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ON THE INVENTORY PROBLEM OF TWO SUBSTITUTE PRODUCTS

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Abstract

An inventory problem stocking two products, with product 1 serving partly as a substitute for product 2, and vice versa, is studied. It is shown here how the optimal inventory level varies with substitute rate $k_1=k_2=k$ and discounted rate $q_1=q_2=q$, when both products have the same and linear cost functions. In the case of $k=1$, we obtain (1) the relation between the optimal inventory level and it for $k=0$, (2) the monotonicity and continuity of the optimal level with respect to a parameter q, and (3) bounds of the optimal inventory level. In the case of $k \neq 1$, we can only get (1) bounds, (2) the continuity of the optimal inventory with respect to k and q, and (3) monotonicity with respect to q for any fixed k under the assumption of same demand distribution of both products. In some examples, we calculate optimal inventory levels.

1. Introduction and Summary

We consider the inventory problem for two substitute products, e.g., butter and margarine or two similar goods with a different design or color. At the beginning of a period, the inventory manager may order stock with no delivery lag and proportional ordering costs. During the period the random demands are satisfied with stocks on hand. When stocks of product 1 are sold out, but those of products 2 remain unsold, $100k_2$ percent of customers who fail to buy product 1, will buy product 2 at a discount of $100(1-q_2)$ percent, and vice versa. When $k_1=k_2=0$, all customers of any product do not buy the other product at all. Storage costs and revenues from sales, stationary over time, are considered. An ordering policy that minimizes expected costs for N periods is sought.

In a recent article [1], a substitute inventory model is treated in the case of $k_1=k_2=1$ and $q_1=q_2=1$. They can be interpreted as multi-echelon, multi-location inventory models, but in those cases, the properties of optimal inventory level are not considered. In [4], bounds on the optimal base stock level are obtained for a special supply policy.

In this paper, we consider how the optimal inventory level (the optimal base stock level) varies with substitute rate $k_1=k_2=k$ and discount rate $q_1=q_2=q$, when both products have the same and linear cost functions. In the case of $k=1$, we obtain (1) the relation between the optimal inventory level and it for $k=0$, (2) the monotonicity and continuity of the optimal level with respect to a parameter q, and (3) bounds of the optimal inventory level. In the case of $k\neq1$, we can only get (1) bounds, (2) the continuity of the optimal inventory level with respect to k and q, and (3) monotonicity with respect to q for any fixed k under the assumption of the same demand distribution of both products. When both products have different probability distributions of demand variables, monotonicity with respect to q or k does not hold. It is partly because the set of these optimal levels does not have Property A,① in general. In some examples, we calculate optimal inventory levels and show that monotonicity with respect to k does not hold.

2. The Model

We consider the following model;

①; it will be defined in §4.

(1) Two types of product, product 1 and product 2, are involved.
(2) The inventory of each product is reviewed and an order is placed for each stock periodically at equal specified intervals of time.
(3) During the period the uncertain demands for each product are satisfied respectively with the stock available for each product and when one product is sold out, but the other product remain unsold, then one product is substituted by stocks of the other product at some specified rate.
(4) At the end of the period any unfilled demand is entirely backlogged to be eventually filled by future.

Furthermore, we make the assumptions as follows;
(1) The demand vectors in successive periods are stochastically independent and identically distributed.
(2) There is no lag in delivery for each product.
(3) There is no fixed charge for placing an order, and ordering cost is proportional.
(4) The storage cost is assumed to be a function of the stock on hand at the end of the period.
(5) Cost functions, revenues from sales, demand distributions and substitute rates do not vary over time.

List of Symbols

In the following, let $i=1,2$ and $j=1,2$.

c_i ; the unit purchase cost for product i.

$h_i(\cdot)$; the holding cost per unit period for product i.

$p_i(z)$; the total sale price of z units for product i.

ξ_i^n ; the demand variable in period n for product i.

$\varphi(\boldsymbol{\xi})$; the joint probability density of $\boldsymbol{\xi}=(\xi_1, \xi_2)$.

$\varphi_i(\xi_i)$; the marginal probability density for product i,

$$\Phi_i(x) \equiv \int_0^x \varphi_i(\xi)\, d\xi \ .$$

x_i^n ; initial inventory of product i on hand at the beginning of period n.

y_i^n ; starting inventory of product i on hand after ordering in period n.

Using vector representation, we denote

$\boldsymbol{\xi}^n=(\xi_1^n,\ \xi_2^n)$, $\quad \boldsymbol{x}^n=(x_1^n,\ x_2^n)$, $\quad \boldsymbol{y}^n=(y_1^n,\ y_2^n)$, $\quad \boldsymbol{c}\cdot\boldsymbol{y}=c_1y_1+c_2y_2$

$T_i(\boldsymbol{y}^n,\ \boldsymbol{\xi}^n)$; the amount of product i on hand at the end of period n.

$$\boldsymbol{T}(\boldsymbol{y}^n,\ \boldsymbol{\xi}^n)=(T_1(\boldsymbol{y}^n,\ \boldsymbol{\xi}^n),\ T_2(\boldsymbol{y}^n,\ \boldsymbol{\xi}^n)),$$

evidently $\boldsymbol{x}^{n+1}=\boldsymbol{T}(\boldsymbol{y}^n,\ \boldsymbol{\xi}^n)$.

k_i ; the substitute rate of product i for product j, that is, $100k_i$ percent of customers who fail to buy product j, will buy product i, $(i\neq j)$, $0\leqq k_i\leqq 1$.

