(s+\mu)-\sum_{k=0}^{i-2}\frac{(-\mu)^k}{k!}f^{(k)}(s+2\mu)\right]\ , \quad (i\geq 2)\ ,$$

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(39) $$2q_{i1}(s)=f(s)-2^{i-1}f(s+\mu)+\sum_{k=0}^{i-2}\frac{(-\mu)^k}{k!}f^{(k)}(s+2\mu)\cdot(2^{i-1}-2^k)\ ,\quad (i>1)\ .$$

Then, from (7), (35) and (37), we have

(40) $$\psi_{n2}(s)=\frac{f(s)f(s+\mu)[(fs+2\mu)]^{n-2}}{{}_2S_{n-1}}\ ,\quad (n>3)\ ,$$

where ${}_2S_{n-1}$ can be given by

(41) $${}_2S_{n-1}=(1-f(s))_2W_{n-1}(s)+f(s+\mu)[f(s+2\mu)]^{n-2}$$

by using (12). The recurrence relation for ${}_2W_i$ is written by

(42) $${}_2W_i(s)=[f(s+2\mu)]^{i-2}\cdot(f(s+\mu)+C^*_{i-1}) + \sum_{j=2}^{i-1}{}_2W_j(s)\cdot[f(s+2\mu)]^{i-j-1}\cdot C_{i-j}\ ,\quad (i>1)$$

from (9), (41) and (35)~(39). Here C_k and $C_k{}^*$ are as follows:

(43) $$C_k=2^k\frac{(-1)^{k+1}}{k!}\cdot\mu^k\cdot f^{(k)}(s+2\mu)\ ,\quad (k>1)$$

(44) $$C_1=1+2\mu f'(s+2\mu)\ ,$$

(45) $$C_k{}^*=-2^k\left[f(s+\mu)-f(s+2\mu)+\sum_{j=1}^{k-1}\frac{(-1)^{j-1}}{j!}\mu^j f^{(j)}(s+2\mu)\right],\quad (k>1)\ ,$$

(46) $$C_1{}^*=1-2(f(s+\mu)-f(s+2\mu))\ .$$

Then, (42) can be solved as

(47) $${}_2W_i(s)=(f(s+\mu)+C^*_{i-1})[f(s+2\mu)]^{i-2} + \sum_{j=2}^{i-1}C_{i-j}(f(s+\mu)+C^*_{j-1})[f(s+2\mu)]^{i-3} + \sum_{j=3}^{i-1}\sum_{k=2}^{j-1}C_{i-j}C_{j-k}(f(s+\mu)+C^*_{k-1})[f(s+2\mu)]^{i-4}+\cdots$$

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$$+\sum_{m_1=i-2}^{i-1}\sum_{m_2=i-3}^{m_1-1}\cdots\sum_{m_{i-3}=2}^{m_{i-4}-1} C_{i-m_1}C_{m_1-m_2}C_{m_2-m_3}\cdots\cdots$$
$$\cdot C_{m_{i-4}-m_{i-3}}(f(s+\mu)+C^*_{m_{i-3}-1})f(s+2\mu)$$
$$+[C_1]^{i-2}\cdot(f(s+\mu)+C_1^*)\ ,\quad (i>1)$$

with ${}_2W_1=1$.

Examples of $\phi_{nr}(s)$ for $n=2, 3$ and 4 are shown as follows:

(48) $$\phi_{21}(s)=\frac{f(s)f(s+\mu)}{1-f(s)+f(s+\mu)}$$

(49) $$\phi_{31}(s)=\frac{f(s)[f(s+\mu)]^2}{(1-f(s))(1+\mu f'(s+\mu)+f(s+\mu))+[f(s+\mu)]^2}$$

(50) $$\phi_{32}(s)=\frac{f(s)f(s+\mu)f(s+2\mu)}{(1-f(s))(1-f(s+\mu)+2f(s+2\mu))+f(s+\mu)f(s+2\mu)}$$

