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SOME EXTENSIONS OF DYNAMIC ECONOMIC LOT SIZE MODEL WITH BACKLOGGING*

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Abstract

This paper considers a deterministic, single product, finite time horizon inventory model with backlogging of unsatisfied demand, which will be called the dynamic economic lot size model with backlogging. In the model production cost in period t consists of setup cost plus marginal production cost. Both setup cost and marginal production cost may vary with time. Also unit holding and unit shortage costs may differ from period to period. An optimal policy is a production schedule that minimizes the total production and inventory cost and satisfies a sequence of known demands.

An essential feature of the model is that it permits backlogging of

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unsatisfied demand and marginal cost to vary with time. It is also a generalization of the Wagner and Whitin's model [7].

From a computational viewpoint it is of much interests to lighten the computational burdens required to find an optimal policy. The main results consist of three theorems that serve to lighten computational burdens, and a theorem that establishes the existence of planning horizen. An algorithm is constructed and an example is also considered.

1. Introduction

Wagner and Whitin's model [7] was a point of departure for the dynamic economic lot size model in which costs and demands at each period are given and deterministic but may differ from period to period. They considered the above model which does not permit backlogging of unsatisfied demand (no backlog case), while marginal cost of production is independent of the amount produced in each period and constant over time. Eppen, Gould and Pashigian [1] extended this model to one which allows marginal production cost to vary with time, while Zangwill [11] studied Wagner and Whitin's model where backlogging of unsatisfied demand is permitted but marginal cost is constant over time.

Now, let $P_t(x_t)$ be the production cost of producing amount x_t at period t. We are used to seeing $P_t(x_t)=s_t\cdot\delta(x_t)+c_tx_t$ ($\delta(0)=0$, $\delta(x_t)=1$ for $x_t>0$). Then we call s_t setup cost and c_t marginal production cost. Let I_t be inventory at period t. Wagner and Whitin [7] discuss a method to find an optimal policy in n periods under the conditions $I_t\geqq 0$ and $P_t(x_t)=s_t\cdot\delta(x_t)+cx_t$ for all t. Eppen, Gould and Pashigian [1] under the conditions $I_t\geqq 0$ and $P_t(x_t)=s_t\cdot\delta(x_t)+c_tx_t$ established an algorithm and theorems that decrease the computational effort required to find an optimal policy, while Zangwill [11] did the same under the conditions $I_t\gtreqqless 0$ and $P_t(x_t)=s_t\cdot\delta(x_t)+cx_t$.

This paper considers a generalized dynamic economic lot size model in which backlogging of unsatisfied demand may be permitted and marginal cost of production is a parameter which may differ from period to period; hence we study the problem of finding an optimal production schedule that minimizes n periods cost under the conditions $I_t \gtreqless 0$ and $P_t(x_t)=s_t\cdot\delta(x_t)+c_t x_t$ for all t. This paper, therefore, is an extension of Eppen, Gould and Pashigian [1] in the sense of permitting backlogging of unsatisfied demand and also an extension of Zangwill [11] in the sense of allowing marginal production cost to vary with time. In other words, this paper is a generalization of Wagner and Whitin [7] which is the original paper of the models of the kind under consideration.

The main results of this paper are three theorems that decrease the computational efforts required to find optimal policy, a planning horizon theorem that establishes the existence of planning horizon, and a forward algorithm that combines these theorems. Furthermore, we prepare a numerical example.

2. Model Formulation

In this section we give the formulation of a dynamic economic lot size model. We assume that items are produced at the beginning of periods and their production rate is infinite. We also suppose that demands of all periods are delivered at the beginning of the period after production is completed and costs with respect to inventory are proportional to the inventory at the end of the period. We use the following symbols;

x_t=production at period t, $x_t \geqq 0$,
d_t=demand at period t, $d_t>0$,
h_t=unit holding cost at period t, $h_t>0$,
s_t=setup cost (ordering cost) at period t, $s_t>0$,
p_t=unit shortage cost at period t, $p_t>0$,
c_t=marginal production cost (unit purchase cost) at period t, $c_t>0$,
$t=1, 2, \cdots, n$.

We assume n is finite. Defining I_t as the inventory at the end of period

t and assuming the initial inventory I_0 to be zero[1], I_t can be expressed as

$$(1) \qquad I_t = \sum_{i=1}^{t} (x_i - d_i) \gtreqless 0 \ .$$

Let $P_t(x_t)$ be the cost of producing x_t units and $H_t(I_t)$ be the cost of holding or backlogging I_t units in period t.