$1-q_i$; the discount rate of the sale price when product i is substituted for product j, $(i\neq j)$, $0\leqq q_i\leqq 1$.

α ; the present value at the beginning of period n of one cost unit at the beginning of period $n+1$.

Model Formulation

Having starting stock $\boldsymbol{y}$, the holding cost and the total sale price in a period become as follows (Table 1).

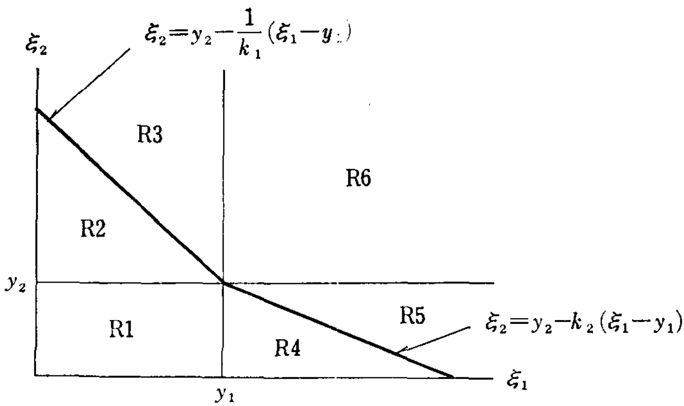

Table 1

demand region	the holding cost	total sale price
R1	$h_1(y_1-\xi_1)+h_2(y_2-\xi_2)$	$p_1(\xi_1)+p_2(\xi_2)$
R2	$h_1(y_1-\xi_1-k_1(\xi_2-y_2))$	$p_1(\xi_1)+p_2(y_2)+q_1p_1(k_1(\xi_2-y_2))$
R3	0	$p_1(\xi_1)+p_2(y_2)+q_1p_1(y_1-\xi_1)$
R4	$h_2(y_2-\xi_2-k_2(\xi_1-y_1))$	$p_1(y_1)+p_2(\xi_2)+q_2p_2(k_2(\xi_1-y_1))$
R5	0	$p_1(y_1)+p_2(\xi_2)+q_2p_2(y_2-\xi_2)$
R6	0	$p_1(y_1)+p_2(y_2)$

Let $L^k(\boldsymbol{y})$ ② be the total expected cost for one period with starting stock $\boldsymbol{y}$ and substitute rate k. Then,

(1)
$$L^k(\boldsymbol{y})=\int_{\xi_1=0}^{y_1}\int_{\xi_2=0}^{y_2}[h_1(y_1-\xi_1)+h_2(y_2-\xi_2)-p_1(\xi_1)-p_2(\xi_2)]\varphi(\boldsymbol{\xi})\,d\boldsymbol{\xi}$$
$$+\int_{\xi_1=0}^{y_1}\int_{\xi_2=y_2}^{y_2+1/k_1\cdot(y_1-\xi_1)}[h_1(y_1-\xi_1-k_1(\xi_2-y_2))-p_1(\xi_1)-p_2(y_2)$$
$$-q_1p_1(k_1(\xi_2-y_2))]\varphi(\boldsymbol{\xi})\,d\boldsymbol{\xi}$$
$$-\int_{\xi_1=0}^{y_1}\int_{\xi_2=y_2+1/k_1\cdot(y_1-\xi_1)}^{\infty}[p_1(\xi_1)+p_2(y_2)+q_1p_1(y_1-\xi_1)]\varphi(\boldsymbol{\xi})\,d\boldsymbol{\xi}$$
$$+\int_{\xi_2=0}^{y_2}\int_{\xi_1=y_1}^{y_1+1/k_2\cdot(y_2-\xi_2)}[h_2(y_2-\xi_2-k_2(\xi_1-y_1))-p_1(y_1)-p_2(\xi_2)$$
$$-q_2p_2(k_2(\xi_1-y_1))]\varphi(\boldsymbol{\xi})\,d\xi$$
$$-\int_{\xi_2=0}^{y_2}\int_{\xi_1=y_1+1/k_2\cdot(y_2-\xi_2)}^{\infty}[p_1(y_1)+p_2(\xi_2)+q_2p_2(y_2-\xi_2)]\varphi(\boldsymbol{\xi})\,d\boldsymbol{\xi}$$
$$-\int_{\xi_1=y_1}^{\infty}\int_{\xi_2=y_2}^{\infty}[p_1(y_1)+p_2(y_2)]\varphi(\boldsymbol{\xi})\,d\boldsymbol{\xi}\ .$$

(2) Denoting $G^k(\boldsymbol{y})$ ③ $=\boldsymbol{c}\cdot\boldsymbol{y}+L^k(\boldsymbol{y})-\alpha c\cdot E(\boldsymbol{T}(\boldsymbol{y},\boldsymbol{\xi}))$,

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② ③ ④; k indicates the substitute rate vector, $k=(k_1, k_2)$. Specially, $k=0$ implies no substitute case.

we assume that $G^k(\boldsymbol{y})$ exists and is finite for all $\boldsymbol{y}$.

At the beginning of period i, assume that the inventory manager knows the history $H_i=(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^i, \boldsymbol{y}^1, \cdots, \boldsymbol{y}^{i-1}, \boldsymbol{\xi}^1, \cdots, \boldsymbol{\xi}^{i-1})$ on the basis of which he chooses $\boldsymbol{y}^i$.

