

## OPTIMIZATION OF EQUIPMENT INVESTMENT\*

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(Received January 11, 1971)

### Abstract

Maximization of the total present net worth of an investment in production equipment is studied. A model which gives rises to the optimum life as well as the optimum production rate is proposed and the optimal decisions are determined by the maximum principle. That there exists three types of investment is a well known fact in the business world. These are (1) investment with a net loss in all phases, (2) investment with loss at an initial period but with a net profit after that period, and (3) investment with a net profit from the beginning to a certain time. The proposed model can take this fact into account.

### Introduction

A problem faced by a manufacturing company when investing in production equipment is that of maximizing the total net worth of such an investment. The sales of goods generate a continuous stream of revenue over the productive life of the equipment. Associated in time with this stream of expenses necessary for the production of these goods. The difference between these two streams represents the return on

\* This study was partly supported by the Office of Saline Water, U.S. Department of Interior, Grant No. 14-01-0001-1283.

investment before deducting capital costs.

In this paper, basic model for profit maximization treated by Preinreich [8] and others [1, 10] is introduced and then a more comprehensive model is proposed. The optimal solution is obtained by means of the maximum principle [2, 3, 5, 7, 9].

### **A Classical Model for Profit Maximization**

From the efficiency point of view, two general kinds of equipment may be distinguished: the "constant efficiency" and the "diminishing efficiency" types. Under the first category we may classify those items whose efficiency remains fairly constant throughout their service lives and whose service terminates abruptly with their first failure. An electric light bulb is the best example of this type of equipment. To the second classification belong those durable goods whose service life may be extended almost indefinitely if their component parts are replaced or repaired as necessary. This type of equipment is characterized by a decline in productivity or an increase in maintenance costs as they are used over time.

The economics of replacement associated with these two types of equipment are very different. For those goods displaying a constant efficiency, a probability distribution for the length of their lives may be obtained from life tests and various replacement policies may be evaluated on the basis of this distribution. Since there is no cost of declining efficiency associated with the problem, the analysis is often reduced to a comparison of the expected values of the several alternatives.

If a simple piece of equipment of the diminishing efficiency type earns revenue according to some function,  $R(t)$ , and incurs a stream of maintenance and operating expenses given by the function  $U(t)$ , the net present value of the investment to the firm is given by [8]

$$(1) \quad V_1 = \int_0^T (R(t) - U(t))e^{-it} dt + D(T)e^{-iT} - B,$$

where

- $V_1$ =net present worth of the investment,
- $B$ =installed cost of the equipment,
- $T$ =economic life of the equipment,
- $D(T)$ =salvage value of the equipment at time  $T$ ,
- $i$ =annual rate of interest.

Note that the expense function,  $U(t)$ , excludes depreciation costs and interest on investment in order to avoid double counting these items in equation (1).

For an infinite chain of similar machines, the present worth formula given by equation (1) becomes [8]

$$(2) \quad V_\infty = \left\{ \int_0^T (R(t) - U(t))e^{-it} dt - D(T)e^{-iT} - B \right\} \frac{1}{(1 - e^{-iT})}.$$

Equations (1) and (2) are very often of the discrete character in which a summation of the discrete revenue and expenditures discounted to the present replaces the integrals of equations (1) and (2).

We shall consider only the continuous case for a single machine. The objective function for the case under consideration can be written as

$$(3) \quad S = V_1.$$

The problem, therefore, becomes that of determining the optimum life of the equipment,  $\bar{T}$ , so that the net present value as given by equation (1) attains its maximum.

Taking the derivative of equation (1) with respect to  $T$  and applying the condition.

$$(4) \quad \frac{dV_1}{dT} = 0$$

given by the differential calculus, we obtain

$$(5) \quad R(T) - U(T) = iD(T) - D'(T)$$

where

$$D'(T) = \frac{dD(T)}{dT}.$$

If the functions for revenue, expenditure and depreciation are known, the optimum service life,  $\bar{T}$ , can be obtained from equation (5) by means of a simple numerical analysis.

### **A Modified Model for Profit Maximization**

In the model discussed above, it has been assumed that the investment time,  $T$ , is solely responsible for the maximization of profits. It is easy to visualize, however, that under actual conditions there are other factors which are equally or more significant than the investment time and which should therefore be brought into the analysis. One such factor is the production rate at which the equipment is operated. In the analysis that follows, the production rate is introduced as the second decision variable which is dependent on time.