The problem is to find an optimal production schedule[2] that minimizes the total cost of n periods, called objective function,

$$(2) \qquad T(x) = \sum_{t=1}^{n} [P_t(x_t) + H_t(I_t)]$$

subject to

$$(3) \qquad x_t \geqq 0, \qquad t = 1, 2. \cdots, n,$$

$$(4) \qquad I_n = 0 \ .$$

But in this paper we especially study the model such that $P_t(x_t) = s_t \cdot \delta(x_t) + c_t x_t$ ($\delta(0)=0$, $\delta(x_t)=1$ for $x_t>0$) and $H_t(I_t) = \max\ (h_t I_t, \ -pI_t)$[3] for all t.

The following theorem (regeneration point theorem) holds in our model which is defined by equations (1), (2), (3), and (4), which was originally given by Manne and Veinott [2], but is presented here in a slightly different expression.

Regeneration Point Theorem[4]

When $x_{\beta_1}>0$ and $x_{\beta_2}>0$ for $\beta_1<\beta_2$, then there exists an optimal policy that has a property $I_\alpha=0$ for some α, $\beta_1 \leqq \alpha < \beta_2$. Therefore, for

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1) Zabel [8] considers this model when backlogging of unsatisfied demand is not permitted and $I_0>0$.

2) Such an optimal production schedule is merely called an optimal policy.

3) The following discussions hold whenever $P_t(x_t)$ is concave on $[0, \infty)$ and $H_t(I_t)$ is also concave on $(-\infty, 0]$ or $[0, \infty)$.

4) This theorem in backlog case, *e.g.* Wagner and Whitin [7] and Eppen, Gould and Pashigian [1], is one where there exists an optimal policy such that $I_{t-1}x_t=0$ for all t. References [2], [6], [9], [10] give the regeneration point theorem under the generalized conditions.

all β, ($\beta=1, 2, \cdots, n$) there exists an optimal policy such that

$$x_\beta=0 \quad \text{or} \quad \sum_{l=\gamma+1}^{\alpha} d_l$$

where $I_\gamma = I_\alpha = 0$ for $\gamma < \beta \leqq \alpha$.
The proof of the theorem is omitted since it is given by Manne and Veinott [3]. This theorem is used for establishing dynamic recursions (functional equations) in order to compute an optimal policy.

3. Extensions of Dynamic Economic Lot Size Model

In this section we establish dynamic programming recursions for finding an optimal policy. Using these recursions we derive three theorems which may be used to lighten the computational burden required to find an optimal policy and a theorem that establishes the existence of planning horizon. For this purpose we employ the following notations:

$f(\beta)$ = sum of minimum cost for periods 1 through $\beta-1$ plus production cost of producing units backlogged at the end of period $\beta-1$ when period β is a production point (period β is called a production point if $x_\beta>0$).

$g(\alpha)$ = minimum cost for periods 1 through α when period α is a regeneration point (period α is called a regeneration point if $I_\alpha=0$).

Then from the above regeneration point theorem and the principle of optimality in dynamic programming $f(\beta)$ and $g(\alpha)$ can be expressed by following equations. (see Fig. 1) In the usage of summation symbol $\sum$ in this paper we suppose that we have $\sum_{h=a}^{b} d_h=0$ if $a>b$.

$$(5) \qquad f(\beta)=s_\beta+\min_{0\leqq\gamma<\beta}\left\{c_\beta \sum_{i=\gamma+1}^{\beta-1} d_i + \sum_{i=\gamma+1}^{\beta-1} \sum_{j=\gamma+1}^{i} p_i d_j + g(\gamma)\right\}$$
$$=s_\beta+\min_{0\leqq\gamma<\beta} F(\gamma, \beta)$$

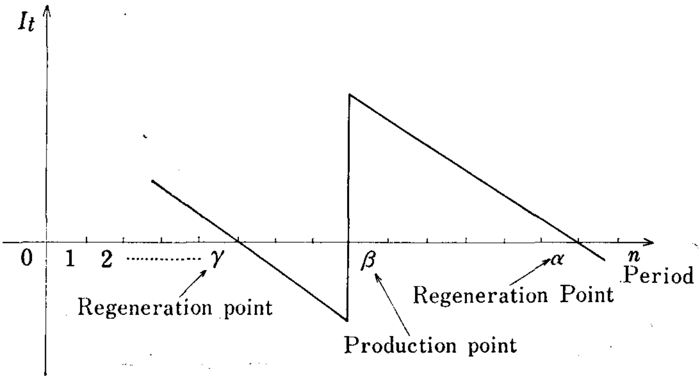

Figure-1 Inventory Situation

where $\beta=1, 2, \cdots, n$, $g(0)=0$, and

$$(6)\qquad F(\gamma, \beta)=c_\beta \sum_{i=\gamma+1}^{\beta-1} d_i + \sum_{i=\gamma+1}^{\beta-1} \sum_{j=\gamma+1}^{i} p_i d_j + g(\gamma)$$
$$= \sum_{j=\gamma+1}^{\beta-1} \left(c_\beta + \sum_{i=j}^{\beta-1} p_i\right) d_j + g(\gamma) .$$