An ordering policy is a sequence of vector valued functions $\bar{Y}=(\bar{Y}_1, \bar{Y}_2, \cdots, \bar{Y}_N)$ such that at the beginning of period i, after having observed the past history H_i, the manager orders $\bar{Y}_i(H_i)-\boldsymbol{x}^i$. Of course $\bar{Y}_i(\cdot)$ must be chosen so that $\bar{Y}_i(H_i)\geqq\boldsymbol{x}^i$ for all H_i. It is convenient to define

$$(3) \qquad f_N{}^k(\boldsymbol{x}^1|\bar{Y}) ④= \sum_{t=1}^{N} \alpha^{t-1}EG^k(\boldsymbol{y}^t)$$

as the expected discounted cost for N periods under a policy $\bar{Y}$ starting with an initial inventory $\boldsymbol{x}^1$ in period 1. The propriety of (2) and (3) is in the spirit of formulations by Veinott [2].

The problem is to choose $\bar{Y}^*$ to minimize $f_N{}^k(\boldsymbol{x}^1|\bar{Y})$. Such a policy is termed optimal.

3. Proportional Cost Functions

We consider the case where all cost functions are proportional. Let

$$(4) \qquad \begin{cases} \boldsymbol{c}(\boldsymbol{z})=c_1z_1+c_2z_2 & \\ h_i(z_i)=h_iz_i & \\ & \text{for } z_i\geqq 0 \\ p_i(z_i)=p_iz_i & \\ c_i(z_i)=h_i(z_i)=p_i(z_i)=0 & \text{for } z_i<0 \end{cases}$$

and $h_i>0$, $p_i>0$, $C_i\geqq 0$, for $i=1, 2$.

Denote by $f_N{}^0(\boldsymbol{x}^1|\bar{Y}^{**})$ the expected discounted cost in no substitute case for initial inventory $\boldsymbol{x}^1$ and the optimal policy $\bar{Y}^{**}$.

Proposition 1. If $h_1+q_1p_1+\alpha c_2-\alpha c_1>0$ and $h_2+q_2p_2+\alpha c_1-\alpha c_2>0$, then $f_N{}^0(\boldsymbol{x}^1|\bar{Y}^{**})\geqq f_N{}^k(\boldsymbol{x}^1|\bar{Y}^*)$ for all $\boldsymbol{x}^1\in X$ and every N where X is the domain of $\boldsymbol{x}^1$.

This is easily proved by comparing $G^0(\boldsymbol{y})$ with $G^k(\boldsymbol{y})$, so we omit the proof. Now, we show some sufficient conditions for which $G^k(\boldsymbol{y})$ be a convex function of $\boldsymbol{y}$.

Theorem 1. If any one of the following assumption holds, then $G^k(\boldsymbol{y})$ becomes a convex function of two dimensional variable $\boldsymbol{y}$.

(i) in the case of $c_1>c_2$,

$$h_1+p_1q_1+\alpha c_2-\alpha c_1>k_2(h_2+p_2q_2+\alpha c_1-\alpha c_2)$$
$$h_2+p_2q_2+\alpha c_1-\alpha c_2>k_1(h_1+p_1q_1+\alpha c_2-\alpha c_1)>0$$
$$p_2(1-q_2)+\alpha(c_2-c_1)>0$$

(ii) in the case of $c_1<c_2$,

$$h_2+p_2q_2+\alpha c_1-\alpha c_2>k_1(h_1+p_1q_1+\alpha c_2-\alpha c_1)$$
$$h_1+p_1q_1+\alpha c_2-\alpha c_1>k_2(h_2+p_2q_2+\alpha c_1-\alpha c_2)>0$$
$$p_1(1-q_1)+\alpha(c_1-c_2)>0$$

(iii) $c_1=c_2$, $h_1=h_2$, $p_1=p_2$, $q_1=q_2$

(iv) $k_1=k_2=1$, $h_1+p_1q_1+\alpha c_2-\alpha c_1=h_2+p_2q_2+\alpha c_1-\alpha c_2$,

and $p_1(1-q_1)>\alpha(c_2-c_1)$ in the case of $c_2>c_1$ and $P_2(1-q_2)>\alpha(c_1-c_2)$ in the case of $c_1>c_2$.

Proof. As $G^k(\boldsymbol{y})$ is a continuously twice differentiable function, we take the first and second order partial derivatives.

Then, using the relation (1) and (2),

$$(5)\qquad \frac{\partial G^k(\boldsymbol{y})}{\partial y_1}=c_1(1-\alpha)-p_1+(h_1+p_1q_1+\alpha c_2-\alpha c_1)\iint_{R1+R2}\varphi(\boldsymbol{\xi})\,d\boldsymbol{\xi}$$
$$+k_2(h_2+p_2q_2+\alpha c_1-\alpha c_2)\iint_{R4}\varphi(\boldsymbol{\xi})\,d\boldsymbol{\xi}+(p_1(1-q_1)+\alpha c_1$$
$$-\alpha c_2)\int_0^{y_1}\varphi_1(\xi_1)\,d\xi_1\ .$$