(51) $$\begin{aligned}\phi_{42}(s)=&f(s)f(s+\mu)[f(s+2\mu)]^2/\{(1-f(s))[f(s+\mu)f(s+2\mu)\\&-4f(s+2\mu)(f(s+\mu)-f(s+2\mu)+\mu f'(s+2\mu))\\&+(1-f(s+\mu)+2f(s+2\mu))(1+2\mu f'(s+2\mu))]\\&+f(s+\mu)[f(s+2\mu)]^2\}\ .\end{aligned}$$

4. MTSF

When MTSF of the n-unit system with r repairmen is denoted by T_{nr}, it is given by

(52) $$T_{nr}=-\left[\frac{d}{ds}\,\phi_{nr}(s)\right]_{s=0}\ .$$

Using (13), and (14), we reduce the equation (52) to

(53) $$T_{nr}=-\frac{f'(0)\left\{{}_rW_{n-1}(0)+\left[\prod_{j=1}^{r}f(j\mu)\right][f(r\mu)]^{n-r-1}\right\}}{\left[\prod_{j=1}^{r}f(j\mu)\right]\cdot[f(r\mu)]^{n-r-1}}$$

Examples of the MTSF for $n=2$, 3 and 4 are shown as follows:

(54) $$T_{21}=-\frac{f'(0)(f(\mu)+1)}{f(\mu)}\ ,$$

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(55) $$T_{31} = -\frac{f'(0)(1+f(\mu)+[f(\mu)]^2+\mu f'(\mu))}{[f(\mu)]^2},$$

(56) $$T_{32} = -\frac{f'(0)}{f(\mu)f(2\mu)}(1-f(\mu)+2f(2\mu)+f(\mu)f(2\mu)),$$

(57) $$T_{42} = -\frac{f(0)}{f(\mu)[f(2\mu)]^2}\{1-f(\mu)+2f(2\mu)$$
$$+2\mu f'(2\mu)(1-f(\mu))-3f(\mu)f(2\mu)+[f(2\mu)]^2\cdot(4+f(\mu))\}.$$

5. Steady State Availability

Steady-state availability of a system is given by $\lim_{t\to\infty} A(t)$ where $A(t)$ is defined as the probability that the system is functioning at time t. Steady-state availability can be obtained from the matrix equation (4) by using the following relation

(58) $${}_rg_{nn}(s) = \frac{r\mu}{s+r\mu}\cdot {}_rg_{n-1,n}(s)\,.$$

This relation is easily derived from the following consideration. The LS transform of the first passage time from the state of system down to the instant of system restored, when one of the units is repaired, is $r\mu/(s+r\mu)$, and the first passage time from the initial instant of system restored to state n is equal to that from state n-1 to state n. Thus, the steady state availability for the n-unit system having r repairmen, A_{nr}, is given by

(59) $$A_{nr} = \frac{\left[\frac{d}{ds}{}_rg_{nn}(s)\right]_{s=0} + \frac{1}{r\mu}}{\left[\frac{d}{ds}{}_rg_{nn}(s)\right]_{s=0}}\,.$$

Therefore, if ${}_rg_{n-1,n}(s)$ is obtained from (4), A_{nr} can be derived from (58) and (59). By solving (4) in consideration of ${}_rq_{ij}=0$ for $j>i+1$, we obtain

(60) $${}_rg_{n-1,n}(s) = \frac{{}_rS_{n-2}}{{}_rS_{n-1}}\cdot {}_rq_{n-1,n}(s)\,,$$

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where ${}_rS_0=1$. Hence, for $r<n$,

(61) $${}_rg_{n-1,\,n}(s)=\frac{{}_rS_{n-2}}{{}_rS_{n-1}}\cdot f(s+r\mu)\ ,$$

and for $r=n$,

(62) $${}_ng_{n-1,\,n}(s)={}_{n-1}g_{n-1,\,n}(s)\ .$$

From (58), (12), (14), (60) and (61), the equation (59), for $r<n$, is reduced to

(63) $$A_{nr}=\frac{-f'(0)[{}_rW_{n-1}(0)-{}_rW_{n-2}(0)f(r\mu)]}{-f'(0){}_r[W_{n-1}(0)-{}_rW_{n-2}(0)f(r\mu)]+\dfrac{\left[\prod_{j=1}^{r}f(j\mu)\right][f(r\mu)]^{n-r-1}}{r\mu}}\ , \quad (r<n)$$

with ${}_rW_0(0)=0$.