The manner in which the production rate affects the operation of the system varies with the market conditions (revenue function), the manufacturing process (expense function) and the type of equipment used (depreciation function). These factors are not completely independent of each other but for computational purposes they may be considered so without lessening the efficiency of the model.

A mathematical model which accounts for all possible forms of variation in the system is obviously unattainable and therefore simplifying assumptions are made here.

1. The company's share of the market,  $M_s$ , remains constant throughout the investment time,  $T$ .

2. The cost of any shortage is negligible\* and no inventory is carried. Consequently, we can write

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\* It will be seen later that, despite of this assumption, the conditions for optimality require a rate of production as close as possible to the market share.

$$(6) \quad 0 \leq P(t) \leq M_s, \quad 0 \leq t \leq T,$$

where  $P(t)$  is the production rate.

3. The amount of maintenance and servicing required per unit time,  $M(P, t)$ , is proportional to the cumulative service obtained from the machine up to time  $t$ ,  $\int_0^t P(t)dt$ , and is inversely proportional to the total expected service of the machine,  $A$ . We may write

$$M(P, t) = m \left( \frac{1}{A} \int_0^t P(t)dt \right)^\gamma E \tag{7}$$

where  $E$  is the fixed overhead cost (\$/time) associated with the machine. The constants  $m$  and  $\gamma$  are positive parameters characteristic of each type of machine and can be determined from the company record (or manufacturer's data) on similar machines in the past.

It can be derived from equation (7) that when the expected production has been obtained from the machine by a certain time  $t$ ,

$$\int_0^t P(t)dt = A \text{ (units produced),}$$

and the rate of maintenance and servicing required becomes, at  $t$ ,

$$M(P, t) = mE \text{ (\$/time).}$$

Figure 1 is a graphical representation of the effect of the value of  $\gamma$  on the maintenance cost function. Both  $m$  and  $\gamma$  must be chosen according to the maintenance conditions dictated by each particular type of machine. In all cases, these parameters as well as all other parameters in the model may be functions of time but, for simplicity, we shall treat them as constants throughout the analysis.

4. The revenue function,  $R(P, t)$ , is proportional to the production rate since we assume that the sale price,  $S_p$  to be constant. Then,

$$(8) \quad R(P, t) = S_p P(t).$$

Similarly, the function

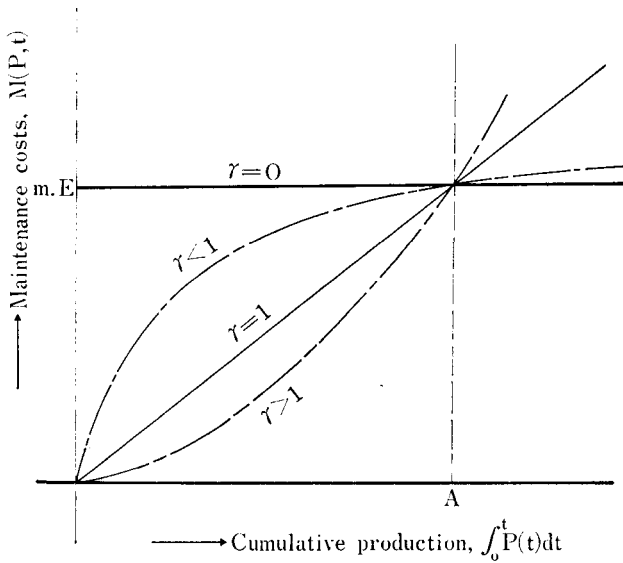


Fig. 1. The effect of  $\gamma$  on the maintenance cost function.

$$(9) \quad VC(P, t) = C_v P(t)$$

represents all various costs other than maintenance cost, overhead cost, and depreciation cost, with  $C_v$  being the pre-unit various cost.

5. With the total installed cost,  $B$ , and a constant rate of depreciation,  $k$ , the salvage value of the machine at time  $t$  is given by

$$(10) \quad D(t) = B e^{-kt}.$$

Using the net present worth as the criteria for optimality we write

$$(11) \quad V = \int_0^T (R(P, t) - VC(P, t) - E - M(P, t)) e^{-it} dt + D(T) e^{-iT} - B$$

The term under the integral sign represents the present worth of revenues minus all expenses except depreciation. The two terms outside

the integral sign may be understood as the net total cost of buying the equipment and selling it at a price  $D(T)$  after  $T$  years of use.