$$(6)\qquad \frac{\partial G^k(\boldsymbol{y})}{\partial y_2}=c_2(1-\alpha)-p_2+(h_2+p_2q_2+\alpha c_1-\alpha c_2)\iint_{R1+R4}\varphi(\boldsymbol{\xi})\,d\boldsymbol{\xi}$$
$$+k_1(h_1+p_1q_1+\alpha c_2-\alpha c_1)\iint_{R2}\varphi(\boldsymbol{\xi})\,d\boldsymbol{\xi}+(p_2(1-q_2)+\alpha c_2$$
$$-\alpha c_1)\int_0^{y_2}\varphi_2(\xi_2)\,d\xi_2\ .$$

$$(7)\qquad \frac{\partial^2 G^k(\boldsymbol{y})}{\partial y_1 \partial y_2}=[h_1+p_1q_1+\alpha c_2-\alpha c_1]\int_0^{y_1}\varphi\left(\xi_1,\ y_2+\frac{1}{k_1}(y_1-\xi_1)\right)d\xi_1$$

$$+[h_2+p_2q_2+\alpha c_1-\alpha c_2]\int_0^{y_2}\varphi\left(y_1+\frac{1}{k_2}(y_2-\xi_2),\ \xi_2\right)d\xi_2\ .$$

Putting

$$(8)\qquad A_1=[h_1+p_1q_1+\alpha c_2-\alpha c_1]\int_0^{y_1}\varphi\left(\xi_1,\ y_2+\frac{1}{k_1}(y_1-\xi_1)\right)d\xi_1\ ,$$

$$A_2=[h_2+p_2q_2+\alpha c_1-\alpha c_2]\int_0^{y_2}\varphi\left(y_1+\frac{1}{k_2}(y_2-\xi_2),\ \xi_2\right)d\xi_2\ .$$

We have

$$(9)\qquad \frac{\partial^2 G^k(\boldsymbol{y})}{\partial y_1^2}=\frac{A_1}{k_1}+k_2A_2+[h_1+p_1q_1+\alpha c_2-\alpha c_1-k_2(h_2+p_2q_2+\alpha c_1$$

$$-\alpha c_2)]\int_0^{y_2}\varphi(y_1,\ \xi_2)\,d\xi_2+[p_1(1-q_1)+\alpha c_1+\alpha c_2]\varphi_1(y_1)\ .$$

$$(10)\qquad \frac{\partial^2 G^k(\boldsymbol{y})}{\partial y_2^2}=\frac{A_2}{k_2}+k_1A_1+[h_2+p_2q_2+\alpha c_1-\alpha c_2-k_1(h_1+p_1q_1+\alpha c_2$$

$$-\alpha c_1)]\int_0^{y_1}\varphi(\xi_1,\ y_2)\,d\xi_1+[p_2(1-q_2)+\alpha c_2-\alpha c_1]\varphi_2(y_2)\ .$$

Using Eqs. (7), (8), (9), and (10) we have

$$(11)\qquad D\equiv t_1^2\frac{\partial^2 G^k(\boldsymbol{y})}{\partial y_1^2}+2t_1t_2\frac{\partial^2 G^k(\boldsymbol{y})}{\partial y_1\partial y_2}+t_2^2\frac{\partial^2 G^k(\boldsymbol{y})}{\partial y_2^2}$$

$$=\frac{A_1}{k_1}(t_1+k_1t_2)^2+\frac{A_2}{k_2}(t_2+k_2t_1)^2+t_1^2\{[h_1+p_1q_1+\alpha c_2-\alpha c_1$$

$$-k_2(h_2+p_2q_2+\alpha c_1-\alpha c_2)]\int_0^{y_2}\varphi(y_1,\ \xi_2)\,d\xi_2+(p_1(1-q_1)+\alpha c_1$$

$$-\alpha c_2)\,\varphi_1(y_1)\}+t_2^2\{[h_2+p_2q_2+\alpha c_1-\alpha c_2-k_1(h_1+p_1q_1+\alpha c_2$$

$$-\alpha c_1)]\int_0^{y_1}\varphi(\xi_1,\ y_2)\,d\xi_1+(p_2(1-q_2)+\alpha c_2-\alpha c_1)\,\varphi_2(y_2)\}\ .$$

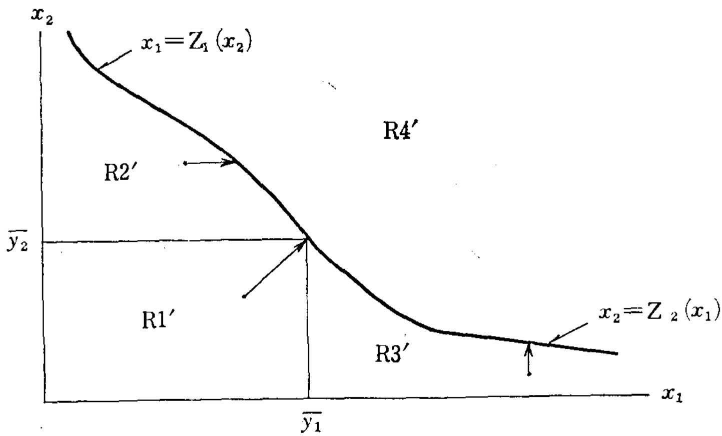
Using any one of the assumptions, we get $D \geqq 0$ for any real number t_1, t_2. Then $G^k(\boldsymbol{y})$ becomes a convex function of the two dimensional variable $\boldsymbol{y}$.

Next, we consider the characterization of the optimal policy of $N=1$ whose proof is obtained similarly as in [2], so omitted.