$$(7)\qquad g(\alpha)= \min_{1\leq\beta\leq\alpha} \left\{c_\beta \sum_{i=\beta}^{\alpha} d_i + \sum_{i=\beta}^{\alpha-1} \sum_{j=i+1}^{\alpha} h_i d_j + f(\beta)\right\}$$
$$\equiv \min_{1\leq\beta\leq\alpha} G(\beta, \alpha)$$

where $\alpha=1, 2, \cdots, n$, and

$$(8)\qquad G(\beta, \alpha)=c_\beta \sum_{i=\beta}^{\alpha} d_i + \sum_{i=\beta}^{\alpha-1} \sum_{j=i+1}^{\alpha} h_i d_j + f(\beta)$$
$$= \sum_{j=\beta}^{\alpha} \left(c_\beta + \sum_{i=\beta}^{j-1} h_i\right) d_j + f(\beta) .$$

Note that $G(\beta, \alpha)$ is simply the sum of the cost of producing the demands for periods β through α, the cost of carrying inventory into periods β through α, and the cost of acting optimally for the first $(\beta-1)$

periods when period α is regeneration point and last setup is performed at period $\beta(\beta\leqq\alpha)$. Also note that $F(\gamma,\beta)$ is the sum of the cost of producing the unsatisfied demands backlogged for periods $\gamma+1$ through $\beta-1$, the cost of shortage for period $\gamma+1$ through $\beta-1$ and the cost of acting optimally for the first γ periods when period β is a production point and period $\gamma(\gamma<\beta)$ is a last regeneration point until period $\beta-1$. We shall define the notations:

$\gamma(\beta)$ = an optimal last regeneration point when period β is production point, $\gamma(\beta)<\beta$.

$\beta(\alpha)$ = an optimal last production point when period α is regeneration point, $\beta(\alpha)\leqq\alpha$.

By means of these notations $f(\beta)$ and $g(\alpha)$ can be rewritten as follows:

$$(5)' \qquad f(\beta)=s_\beta+\min_{0\leq\gamma<\beta} F(\gamma,\beta)=s_\beta+F(\gamma(\beta),\beta),$$

$$(7)' \qquad g(\alpha)=\min_{1\leq\beta\leq\alpha} G(\beta,\alpha)=G(\beta(\alpha),\alpha).$$

We now present three theorems which will lead to reduced computational effort in the forward algorithm.

Theorem 1. The following relations hold;

a) If there exists $\beta_1(<\beta)$ such that $c_{\beta_1}\leqq c_\beta+\sum_{i=\beta_1}^{\beta-1} p_i$, then

$$\min_{0\leqq\gamma<\beta} F(\gamma,\beta)=\min_{\gamma\in\Gamma} F(\gamma,\beta)$$

where $\Gamma=\{\gamma\,|\,\gamma(\beta_1)\leqq\gamma<\beta\}$.

b) If there exists $\beta_1(<\beta)$ such that $c_{\beta_1}\geqq c_\beta+\sum_{i=\beta_1}^{\beta-1} p_i$, then

$$\min_{0\leqq\gamma<\beta} F(\gamma,\beta)=\min_{\gamma\in\Gamma} F(\gamma,\beta)$$

where $\Gamma=\{\gamma\,|\,0\leqq\gamma\leqq\gamma(\beta_1)\ \text{or}\ \beta_1\leqq\gamma<\beta\}$.

Proof: From the definition of $\gamma(\beta_1)$ the relation

$$F(\gamma(\beta_1), \beta_1) \leqq F(\gamma, \beta_1)$$

holds for $0 \leqq \gamma < \beta_1$. This relation may be rewritten as

$$F(\gamma(\beta_1)\gamma, \beta) + A \leqq F(\gamma, \beta)$$

where $A = [F(\gamma, \beta) - F(\gamma, \beta_1)] - [F(\gamma(\beta_1), \beta) - F(\gamma(\beta_1), \beta_1)]$.