In particular, in the case of $r=n-1$, and n, from (12), (24), (62) and (63), we have

(64) $$A_{nr}=\frac{-f'(0)D_n}{-f'(0)D_n+\dfrac{1}{r\mu}}$$

where

(65) $$D_n=1+\sum_{j=2}^{n-1}\binom{n-1}{j-1}\prod_{k=1}^{j-1}\frac{1-f(k\mu)}{f(k\mu)}+\prod_{k=1}^{n-1}\frac{1-f(k\mu)}{f(k\mu)}\ .$$

Examples for $n=2$, 3 and 4 are shown as follows:

(66) $$A_{2r}=\frac{-f'(0)}{-f'(0)+\dfrac{f(\mu)}{r\mu}}\ ,$$

(67) $$A_{31}=\frac{-f'(0)(1+\mu f'(\mu))}{-f'(0)(1+\mu f'(\mu))+[f(\mu)]^2/\mu}\ ,$$

(68) $$A_{3r}=\frac{-f'(0)(1-f(\mu)+f(2\mu))}{-f'(0)(1-f(\mu)+f(2\mu))+\dfrac{f(\mu)f(2\mu)}{r\mu}}\ , \quad (r>1)$$

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(69) $A_{42}=$

$$\frac{-f'(0)[(1+2\mu f'(2\mu))(1-f(\mu))+f(2\mu)(1-2f(\mu)+2f(2\mu))]}{-f'(0)[(1+2\mu f'(2\mu))(1-f(\mu))+f(2\mu)(1-2f(\mu)+2f(2\mu))]+\dfrac{f(\mu)[f(2\mu)]^2}{2\mu}}.$$

6. Examples of Numerical Calculations

6.1 MTSF

Figure 2 shows the MTSF of a 4-unit system with 2 repairmen calculated for various failure time distributions as a function of repair rate μ. In the cases where failure time distributions of the unit are exponential, gamma, and Weibull, values of MTSF are plotted. Here failure probability density functions $f_g(t)$ and $f_w(t)$ for the gamma and the Weibull distributions are defined by

(70) $$\bar{f}_g(t)=\frac{\lambda}{\Gamma(k)}(\lambda t)^{k-1}\cdot e^{-\lambda t},$$

and

(71) $$\bar{f}_w(t)=\frac{m}{a}t^{m-1}\cdot\exp\left(-\frac{t^m}{a}\right).$$

Among the curves of Fig. 2, the gamma distribution with $k=2$ and the Weibull distribution with $m=1.44$, as well as the gamma distribution with $k=10$ and the Weibull distribution with $m=3.55$, have the same means and variances. Nevertheless, in these cases, values of MTSF for the gamma distribution are larger than those of the Weibull distribution for increasing failure rate, and the difference is relatively large. On the whole, the MTSF increases as m and k increase, namely as the variance decreases. The reason is that the system failure occurs only when many units fail in a short time successively, for the life time of a repairable standby system is much longer than the MTBF of the unit Therefore, the MTSF depends markedly upon the failure rate of the unit for the case where t is near 0. When t is close to zero, exponential

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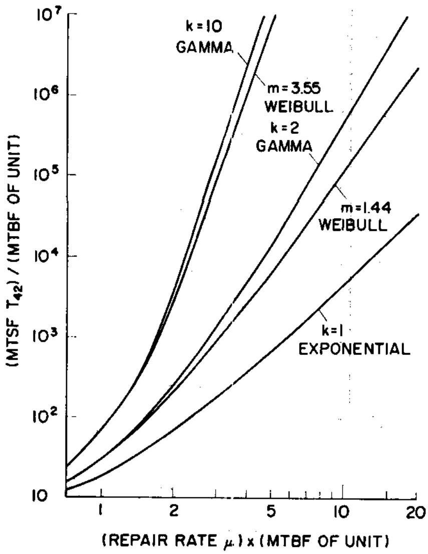

Fig. 2. MTSF for 4-Unit Systems with 2 Repairmen as a Function of Repair Rate.

parts of $\bar{f}_g(t)$ and $\bar{f}_w(t)$ are nearly equal to 1. In such a case, t^{k-1} is smaller than t^{m-1} for $k=2$ and $m=1.44$, as well as for $k=10$ and $m=3.55$. This explains the difference between the gamma and the Weibull distributions with the same means and variances. Therefore, if a general distribution is approximated to be exponential, it is possible that a large error will take place in the calculation of MTSF.