Theorem 2. If the assumption of Theorem 1 is satisfied and

$$0<\frac{p_1-c_1(1-\alpha)}{h_2+p_2q_2+\alpha c_1-\alpha c_2}<k_2 \quad \text{and} \quad 0<\frac{p_2-c_2(1-\alpha)}{h_1+p_1q_1+\alpha c_2-\alpha c_1}<k_1 ,$$

then an optimal policy for $N=1$ is characterized as follows.



Optimal ordering policy for one period

Region	Product 1	Order quantity	Product 2	Order quantity
R1′	Raise inventory to $\bar{y}_1$	$\bar{y}_1-x_1$	Raise inventory to $\bar{y}_2$	$\bar{y}_2-x_2$
R2′	Raise inventory to $Z_1(x_2)$	$Z_1(x_2)-x_1$	Do not order	0
R3′	Do not order	0	Raise inventory to $Z_2(x_1)$	$Z_2(x_1)-x_2$
R4′	Do not order	0	Do not order	0

$(\bar{y}_1, \bar{y}_2)$ minimizes $G^k(y_1, y_2)$. $Z_i(y_j)$ is the unique root of the equation $\frac{\partial G^k(y_1, y_2)}{\partial y_i}=0$, $i\neq j$, $i, j=1, 2$. $(\bar{y}_1, \bar{y}_2)$ is also the unique root of the simultaneous equations of $\frac{\partial G^k(y_1, y_2)}{\partial y_1}=0$, $\frac{\partial G^k(y_1, y_2)}{\partial y_2}=0$ and is the point of intersection of the two functions $Z_1(x_2)$ and $Z_2(x_1)$ each of which is nonincreasing function of x_2 and x_1, respectively.

Now, we mention the relation between optimal policies for one period ($N=1$) and those for $N\geqq 2$. Let $\bar{\boldsymbol{y}}(\boldsymbol{x})$ be the unique minimum of $G(\boldsymbol{w})$ over $\boldsymbol{w}\geqq\boldsymbol{x}$,[5] then $\bar{\boldsymbol{y}}(\boldsymbol{x})$ is the optimal one period ordering rule.

Proposition 2. Under the assumption of Theorem 1, the following assertions hold.

(i) If $\boldsymbol{x}^1\in R1'$ (that is, $\boldsymbol{x}^1=(x_1^1, x_2^1)\leqq(\bar{y}_1, \bar{y}_2)$), then the optimal policy is given by $\bar{Y}_i(H_i)=(\bar{y}_1, \bar{y}_2)$ $(i=1, \cdots, N)$.

(ii) If $c_1=c_2$, $h_1=h_2$, $p_1=p_2$, $q_1=q_2$, $k_1=k_2=1$, then the optimal policy is given by $\bar{Y}_i(H_i)=\bar{\boldsymbol{y}}(\boldsymbol{x}^i)$ $(i=1, \cdots, N)$, the policy in which $\bar{\boldsymbol{y}}(\boldsymbol{x}^i)$ is used in each period.

(iii) $c_1=c_2$ and any stocks of one product which are used to satisfy demands of another product in a period must be replaced from exogeneous sources at the beginning of the following period,[6] then the same policy as (ii) is optimal.

Proof. The assertion in the case of (i), (ii) and (iii) are justified by Theorem 1, 2, and 4 in Veinott [3], respectively.

Then, in the sequel, we treat only the base stock level $(\bar{y}_1, \bar{y}_2)$ as an optimal inventory level and properties of $(\bar{y}_1, \bar{y}_2)$ are studied.

4. Symmetric (cost functions) case

We consider the case in which all cost functions of both products are same and only the demand densities are different. In this case, the assumption (iii) of Theorem 1 is satisfied.

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⑤; We write $\boldsymbol{w}\geqq\boldsymbol{x}$ when $w_i\geqq x_i$ for $i=1, 2$ and $\boldsymbol{w}=\boldsymbol{x}$ when $w_i=x_i$ for $i=1, 2$.
⑥; this implies $\boldsymbol{T}(\boldsymbol{y}^n, \boldsymbol{\xi}^n)=\boldsymbol{y}^n-\boldsymbol{\xi}^n$ for all n.

We assume, in the sequel, that $0<\frac{p-c(1-\alpha)}{h+pq}<k$ and the two demand variables of product 1 and product 2 are stochastically independent.

4.1. The case of $k_1=k_2=1$

First, we consider the case of $k_1=k_2=1$, and let $\bar{y}_1(q)$ and $\bar{y}_2(q)$ be the optimal inventory levels in this substitute case for discount rate $(1-q)$. Then, putting Eqs. (5) and (6) be zero, $\bar{y}_1(q)$ and $\bar{y}_2(q)$ are the unique solution of the following simultaneous equations.

(12) $$c'-p+(h+pq)\Phi_1*\Phi_2(\bar{y}_1(q)+\bar{y}_2(q))+p(1-q)\Phi_1(\bar{y}_1(q))=0$$

(13) $$c'-p+(h-pq)\Phi_1*\Phi_2(\bar{y}_1(q)+\bar{y}_2(q))+p(1-q)\Phi_2(\bar{y}_2(q))=0\ ,$$

where $\Phi_1*\Phi_2$ is the convolution of Φ_1 and Φ_2, and $c'\equiv c(1-\alpha)$. From Eqs. (12), (13), we get

(14) $$\Phi_1(\bar{y}_1(q))=\Phi_2(\bar{y}_2(q)) \qquad \text{for } q\neq 1\ .$$

Even though when $q=1$, we assume (14), in order to preserve continuity of $\bar{y}_i(q)$ at $q=1$.