First of all we prove part *a*). For $\gamma < \gamma(\beta_1)$, A is rearranged as

$$\begin{aligned} A = & \left[(c_\beta - c_{\beta_1}) \sum_{i=\gamma+1}^{\beta_1 - 1} d_i + c_\beta \sum_{i=\beta_1}^{\beta-1} d_i + \sum_{i=\beta_1}^{\beta-1} \sum_{j=\gamma+1}^{i} p_i d_j \right] \\ & - \left[(c_\beta - c_{\beta_1}) \sum_{i=\gamma(\beta_1)+1}^{\beta_1} d_i + c_\beta \sum_{i=\beta_1}^{\beta-1} \sum_{j=\gamma(\beta_1)+1}^{i} p_i d_j \right] \\ = & (c_\beta - c_{\beta_1}) \sum_{i=\gamma+1}^{\gamma(\beta_1)} d_i + \sum_{i=\beta_1}^{\beta-1} \sum_{j=\gamma+1}^{\gamma(\beta_1)} p_i d_j \\ = & \left(c_\beta - c_{\beta_1} + \sum_{i=\beta_1}^{\beta-1} p_i \right) \sum_{j=\gamma+1}^{\gamma(\beta_1)} d_j \, . \end{aligned}$$

Thus A is nonnegative by our condition. Therefore for $\gamma < \gamma(\beta_1)$ we have $F(\gamma(\beta_1), \beta) \leqq F(\gamma, \beta)$, $\beta_1 < \beta$. This completes the proof of part *a*).

Secondly we prove part *b*). As in case of proof of part *a*) for $\gamma(\beta_1) < \gamma < \beta_1$ A is rearranged as

$$A = \left(c_{\beta_1} - c_\beta - \sum_{i=\beta_1}^{\beta-1} p_i \right) \sum_{j=\gamma(\beta_1)+1}^{\gamma} d_j \, .$$

By our condition we obtain $A \geqq 0$. Thus we have $F(\gamma(\beta_1), \beta) \leqq F(\gamma, \beta)$ for $\gamma(\beta_1) < \gamma < \beta_1$ and this result completes the proof of part *b*) (Q.E.D.)

Part *a*) of theorem 1, asserts that if its condition holds, there exists an optimal policy such that the relation $\gamma(\beta_1) \leqq \gamma(\beta)$ holds for $\beta_1 < \beta$, *i.e.*, an optimal regeneration point for period β takes place at the period after an optimal regeneration point for period $\beta_1 < \beta$. Part *b*) of theorem 1, if the condition of part *a*) were not satisfied, asserts that an optimal regeneration point for period β does not take place at the periods between $\gamma(\beta_1)+1$ and $\beta_1 - 1$. We can give the geometric interpretation

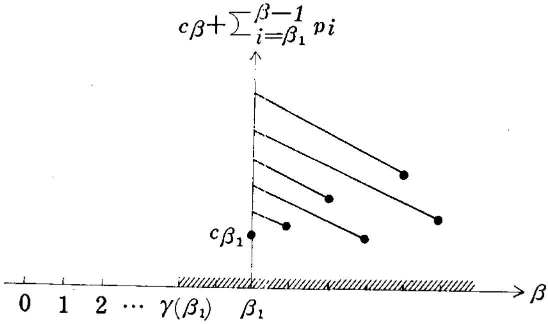

Figure-2 a). Part a) of Theorem I

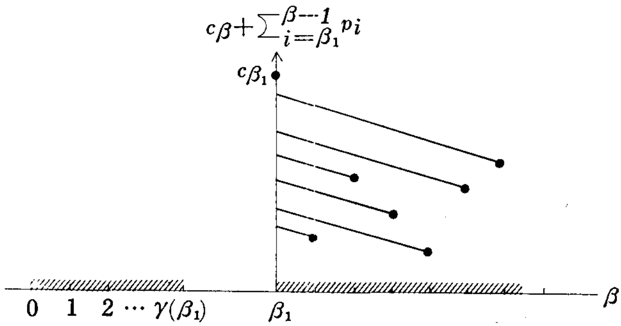

Figure-2 b) Part b) of Theorem 1

(see Fig. 2) for part *a*) and part *b*) of theorem 1 (where we assume $p_t=p$ for all t).

Theorem 2.

a) For $\alpha_1<\alpha_2$ we have

$$g(\alpha_2)=\min_{\beta\in B} G(\beta, \alpha_2)$$

where $B=U-(V_1\cup V_2)$, $U=\{\beta|1\leqq\beta\leqq\alpha_2\}$, $V_1=\left\{\beta|1\leqq\beta\leqq(\alpha_1),\right.$

$\left. c_\beta\geqq c_{\beta(\alpha_1)}-\sum_{i=\beta}^{\beta(\alpha_1)-1} h_i\right\}$, and $V_2=\left\{\beta|\beta(\alpha_1)<\beta<\alpha_2,\ c_\beta\geqq c_{\beta(\alpha_1)}+\sum_{i=\beta(\alpha_1)}^{\beta-1} h_i\right\}$

b) If $g(\alpha)\leqq f(\alpha+1)$ and $c_{\beta(\alpha)}+\sum_{i=\beta(\alpha)}^{\alpha} h_i\leqq c_{\alpha+1}$, we have

$$g(\alpha+1)=\min_{1\leqq\beta\leqq\alpha} G(\beta,\alpha+1).$$

c) If $f(\alpha+1)\leqq g(\alpha)$ and $c_{\alpha+1}\leqq c_\beta+\sum_{i=\beta}^{\alpha} h_i$ for all $1\leqq\beta\leqq\alpha$, we have

$$g(\alpha+1)=G(\alpha+1,\alpha+1).$$

Proof: First, we prove part *a*). From definition of $\beta(\alpha_1)$ the relation $G(\beta(\alpha_1),\alpha_1)\leqq G(\beta,\alpha_1)$ holds for all α_1. This relation is rearranged into the following form.