6.2 System Availability

Steady-state availability for the case that $F(t)$ is Erlang (the gamma distribution with integer parameter k), is investigated. For the Erlang distribution, the LS transform $f(s)$ can be obtained analytically with ease.

For $n=3$, $r=1$, using (67) and (70), we have

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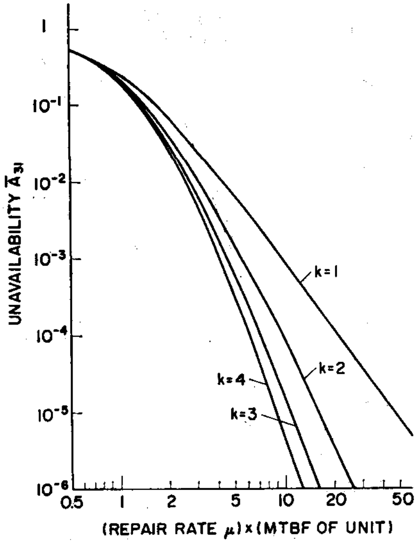

Fig. 3. Unavailability for 3-Unit Systems with 1 Repairman as a Function of Repair Rate.

$$(72)\qquad A_{31}=\frac{\frac{k}{\lambda}\left(1-\frac{k\mu}{\lambda}\left(\frac{\lambda}{\lambda+\mu}\right)^{k+1}\right)}{\frac{k}{\lambda}\left(1-\frac{k\mu}{\lambda}\left(\frac{\lambda}{\lambda+\mu}\right)^{k+1}\right)+\frac{1}{\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{2k}}\,.$$

In Fig. 3, the curves of unavailability $\bar{A}_{31}$ defined by

$$(73)\qquad \bar{A}_{31}=1-A_{31}$$

are shown for various values of k as a function of repair rate μ. By comparison, large differences are observed among the curves. In particular, it is observed that the unavailability decreases faster for the larger value of k, as μ increases.

7. Limiting Distribution of System Failure Time

Gnedenko [3] has studied a repairable parallel redundant system

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which consists of units with exponential failure time distribution and of one repairman with general repair time distribution. He has investigated the limiting distribution of the system failure time under the condition that the repair rate increases indefinitely. Here, the limiting failure time distribution for the standby redundant system with repair is obtained.

For example, we consider the system of $n=2$ and $r=1$. As obtained previously, the LS transform $\phi_{21}(s)$ of system failure time distribution is given by (48). A parameter α_μ is considered so that $\alpha_\mu \to 0$ as $\mu \to \infty$. Then, m_i is defined as ith moment of the failure time distribution of a unit. Since

$$(74) \qquad 1-f(\alpha_\mu s)=m_1\alpha_\mu s-\frac{m_2}{2!}\alpha_\mu{}^2 s^2+\cdots\cdots,$$

then

$$(75) \qquad \frac{1}{\phi_{21}(\alpha_\mu s)}-1=\frac{1-f(\alpha_\mu s)}{f(\alpha_\mu s)}\cdot\frac{1+f(\alpha_\mu s+\mu)}{f(\alpha_\mu s+\mu)} \longrightarrow m_1\alpha_\mu s\frac{1+f(\mu)}{f(\mu)}$$

as $\mu \to \infty$. If the condition that

$$(76) \qquad \frac{\alpha_\mu}{f(\mu)}=c=\text{const}$$

exists, the limit of (75) becomes s/a where $a=1/(m_1c)$. Thus, letting η_μ be the life time of the system, under the above condition, we have

$$(77) \qquad Pr\{\alpha_\mu\eta_\mu<t\}=1-\exp-\left(\frac{t}{m_1c}\right)$$

as $\mu \to \infty$. Namely, the limiting distribution becomes exponential.