Note. From Eq. (14), we obtain that if the demand distribution of product i is stochastically smaller than that of product j, then the optimal stock level of product i is less than or equal to that of product j for any fixed q. $(i,j=1,2)$

Let $\hat{y}_1$, $\hat{y}_2$ be the optimal inventory levels for product 1 and product 2, in no substitute case. Define $\Delta_1(>0)$, $\Delta_2(>0)$ as follows.

(15) $$\Phi_1*\Phi_2(\hat{y}_1+\Delta_2)=\frac{p-c'}{h+p}=\Phi_1*\Phi_2(\Delta_1+\hat{y}_2)\ .$$

Put,

$$\boldsymbol{\Delta}\equiv(\Delta_1,\ \Delta_2),\quad \hat{\boldsymbol{y}}\equiv(\hat{y}_1,\ \hat{y}_2),\quad \bar{\boldsymbol{y}}(q)\equiv(\bar{y}_1(q),\bar{y}_2(q))$$

In the sequel, we assume, for simplicity, the strict monotonicity of $\Phi_i(\xi)$, but similar results with a slight modification are obtained without this assumption.

Now, we define Property A which we call sometimes linearly ordered.

Let Y be a set of two dimensional vectors of real numbers.

Definition 1. For some $\boldsymbol{y}=(y_1, y_2)$, $\boldsymbol{y}'=(y'_1, y'_2)\in Y$, a pair $[\boldsymbol{y}, \boldsymbol{y}']$ has Property A if $y'_i \leqq y_i$ whenever $y'_j \leqq y_j$, for $i \neq j$, $i, j=1, 2$.

Definition 2. Y has Property A if any pair $[\boldsymbol{y}, \boldsymbol{y}']$, $\boldsymbol{y}$ and $\boldsymbol{y}' \in Y$ has Property A.

Lemma 1. Following pairs and the set have Property A.

1. $[\boldsymbol{\Delta}, \hat{\boldsymbol{y}}]$, $[\bar{\boldsymbol{y}}(q), \hat{\boldsymbol{y}}]$ aud $[\boldsymbol{\Delta}, \bar{\boldsymbol{y}}(q)]$, for any fixed q.
2. $Y=\{\bar{\boldsymbol{y}}(q)\,;\ 0\leqq q\leqq 1\}$

Proof. It is well known that $\hat{y}_1(\hat{y}_2)$ is the unique solution of the following equation.

$$\Phi_1(\hat{y}_1)=(\Phi_2(\hat{y}_2))=\frac{p-c'}{h+p} \tag{16}$$

In Eq. (14), we put $\Phi_1(\bar{y}_1(q))=\Phi_2(\bar{y}_2(q))=a$, then we get the following relations.

$$\text{(17)}\quad \begin{cases} \text{If } \dfrac{p-c'}{h+p}>a, \text{ then } \hat{y}_1>\bar{y}_1(q) \text{ and } \hat{y}_2>\bar{y}_2(q), \\[2ex] \text{if } \dfrac{p-c'}{h+p}<a, \text{ then } \hat{y}_1<\bar{y}_1(q) \text{ and } \hat{y}_2<\bar{y}_2(q) \\[2ex] \text{and} \\[2ex] \text{if } \dfrac{p-c'}{h+p}=a, \text{ then } \hat{y}_1=\bar{y}_1(q) \text{ and } \hat{y}_2=\bar{y}_2(q), \end{cases}$$

because $\Phi_i(\xi)$ is strictly monotone increasing and continuous for $i=1, 2$.

So, $[\hat{\boldsymbol{y}}, \hat{\boldsymbol{y}}(q)]$ has Property A.

Now, the relation $\Delta_1 \gtrless \hat{y}_1$⑦ implies that $\Delta_2 \gtrless \hat{y}_2$, since

$$\Phi_1 * \Phi_2(\hat{y}_1+\Delta_2)=\Phi_1 * \Phi_2(\Delta_1+\hat{y}_2) \gtrless \Phi_1 * \Phi_2(\hat{y}_1+\hat{y}_2)$$

Thus, $[\boldsymbol{\Delta}, \hat{\boldsymbol{y}}]$ has also Property A.

---

⑦; in the sequel, double signs are to be read in the same order.

Next, we will verify that $[\boldsymbol{\Delta}, \bar{\boldsymbol{y}}(q)]$ has Property A.

When $\Delta_1 \leqq \bar{y}_1(q)$, we show $\bar{y}_1(q) \leqq \hat{y}_1$ at first.

On the contrary, if $\bar{y}_1(q) \gneqq \hat{y}_1$ holds, then using Eq. (12)

$$\Phi_1 * \Phi_2(\bar{y}_1(q)+\bar{y}_2(q)) = \frac{1}{h+pq}\{p-c'-p(1-q)\Phi_1(\bar{y}_1(q))\}$$

$$\gneqq \frac{p-c'}{h+p} = \Phi_1 * \Phi_2(\Delta_1+\hat{y}_2) \leqq \Phi_1 * \Phi_2(\bar{y}_1(q)+\hat{y}_2)$$

so, we get $\bar{y}_2(q) \leqq \hat{y}_2$, this contradicts that $[\bar{\boldsymbol{y}}(q), \hat{\boldsymbol{y}}]$ has Property A. Therefore, $\bar{y}_1(q) \leqq \hat{y}_1$ holds.