$$G(\beta(\alpha_1),\alpha_2)+B\leqq G(\beta,\alpha_2),\qquad \alpha_1<\alpha_2,$$

where

$$B=[G(\beta,\alpha_2)-G(\beta,\alpha_1)]-[G(\beta(\alpha_1),\alpha_2)-G(\beta(\alpha_1),\alpha_1)].$$

case (*i*) if $1\leqq\beta\leqq\beta(\alpha_1)$,

$$B=\left[c_\beta-c_{\beta(\alpha_1)}+\sum_{i=\beta}^{\beta(\alpha_1)-1} h_i\right]\sum_{j=\alpha_1+1}^{\alpha_2} d_j\ .$$

case (*ii*) if $\beta(\alpha_1)<\beta\leqq\alpha_1$,

$$B=\left[c_\beta-c_{\beta(\alpha_1)}-\sum_{i=\beta(\alpha_1)}^{\beta-1} h_i\right]\sum_{j=\alpha_1+1}^{\alpha_2} d_j\ .$$

In both cases $B\geqq 0$ if the contents of the brackets are nonnegative and then we obtain $G(\beta(\alpha_1),\alpha_2)\leqq G(\beta,\alpha_2)$. These results imply part *a*) of theorem. Secondly, we begin to prove part *b*) and *c*). $g(\alpha+1)$ is rewritten as follows;

$$g(\alpha+1)=\min\left\{\begin{array}{l}\min\limits_{1\leqq\beta\leqq\alpha}\left[G(\beta,\alpha)+c_\beta d_{\alpha+1}+\sum\limits_{i=\beta}^{\alpha} h_i\, d_{\alpha+1}\right],\\ c_{\alpha+1}d_{\alpha+1}+f(\alpha+1)\end{array}\right\}$$

proof of b):

$$\min_{1\leqq\beta\leqq\alpha}\left\{G(\beta,\alpha)+c_\beta d_{\alpha+1}+\sum_{i=\beta}^{\alpha} h_i d_{\alpha+1}\right\}\leqq g(\alpha)+c_{\beta(\alpha)}d_{\alpha+1}+\sum_{i=\beta(\alpha)}^{\alpha} h_i d_{\alpha+1}$$
$$\leqq g(\alpha)+c_{\alpha+1}d_{\alpha+1}\leqq f(\alpha+1)+c_{\alpha+1}d_{\alpha+1}\,.$$

proof of c)

$$\min_{1\leqq\beta\leqq\alpha}\left\{G(\beta,\alpha)+c_\beta d_{\alpha+1}+\sum_{i=\beta}^{\alpha} h_i d_{\alpha+1}\right\}\geqq g(\alpha)+c_{\alpha+1}d_{\alpha+1}$$
$$\geqq f(\alpha+1)+c_{\alpha+1}d_{\alpha+1}\,.$$

These results complete the proof of the theorem. (Q.E.D.)

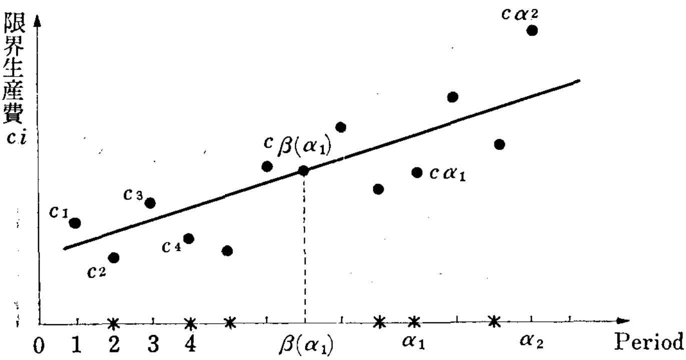

Figure-3 Two ways extrapolation
(asteriskes ✳ are elements of B)

Giving a geometric interpretation for part *a*) of theorem 2, we can obtain Fig. 3. But we suppose $h_i=h$ (constant) for all i in order to simplify the figure. Given an optimal last production point $\beta(\alpha_1)$ for period α_1, an optimal last production point $\beta(\alpha_2)$ for period α_2 (see Fig. 3) is an element of the set of periods that have marginal cost lying on or below two ways extrapolation from $c_{\beta(\alpha_1)}$. Since in the model of Wagner and Whitin [7] and Zangwill [11] marginal cost is constant through all periods, there does not exist a period for which marginal cost lies on or below backward extrapolation from $c_{\beta(\alpha_1)}$. Then there exists an optimal policy such that the relation $\beta(\alpha_1)\leqq\beta(\alpha_2)$ holds for $\alpha_1<\alpha_2$. This relation is just the one that they derived in their models.