Such a codition where the limiting distribution is exponential exists for any possible n and r. The reason is that (13) can be reduced to

$$(78) \qquad \frac{1}{\phi_{nr}(s)}-1=\frac{(1-f(s))({}_rW_{n-1}(s)+U_{n-1,\,r}(s))}{f(s)U_{n-1,\,r}(s)}.$$

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From this relation, the condition where $\phi_{nr}(\alpha_\mu s)$ is $a/(s+a)$ as $\mu\to\infty$, is given by

$$\alpha_\mu\cdot\frac{{}_rW_{n-1}(\alpha_\mu s)+U_{n-1,r}(\alpha_\mu s)}{U_{n-1,r}(\alpha_\mu s)}=\text{const},\tag{79}$$

where a is a constant. As $\mu\to\infty$, ${}_rW_{n-1}(\alpha_\mu s)\to 1$ and $U_{n-1,r}(\alpha_\mu)\to 0$ because ${}_rq_{ij}\to 0$ in (11) except for $j=1$. Thus, by using (14), the condition (79) is reduced to

$$\frac{\alpha_\mu}{\left[\prod_{i=1}^{r} f(i\mu)\right]\cdot[f(r\mu)]^{n-r-1}}=c_{nr}\,.\tag{80}$$

Under this condition,

$$Pr\{\alpha_\mu\eta_\mu<t\}=1-\exp\left(-\frac{t}{m_1c_{nr}}\right)\tag{81}$$

holds as $\mu\to\infty$ for the n unit system with r repairmen.

8. Limiting Form of MTSF and Availability

From the above discussions, when μ becomes very large, the limiting form of the MTSF T_{nr} is given by

$$T_{nr}\longrightarrow\frac{-f'(0)}{\left[\prod_{k=1}^{r} f(k\mu)\right]\cdot[f(r\mu)]^{n-r-1}}\tag{82}$$

by using (53). Furthermore, from (63), (64) and (65), when μ becomes sufficiently large, the limiting form of the steady state unavailability $\bar{A}_{nr}$ can be written by

$$\bar{A}_{nr}\longrightarrow\frac{\left[\prod_{i=1}^{r} f(i\mu)\right]\cdot[f(r\mu)]^{n-r-1}}{-f'(0)r\mu}\,.\tag{83}$$

9. Conclusion

The Pyke's matrix method is applied to the reliability analysis of

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repairable standby redundant systems having general failure and exponential repair time distributions. For the system with any possible n and r, a method of finding the LS transforms of failure function, the MTSF and the steady state availability is shown. The results obtained by the method indicate that these values are markedly dependent upon the shape of the failure time distribution of the unit.

The limiting distribution of system life time, when μ becomes indefinitely large, investigated, and the condition under which the system failure time has exponential limiting distribution is shown for any n and r. Furthermore, the limiting forms of MTSF and steady state unavailability, when μ becomes very large, are obtained.

Besides, in case a failure to standby units is considered, it is thought that the Pyke's matrix method can also be applied when the failure time distribution of standby units is exponential.

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References

[1] Barlow, R., "Repairman Problems," Studies in Applied Probability and Management Science, Stanford University Press, California (1962).

[2] Downton, F., "The Reliability of Multiplex Systems with Repair," J. Roy. Statist. Soc., ser. B, **28** (1966), 459-476.

[3] Gnedenko, B.V., "Some Theorems on Standbys," Proc. 5th Berkeley Symposium on Mathematical Statistics and Probability, Vol. 3, University of California Press, Berkeley (1967), 285-291.

[4] Natarajan, R., "A Reliability Problem with Spares and Multiple Repair Facilities," Operations Research, **16** (1968), 1041-1057.

[5] Osaki, S., "A Note on a Two-unit Standby Redundant System," J. Operations Research Soc. Japan, **12** (1970), 43-51.

[6] Pyke, R., "Markov Renewal Processes with Finitely Many State States," Ann. Math. Statist., **32** (1961), 1243-1259.

[7] Srinivasan, V.S., "First Emptiness in the Spare Parts Problem for Repairable Components," Operations Research, **16** (1968), 407-415.

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