Using this relation and Eq. (12),

$$\Phi_1 * \Phi_2(\bar{y}_1(q)+\bar{y}_2(q)) \gtreqless \frac{p-c'}{h+p} = \Phi_1 * \Phi_2(\hat{y}_1+\Delta_2)$$

Then, $\bar{y}_1(q)+\bar{y}_2(q) \gtreqless \hat{y}_1+\Delta_2$

Together with $\bar{y}_1(q) \leqq \hat{y}_1$, we obtain $\bar{y}_2(q) \gtreqless \Delta_2$ which is to be required.

Finally, from Eq. (14) and the assumption of the strict monotonicity of $\Phi_i(\xi)$, we immediately obtain that Y has Property A.

Theorem 3. For all q; $(0 \leqq q \leqq 1)$

if $\boldsymbol{\Delta} < \hat{\boldsymbol{y}}$, then $\boldsymbol{\Delta} < \bar{\boldsymbol{y}}(q) < \hat{\boldsymbol{y}}$

if $\boldsymbol{\Delta} = \hat{\boldsymbol{y}}$, then $\boldsymbol{\Delta} = \bar{\boldsymbol{y}}(q) = \hat{\boldsymbol{y}}$

if $\boldsymbol{\Delta} > \hat{\boldsymbol{y}}$, then $\boldsymbol{\Delta} > \bar{\boldsymbol{y}}(q) > \hat{\boldsymbol{y}}$

Proof. From Eq. (5), we get

$$\left[\frac{\partial G^1(y)}{\partial y_1}\right]_{y_1=\hat{y}_1} = c'-p+(h+pq)\Phi_1 * \Phi_2(\hat{y}_1+y_2)+p(1-q)\frac{p-c'}{h+p}$$

$$=(h+pq)\left\{\Phi_1 * \Phi_2(\hat{y}_1+y_2)-\frac{p-c'}{h+p}\right\}$$

Hence, $\left[\frac{\partial G^1(y)}{\partial y_1}\right]_{y_1=\hat{y}_1} \gtreqless 0$ according with $y_2 \gtreqless \Delta_2$.

And also, from the convexity of $G^1(\boldsymbol{y})$, $\dfrac{\partial G^1(\boldsymbol{y})}{\partial y_1}$ is a monotone increasing function of y_1 for any fixed y_2.

Thus, when $y_1>\hat{y}_1$ and $y_2>\Delta_2$ or when $y_1<\hat{y}_1$ and $y_2<\Delta_2$, we have

$$\frac{\partial G^1(\boldsymbol{y})}{\partial y_1}\geqq\left[\frac{\partial G^1(\boldsymbol{y})}{\partial y_1}\right]_{y_1=\hat{y}_1}>0 \text{ or } \frac{\partial G^1(\boldsymbol{y})}{\partial y_1}\leqq\left[\frac{\partial G^1(\boldsymbol{y})}{\partial y_1}\right]_{y_1=\hat{y}_1}<0$$

and so,

$$G^1(y_1,\ y_2)\geqq G^1(\hat{y}_1,\ y_2) \tag{18}$$

Similarly, when $y_2>\hat{y}_2$ and $y_1>\Delta_1$, or when $y_2<\hat{y}_2$ and $y_1<\Delta_1$, we have

$$G^1(y_1,\ y_2)\geqq G^1(y_1,\ \hat{y}_2) \tag{19}$$

Thus, when $\Delta<\hat{\boldsymbol{y}}$, using (17) and Property A, we get

$$\boldsymbol{\Delta}\leqq\bar{\boldsymbol{y}}(q)\leqq\hat{\boldsymbol{y}}\ .$$

Equalities of this relation are excluded. If $\bar{\boldsymbol{y}}(q)=\hat{\boldsymbol{y}}$, then from (12),

$$\Phi_1*\Phi_2(\bar{y}_1(q)+\bar{y}_2(q))=\frac{p-c'}{h+p}=\Phi_1*\Phi_2(\hat{y}_i+\Delta_j)$$

for $i\neq j$, $i,j=1,2$ and we have $\hat{\boldsymbol{y}}(=\bar{\boldsymbol{y}}(q))=\boldsymbol{\Delta}$ which contradicts the assumption. Equality of left side is excluded in the same manner.

Similarly, when $\boldsymbol{\Delta}>\hat{\boldsymbol{y}}$, we have $\boldsymbol{\Delta}>\bar{\boldsymbol{y}}(q)>\hat{\boldsymbol{y}}$.

In the sequel, we put $\gamma=\dfrac{p-c'}{h+p}$

Note that a rough range of $\boldsymbol{\Delta}$ and $\hat{\boldsymbol{y}}$ are obtained as follows. Using Chebychev inequality, we have

$$E(\xi_i)-\sqrt{\frac{V(\xi_i)}{1-\gamma}}<\hat{y}_i<E(\xi_i)+\sqrt{\frac{V(\xi_i)}{1-\gamma}} \qquad \text{for } i=1,2$$

and

$$E(\xi_1+\xi_2)-\sqrt{\frac{V(\xi_1+\xi_2)}{1-\gamma}}<\Delta_i+\hat{y}_j<E(\xi_1+\xi_2)+\sqrt{\frac{V(\xi_1+\xi_2)}{1-\gamma}}$$

$$\text{for } i\neq j,\ i,j=1,2.$$

Example 4.1.1. The demand distribution is $N(m_i, \sigma^2)$ for $i=1,2$. From Eq. (16), we get

$$\hat{y}_1=m_1+Z_\gamma\sigma \ , \qquad \hat{y}_2=m_2+Z_\gamma\sigma \ ,$$

where Z_γ is the γ-th fractile of the standard Normal distribution.