Let us, for simplicity, assume  $\gamma=2$ . Substituting equations (6) through (10) into equation (11) and rearranging, we obtain

$$V = \int_0^T \left\{ (s_p - C_v)P(t) - E \left( 1 + m \left( \frac{1}{A} \int_0^t P(\tau) d\tau \right)^2 \right) \right\} e^{-it} dt + B(e^{-(k+i)T} - 1). \tag{12}$$

Our objective is to maximize the net present value of the investment as given in equation (12) by choosing the most profitable rate of production,  $\bar{P}(t)$ , during the optimum investment time,  $\bar{T}$ . We shall accomplish this through the use of the maximum principle.

*Optimization based on the modified model*

The maximization of the net present value of the investment,  $V$ , given by equation (12) with respect to the optimal life,  $\bar{T}$ , as well as the optimal production rate,  $\bar{P}$ , is difficult, if not impossible, to obtain by use of the differential calculus alone.

To apply the maximum principle let the production rate be the decision variable, *i.e.*,

$$(13) \quad \theta(t) = P(t), \quad 0 \leq \theta(t) \leq \theta_{\max}$$

The state variables are defined as follows:

$$(14) \quad x_1(t) = \frac{1}{A} \int_0^t \theta(\tau) d\tau,$$

$$(15) \quad \frac{dx_1}{dt} = \frac{\theta(t)}{A}, \quad x_1(0) = 0,$$

$$(16) \quad x_2(t) = B(e^{-(k+i)t} - 1),$$

$$(17) \quad \frac{dx_2}{dt} = -(k+i)Be^{-(k+i)t}, \quad x_2(0) = 0,$$

$$(18) \quad x_3(t) = \int_0^t (q\theta(t) - E(1 + mx_1^2))e^{-it} dt,$$

$$(19) \quad \frac{dx_3}{dt} = (q\theta(t) - E(1 + mx_1^2))e^{-it}, \quad x_3(0) = 0,$$

where

$$(20) \quad q = (S_p - C_v) > 0 .$$

$q$  is the unit logistic margin, that is the scale price minus the variable cost per unit.

Since the system defined by equations (15), (17), and (19) is non-autonomous (the right hand sides of equations depend explicitly on time), we shall introduce an additional state variable  $x_4$ , defined by [3]

$$(21) \quad \begin{aligned} x_4(t) &= t , \\ \frac{dx_4}{dt} &= 1 , \quad x_4(0) = t_0 = 0 . \end{aligned}$$

The objective function to be maximized now becomes

$$(22) \quad S = \sum_{i=1}^4 c_i x_i(T) = x_2(T) + x_3(T) .$$

Therefore,

$$c_1 = c_4 = 0 , \quad c_2 = c_3 = 1 .$$

The Hamiltonian function and adjoint variables of the system can be written as [2, 3, 5, 9]

$$(23) \quad \begin{aligned} H = z_1 \left\{ \frac{\theta}{A} \right\} &+ z_2 \{ -(k+i) B e^{-(k+i)x_4} \} \\ &+ z_3 \{ [q\theta - E(1+m x_1^2)] e^{-ix_1} \} + z_4 \{ 1 \} , \end{aligned}$$

$$(24) \quad \frac{dz_1}{dt} = - \frac{\partial H}{\partial x_1} = 2z_3 E m x_1 e^{-ix_1} ,$$

$$(24a) \quad z_1(T) = c_1 = 0 ,$$

$$(25) \quad \frac{dz_2}{dt} = - \frac{\partial H}{\partial x_2} = 0 ,$$

$$(25a) \quad z_2(T) = c_2 = 1 ,$$

$$(26) \quad \frac{dz_3}{dt} = - \frac{\partial H}{\partial x_3} = 0 ,$$

$$(26a) \quad z_3(T) = c_3 = 1 ,$$



$$\frac{dz_4}{dt} = -\frac{\partial H}{\partial x_4} = -z_2(k+i)^2 Be^{-(k+i)x_4} + z_3 i(q\theta - E(1+mx_1^2))e^{-ix_4},$$

(27)

(27a)  $z_4(T) = c_4 = 0.$

It is worth noting that the optimal control problem given by the set of equations, equations (15), (17), (19), and (21), is a problem which has the trajectory with a free right end and the free terminal time. Very little has appeared in literature concerning this class of optimal control problems. However, the validity of the boundary condition of the adjoint variables given by equations (24a), (25a), (26a), and (27a) for this class of problems are described in reference [9].