Theorem 3. Suppose we obtain $g(\gamma)=G(\beta(\gamma),\gamma)$ for some γ. If $c_\beta+\sum_{i=\gamma+1}^{\beta-1}p_i\geqq c_{\beta(\gamma)}+\sum_{i=\beta(\gamma)}^{\gamma}h_i$, then $F(\gamma,\beta)\geqq F(\gamma+1,\beta)$ holds for some β.

Proof; For some γ the following relation holds;

$$g(\gamma+1)\leqq\min_{1\leqq\beta\leqq\gamma}\left\{G(\beta,\gamma)+c_\beta d_{\gamma+1}+\sum_{i=\beta}^{\gamma}h_i d_{\gamma+1}\right\}$$

$$\leqq g(\gamma)+c_{\beta(\gamma)}d_{\gamma+1}+\sum_{i=\beta}^{\gamma}h_i d_{\gamma+1}\,.$$

Therefore, since $g(\gamma)-g(\gamma+1)\geqq -c_{\beta(\gamma)}d_{\gamma+1}-\sum_{i=\beta(\gamma)}^{\gamma}h_i d_{\gamma+1}$,

$$F(\gamma,\beta)-F(\gamma+1,\beta)=\left(c_\beta+\sum_{i=\gamma+1}^{\beta-1}p_i\right)d_{\gamma+1}+g(\gamma)-g(\gamma+1)$$

$$\geqq\left(c_\beta+\sum_{i=\gamma+1}^{\beta-1}p_i-c_{\beta(\gamma)}-\sum_{i=\beta(\gamma)}^{\gamma}h_i\right)d_{\gamma+1}\geqq 0\,. \qquad \text{(Q.E.D.)}$$

The cost that is increased by producing one more unit at period $\beta(\gamma)$ is $c_{\beta(\gamma)}+\sum_{i=\beta(\gamma)}^{\gamma}h_i$ and the cost decrease is $\sum_{i=\gamma+1}^{\beta-1}p_i+c_\beta$. If the decreased cost is greater than the increased one, theorem 3 implies that an optimal regeneration point moves from γ to $\gamma+1$.

The above three theorems are used for the purpose of narrowing down the number of periods which must be considered as candidates for optimal regeneration point and production point.

Next we discuss a theorem that may divide the whole planning horizon into shorter ones, namely a theorem that establishes the existence of planning horizon. Now we give a definition of planning horizon.

Definition: Let $x^*=(x_1{}^*,\cdots,x_n{}^*)$ be an optimal production schedule (optimal policy) minimizing the total cost for the whole planning horizon. If sub-optimal schedule $(x_1^*, \cdots, {}^*_{t-1})$, $t-1<n$, of this whole optimal schedule x^* is decided by information only with respect to costs and demands for periods 1 through t and is independent of information after period $t+1$, we say that periods 1 through $t-1$ are a planning horizon.

Planning Horizon Theorem: If for $j<k$ $\beta(k)=\beta(j)$ or $\beta(k)>j$ and $\gamma(k)=\gamma(\beta(j))$ or $\gamma(k)\geqq j$, then periods 1 through $\beta(j)-1$ are a planning horizon.

Proof: we may assume that $\beta(j)$ and $\gamma(\beta(j))$ are given for some j. At first we consider period $j+1$. If $\beta(j+1)=\beta(j)$,

$$g(j+1)=G(\beta(j), j+1)$$
$$=F(\gamma(\beta(j)), \beta(j))+s_{\beta(j)}+\sum_{l=\beta(j)}^{j}\sum_{m=l+1}^{j+1} h_l d_m+c_{\beta(j)}\sum_{l=\beta(j)}^{j+1} d_l .$$

Otherwise if $\beta(j+1)=j+1$,

$$g(j+1)=G(j+1, j+1)$$
$$=\min\ \{G(\beta(j), j)+s_{j+1}+c_{j+1}d_{j+1}, F(\gamma(\beta(j)), j+1)$$
$$+s_{j+1}+c_{j+1}d_{j+1}\}.$$

In both cases the policy of periods 1 through $\beta(j)-1$ does not change at all. Secondly we consider period $j+2$. If $\beta(j+2)=\beta(j)$,

$$g(j+2)=G(\beta(j), j+2)$$
$$=F(\gamma(\beta(j)), \beta(j))+s_{\beta(j)}+c_{\beta(j)}\sum_{l=\beta(j)}^{j+2} d_l+\sum_{l=\beta(j)}^{j+1}\sum_{m=l+1}^{j+2} h_l d_m .$$