Solving equations (25) and (26), we obtain

$$(28) \quad z_2(t) = 1, \quad 0 \leq t \leq T,$$

$$(29) \quad z_3(t) = 1, \quad 0 \leq t \leq T.$$

Substituting equations (28) and (29) into equation (23) and separating terms, we obtain

$$(30) \quad H = H^* - (k+i)Be^{-(k+i)x_4} - E(1+mx_1^2)e^{-ix_4} + z_4$$

where

$$(31) \quad H^* = \left( \frac{z_1}{A} + qe^{-ix_4} \right) \theta(t)$$

is the variable part, with respect to  $\theta(t)$ , of the Hamiltonian.

It is now apparent from equation (31) that the optimal control associated with this problem is of the "on-off" or "bang-bang" type, or of the combination of this type of control with the singular control, in which the variable part of the Hamiltonian function takes the form [3, 4, 6]

$$(32) \quad H^* = h\theta.$$

This type of control is characterized by the variation of the decision variable,  $\theta$ , which may take its maximum value (when  $h$  is positive) or

its minimum value (when  $h$  is negative) in order to maximize the Hamiltonian function. When  $h=0$ , the optimal decision,  $\bar{\theta}$ , is either unspecified or singular. ( $h$  is often called the switching function).

Let  $h$  be the coefficient of  $\theta$  in equation (31), that is

$$(33) \quad h = \frac{z_1}{A} + qe^{-it}.$$

Then the optimal control which renders the Hamiltonian its maximum value will be

$$(34) \quad \bar{\theta} = \begin{cases} \theta_{\max} & \text{(Production at the maximum rate) if } h > 0 \\ \theta_{\min} = 0 & \text{(no production at all) if } h < 0 \end{cases}$$

where  $\bar{\theta}$  is the optimal decision policy (optimum production rate) which will maximize the objective function. Recall that  $\theta \geq 0$ . We shall now find the switching time,  $t_s$ , at which  $h$  changes sign. The switching time may be found from the condition

$$(35) \quad h(t_s) = 0.$$

From the optimality condition obtained in equation (34) it is seen that  $\theta$  is not a continuous function of time and that it may take only one of the extreme values. For computational purposes,  $\theta$  may be assumed to be a constant,  $\phi$ , *i.e.*,

$$(36) \quad \theta = \phi \quad \text{where } \phi = \begin{cases} \theta_{\max} \\ \text{or} \\ 0 \end{cases}.$$

Using equation (36) and solving for  $x_1$  and  $z_1$  in equations (15) and (24) with the boundary conditions,  $x_1(0)=0$ , and  $z_1(T)=0$ , we obtain

$$(37) \quad x_1(t) = \frac{\phi t}{A},$$

$$(38) \quad z_1(t) = \frac{2mE\phi}{Ai^2} ((iT+1)e^{-iT} - (it+1)e^{-it}),$$

and  $h$  can now be written

$$(39) \quad h = \frac{2mE\phi}{A^2i^2} ((iT+1)e^{-iT} - (it+1)e^{-it}) + qe^{-it} .$$

Since  $q > 0$  in the practical situation, it follows from equation (39) that

$$(40) \quad h > 0 \quad \text{for} \quad 0 \leq t \leq T$$

and consequently,

$$(41) \quad t_s > T .$$

It appears that the singular solution (when  $h=0$  during the finite time interval) [4, 6] may possibly occur only when the unit logistic margin,  $q = S_p - C_v$  is non positive. This corresponds to a trivial problem of the production with the obvious loss.

Since we are concerned only with the interval  $0 \leq t \leq T$  at the end of which the service life of the machine is terminated, the optimum production policy for this period is

$$(42) \quad \bar{P}(t) = \bar{\theta}(t) = \theta_{\max} = \min \left\{ \begin{array}{l} \text{Maximum Plant Capacity} \\ M_s, \text{ the market share} \end{array} \right\},$$

$$0 \leq t \leq T .$$

In order to maximize the total present worth of the investment, then, the maximum possible rate of production should be maintained throughout the service life of the machine. The rate of production, however, should not exceed the market share of the company since inventories are not allowed. The optimal condition given by equation (42) precludes the first part of assumption number two since the optimal condition minimizes shortages regardless of how inexpensive they may be. The assumption, however, is not redundant since the introduction of a shortage cost and its effect on the optimality condition were not tested.

It only remains to be determined what the optimum investment time  $\bar{T}$  should be. According to the maximum principle, the optimal time is determined from the condition that  $\max H = 0$  for  $t_0 \leq t \leq T$  [3, 7, 9].

Solving equation (27) for  $z_4$ , we obtain

$$(43) \quad z_4(t) = (k+i)B(e^{-(k+i)t} - e^{-(k+i)T}) + (q\theta - E)(e^{-iT} - e^{-it}) + \frac{mE\theta^2}{A^2i^2} ((i^2t^2 + 2it + 2)e^{-it} - (i^2T^2 + 2iT + 2)e^{-iT}) .$$

Substituting expressions for  $z_1, z_2, z_3, z_4$  and  $q$  into equation (23), the Hamiltonian function becomes

$$(44) \quad H = -(k+i)Be^{-(k+i)T} + (q\theta - E)e^{-iT} - \frac{mE\theta^2}{A^2} T^2 e^{-iT} .$$

Letting  $H=0$  in equation (44), we obtain

$$(45) \quad e^{-k\bar{T}} = \frac{(S_P - C_V)\bar{\theta} - E}{(k+i)B} - \frac{mE\bar{\theta}^2}{(k+i)A^2B} \bar{T}^2$$

from which the optimum investment time  $\bar{T}$  can be found.

Equation (45) can also be obtained by taking the derivative of  $V$  given by equation (12) with respect to  $T$  and equating to zero once we know that the optimal production rate,  $\bar{\theta}(t)$ , is a constant which is given by equations (34) and (36).

Let us define

$$(46) \quad \alpha = \frac{(S_P - C_V)\bar{\theta} - E}{(k+i)B} ,$$

$$(47) \quad \beta = \frac{mE\bar{\theta}^2}{(k+i)A^2B} ,$$

$$(48) \quad F_1 = e^{-kT} ,$$

$$(49) \quad F_2 = \alpha - \beta \bar{T}^2 .$$

Equation (45), then, can be written as

$$(50) \quad F_1 = F_2 .$$

Note that the maximum values of  $F_1$  and  $F_2$  are 1 and  $\alpha$  respectively, which occur at  $\bar{T}=0$  and both are monotonically decreasing functions of  $\bar{T}$ . As shown in Fig. 2, therefore, three situations must be considered

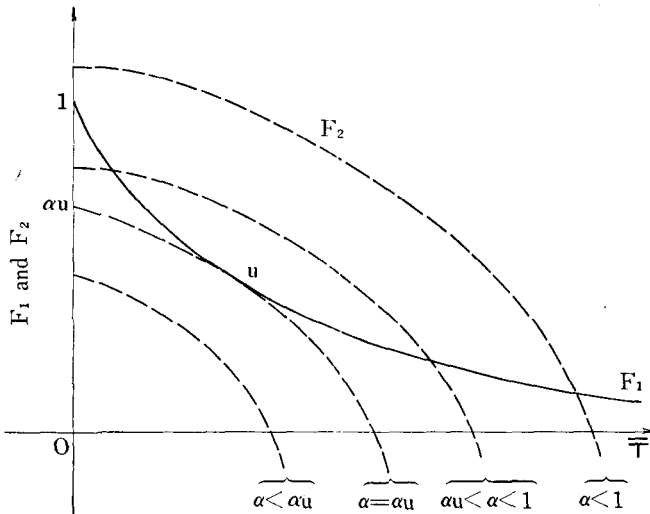


Fig. 2. Three situations in solving for  $\bar{T}$ .

in solving  $\bar{T}$  from equation (50).

When  $\alpha > 1$  only one real and positive root occurs at which the objective function (net present worth) attains a unique extremum.