If $\beta(j+2)=j+1$, $g(j+2)=G(j+1, j+2)=\min\{G(\beta(j), j)+c_{j+1}(d_{j+1}+d_{j+2})+h_{j+1}d_{j+2}, F(\gamma(\beta(j)), j+1)+s_{j+1}+c_{j+1}(d_{j+1}+d_{j+2})+h_{j+1}d_{j+2}\}$. If $\beta(j+2)=j+2$, $g(j+2)=G(j+2, j+2)=\min\{G(\beta(j), j)+p_{j+1}d_{j+1}+s_{j+2}+s_{j+2}+c_{j+2}(d_{j+1}+d_{j+2}), G(\beta(j), j)+s_{j+1}+s_{j+2}+c_{j+1}d_{j+1}+c_{j+2}d_{j+2}, F(\gamma(\beta(j)), j+2)+s_{j+2}+c_{j+2}d_{j+2}\}$. In all cases the policy of periods 1 through $\beta(j)-1$ does not change at all. By extending this argument to period $j+3$, $j+4, \cdots,$ the theorem is proved for all $k>j$. (Q.E.D.)

Corollary: When one of the following conditions is satisfied, the above planning horizon theorem holds;

(*i*) period j is a regeneration point and $\beta(j)$ is a period such that minimizes $c_{\beta(j)}+\sum_{l=\beta(j)}^{j} h_l$,

(*ii*) the relation $\gamma(j+1)=j$ holds,

(*iii*) $c_i+\sum_{l=j+1}^{i-1} p_l \geqq c_{j+1}$ for each $i=j+1, j+2, \cdots$,

The proof of the corollary is simple and similar to the one of the previous theorem and therefore omitted.

4. Algorithm and Example

In this section we construct the algorithm and apply it to a numerical example. Before going further, however, we summarize the above results by a skillful method. This method facilitates the descriptions of the algorithm that is presented below.

Now let $a_{ij}=c_i+\sum_{k=i}^{j-1} h_k$ for $i<j$ and $a_{ij}=c_i$ for $i=j$. Let also $b_{ij}=c_i+\sum_{k=j}^{i-1} p_k$ for $j<i$ and $b_{ij}=c_i$ for $j=i$. If $i=j$, then $a_{ij}=b_{ij}=c_i$. a_{ij} and b_{ij} may be arrayed in the following matrix form and we denote it by M.

$$M=\begin{pmatrix} c_1 & a_{12} & \cdots & a_{1n} \\ b_{21} & & & \\ \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ b_{n1} & & & c_n \end{pmatrix}$$

Using these notations, equations (5), (6), (7), and (8) are rewritten as

$$(5)'' \qquad f(i)=s_i+\min_{0\leq k<i}\left\{g(k)+\sum_{j=k+1}^{i-1} b_{ij}d_j\right\},$$

$$(6)' \qquad F(k,i)=g(k)+\sum_{j=k+1}^{i-1} b_{ik}d_j ,$$

(7)″ $$g(i)=\min_{1\leqq k\leqq i}\left\{f(k)+\sum_{j=k}^{i} a_{ik}d_j\right\},$$

(8)′ $$G(k,i)=f(k)+\sum_{j=k}^{i} a_{ij}d_j,$$

where $i=1,2,\cdots n$, and $g(0)=0$. We also employ the following notations:

Γ_i = set of periods that includes an optimal regeneration point when period i is a production point. Thus $\gamma(i)\in\Gamma_i$.

B_i = set of periods that includes an optimal production point when period i is a regeneration point. Thus $B(i)\in B_i$.

Using these new notations and restating the previous theorems, we have the following:

Theorem 1. a) If $b_{ij}\geqq c_j$, $\Gamma_i=\{k|\gamma(j)\leqq k<i\}$. (Periods 1 through $\gamma(j)-1$ may be eleminated.)

b) If $b_{ij}\leqq c_j$, $\Gamma_i=\{k|0\leqq k\leqq\gamma(j)$ or $j\leqq k<i\}$.

(Periods $\gamma(j)+1$ through $j-1$ may be eliminated.)

Theorem 2. a) $B_j=\{i|1\leqq i<j,\ a_{ij}<a_{\beta(j-1)j}\}\cup\{B(j-1)\}\cup\{j\}$.

b) If $f(j)\geqq g(j-1)$ and $c_j\geqq a_{\beta(j-1)j}$, we may eliminate j from B_j.

c) If $f(j)\leqq g(j-1)$ and $c_j=\min\limits_{1\leqq i\leqq j} a_{ij}$, $j=B_j$.

Theorem 3. If $b_{ij}\geqq a_{\beta(j-1)j}$, $F(j,i)\leqq F(j-1,i)$. We may eliminate $j-1$ from Γ_i.

Table—1.

j	1	2	3	4	5	6
d_j	120	20	140	120	200	100
s_j	15	50	900	600	100	60
c_j	8	10	4	3	8	4
h_j	1	1	1	1	1	1
p_j	5	5	5	5	5	5

Table—2. Computations for an optimal policy

j		1	2	3	4	5	6
d_j		120	30	140	120	200	100
$M=\begin{pmatrix} c_1 & a_{ij} \\ b_{ij} & c_6 \end{pmatrix}$	1	8	9*	10*	11*	(12)	(13)
	2	15	(10)	(11)	(12)	(13)	(14)
	3	14	(9)	4	5	6*	(7)
	4	18	(13)	8	3	4	5*
	5	28	(23)	(18)	(13)	(8)	(9)
	6	29	(24)	(19)	(14)	(9)	4
Γ_j		0	0, 1	0, 2	2, 3	4	4, 5
$F(i, j)$		$F(0,1)$ $=0$	$F(0,2)=$ 2100 $F(1,2)$ $=975$	$F(1,3)$ $=975$ $+9(30)$ $F(2,3)$ $=1245$	$F(2,4)$ $=1245$ $+8(140)$ $F(3,4)$ $=2645$	$F(4,5)$ $=3305$	$F(4,6)$ $=3305$ $+9(200)$ $F(5,6)$ $=4125$
s_j		15	50	900	600	100	60
$f(j)$		15	1025	2145	2965	3405	4185
$\gamma(j)$		0	0	2 (or 1)	2	4	5
B_j		1	1	1, 3	1, 3, 4	3, 4	4, 6
$G(i, j)$		$G(1,1)$ $=15+$ $8(120)$	$G(1,2)$ $=975$ $+9(30)$	$G(1,3)$ $=1245$ $+10(14)$ $G(3,3)$ $=2145$ $+4(140)$	$G(1,4)$ $=2645$ $+11(120)$ $G(3,4)$ $=2705$ $+5(120)$ $G(4,4)$ $=2965$ $+3(120)$	$G(3,5)$ $=3305$ $+6(200)$ $G(4,4)$ $=3325$ $+4(200)$	$G(4,6)$ $=4125$ $+5(100)$ $G(6,6)$ $=4185$ $4(100)$
$g(j)$		975	1245	2645	3305	4125	4585
$\beta(j)$		1	1	1	3	4	6

The algorithm is simply presented as the following sequence of steps. We suppose that calculations have been done for periods 1 through $j-1$. Then for period j,

step 1: By part *a*) or *b*) of theorem 1 and theorem 3 tabulate elements of Γ_j,

step 2: Calculate $F(i,j)$ for all $i \in \Gamma_j$,

step 3: Find $f(j)=s_j+\min_{i\in\Gamma_j} F(i,j)$ and $\gamma(j)$,

step 4: By theorem 2 tabulate elements of B_j,

step 5: Calculate $G(i,j)$ for all $i \in B_j$,

step 6: Find $g(j)=\min_{i\in B_j} G(i,j)$ and $\beta(j)$.

Return to step 1 for period $j+1$ and stop at period n. Then we obtain an optimal policy and minimum cost $g(n)$ for n periods problem.

In order to understand the concrete effects of forward algorithm we consider the following example with $n=6$. The necessary datum is shown in Table 1. Unit of cost is one dollar. The calculations of optimal policy for six periods problem are summarized in Table 2. The main diagonal elements c_i of M is designeted by underlining. (2) Calculated from top to bottom from the first column. (2) As soon as we obtain $\beta(j)$ we give the sign $*$ to $a_{\beta(j-1)j}$ of the successive period. (3) If the part *a*) of theorem 2 and theorem 3 hold, we bracket a_{ij} and b_{ij}. Hence bracket for some column those numbers which are greater than or equal to the one with $*$, except for numbers with $*$ and underlined. (4) If the condition of part *a*) of theorem 2 is satisfied, then bracket the c_j. (5) Decide elements of B_j by a_{ij} without bracket. (6) The part *b*) of theorem 1 eliminates period 1 from Γ_3. (7) By the part *a*) of theorem 1 $\Gamma_4=\{2, 3\}$. (8) $\Gamma_5=4$ by the part *a*) of theorem 1 and theorem 3 and Periods 1, 2, and 3 are eliminated from consideration. (9) $\Gamma_6=\{4, 5\}$ by the part *a*) of theorem 1.

The optimal policy discovered is $x_6=100$, $x_5=0$, $x_4=460$, $x_3=0$, $x_2=0$ and $x_1=150$. Then the minimum cost is 4,585,000 dollars.

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