When  $\alpha_u \leq \alpha \leq 1$ , there exist two positive real roots, which satisfy equation (50).  $\alpha_u$  is the value of  $\alpha$  at which the two roots coincide. In other words, when  $\alpha = \alpha_u$ , the curves representing  $F_1$  and  $F_2$  are tangential to each other.

When  $\alpha < \alpha_u$ , there is no real value solution to equation (50).

The tangential point of  $F_1$  and  $F_2$  where  $\alpha = \alpha_u$  and  $\bar{T} = \bar{T}_u$ , can be determined by simultaneously solving equation (50) and the condition,

$$(51) \quad \left. \frac{dF_1}{d\bar{T}} \right|_{\bar{T} = \bar{T}_u} = \left. \frac{dF_2}{d\bar{T}} \right|_{\bar{T} = \bar{T}_u}$$

Equations (50) and (51) can be written respectively as

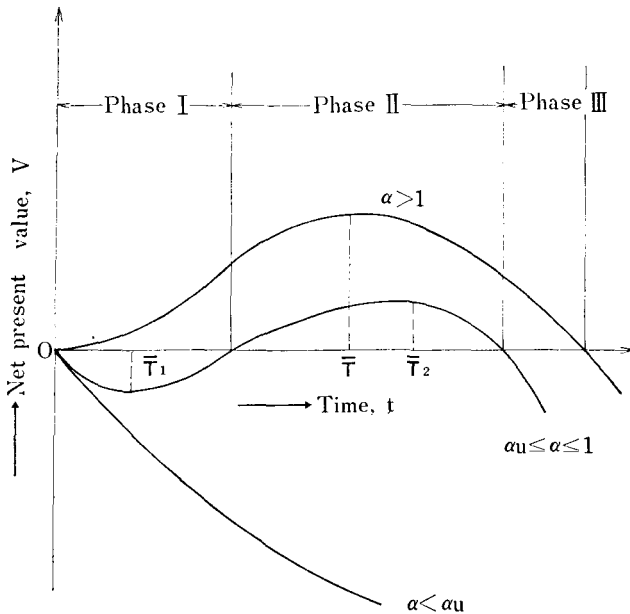


Fig. 3. Net present value under three conditions for  $\alpha$ .

$$(52) \quad e^{-k\bar{T}} = \alpha_u - \beta \bar{T}^2,$$

$$(53) \quad ke^{-k\bar{T}}u = 2\beta \bar{T}_u,$$

And the solution for  $\bar{T}_u$  can be carried out numerically.

*Summary of results*

It has been shown that, in order to maximize the total present worth of the investment according to the modified model, the maximum possible rate of production which is either the maximum plant capacity or the market share should be maintained throughout the service life of the machine.

That there exists three types of investment is a well known fact in

the business world. Figure 3 is a graphical representation of the behavior of the net present worth function under these three conditions. These are (1) investment with a net loss in all phases, (2) investment with a loss at a certain initial period in Phase I but a net profit after the initial period in Phase II, and (3) investment with a net profit from the beginning to the end of Phase III. Optimization based on the more realistic model can take into account this fact. The determining factor of these three situations is  $\alpha$  (see equation (46)) which is a function of parameters including the sale price,  $S_p$ , the various cost per unit,  $C_v$ , the optimal production rate,  $\bar{\theta}$ , the fixed overhead cost,  $E$ , the rate of depreciation,  $k$ , the annual interest rate,  $i$ , and the installment cost of the equipment,  $B$ . The three situations are summarized as follows:

1.  $\alpha > 1$ . In this case equation (45) generates only one root at which a positive extremum is attained by the net present worth function.

2.  $\alpha_u \leq \alpha \leq 1$ . Two roots,  $\bar{T}_1$  and  $\bar{T}_2$ , where  $\bar{T}_2 > \bar{T}_1$ , are obtained.  $\bar{T}_1$  occurs before the break-even point indicating the time at which the maximum loss occurs. At  $\bar{T}_2$  the net present worth of the investment is maximized.

3.  $\alpha < \alpha_u$ . Equation (45) fails to have a root and the net present worth function does not have an extremum. Losses increase indefinitely.

Figure 3 is a graphical representation of the net present worth function under these three conditions. Extensive numerical simulation parameters have confirmed the validity of analytical results presented.

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