Since the convolution of Φ_1 and Φ_2 is $N(m_1+m_2, 2\sigma^2)$, we get

$$\Delta_1+\hat{y}_2=\hat{y}_1+\Delta_2=m_1+m_2+Z_\gamma\sqrt{2}\cdot\sigma \ ,$$

then, $\Delta_1=m_1+(\sqrt{2}-1)\sigma Z_\gamma$, $\Delta_2=m_2+(\sqrt{2}-1)\sigma Z_\gamma$

Thus, we have $\boldsymbol{\Delta}<\hat{\boldsymbol{y}}$. Then, from the Theorem 3,

$$\boldsymbol{\Delta}<\bar{\boldsymbol{y}}(q)<\hat{\boldsymbol{y}} \quad \text{for all} \quad q\,(0\leqq q\leqq 1) \ .$$

Also, directly from Eqs. (12) and (14), we have

$$\Delta_1<\bar{y}(1)=m_1+Z_\gamma\frac{\sigma}{\sqrt{2}}<\hat{y}_1 \ , \qquad \Delta_2<\bar{y}_2(1)=m_2+Z_\gamma\frac{\sigma}{\sqrt{2}}<\hat{y}_2 \ ,$$

in the case of $q=1$.

Example 4.1.2. The demand distributions of both products belong to the same type distribution whose convolution belongs to the same distribution type.

Let mean and variance of product i be m_i and σ_i^2

Then,

$$\begin{aligned} &\hat{y}_i=m_i+Z_\gamma\sigma_i \ , \\ &\Delta_i=m_i+(\sqrt{\sigma_1^2+\sigma_2^2}-\sigma_j)Z_\gamma \ , \end{aligned} \qquad \text{for } i\neq j, \ i,j=1,2$$

where Z_γ is the γ-th fractile of the Normalized distribution of Φ whose mean is 0 and variance is 1.

Then, always $\boldsymbol{\Delta}<\hat{\boldsymbol{y}}$ holds, and from Theorem 3,

$$\boldsymbol{\Delta}<\hat{\boldsymbol{y}}(q)<\hat{\boldsymbol{y}} \quad \text{for all} \quad q\,(0\leqq q\leqq 1) \ .$$

Example 4.1.3. The demand distribution $\Phi_i(\xi)$ is negative exponential with mean μ_i for $i=1,2$. From Eq. (16), we get

$$\hat{y}_i=-\mu_i\ln(1-\gamma) \ , \qquad \text{for } i=1,2.$$

In order to calculate Δ_1 and Δ_2, we take $\mu_1=\frac{1}{2}\mu_2$ specially.

Then, $\Delta_1=2\mu_1\ln(1+\sqrt{\gamma})$, $\quad \Delta_2=\mu_2\ln\frac{1+\sqrt{\gamma}}{\sqrt{1-\gamma}}$

Hence, some $\gamma'(0<\gamma'<1)$ exists, and

$$\text{if } 0<\gamma<\gamma', \quad \text{then } \hat{\boldsymbol{y}}<\boldsymbol{\Delta}$$

$$\text{if } 1>\gamma>\gamma', \quad \text{then } \hat{\boldsymbol{y}}>\boldsymbol{\Delta}$$

Directly from Eqs. (12) and (14), we also have

$$\Delta_i<\bar{y}_i(1)=\frac{2}{3}\mu_i\ln\frac{1+\sqrt{\gamma}}{1-\gamma}<\hat{y}_i \quad \text{if } 1>\gamma>\gamma'$$

$$\Delta_i>\bar{y}_i(1)= \quad \text{〃} \quad >\hat{y}_i \quad \text{if } 0<\gamma<\gamma'$$

for $i=1, 2$ in the case of $q=1$ and $\mu_1=\frac{1}{2}\mu_2$.

Theorem 4. $\bar{y}_i(q)$ is a continuous function of q for $0\leqq q\leqq 1$, $i=1, 2$. We prove it in the Appendix.

Theorem 5. $\bar{\boldsymbol{y}}(q)$ is a monotone function of $q(0\leqq q\leqq 1)$. More precisely,

(22) when $\boldsymbol{\Delta}>\hat{\boldsymbol{y}}$, $\boldsymbol{\Delta}>\bar{\boldsymbol{y}}(q)>\bar{\boldsymbol{y}}(q')>\hat{\boldsymbol{y}}$ if $q>q'$

(23) when $\boldsymbol{\Delta}<\hat{\boldsymbol{y}}$, $\boldsymbol{\Delta}<\bar{\boldsymbol{y}}(q)<\bar{\boldsymbol{y}}(q')<\hat{\boldsymbol{y}}$ if $q>q'$

Proof. We will show (22) only. (23) may be obtained quite analogously. From Eq. (5), we get

(24) $$\left[\frac{\partial G^1{}_{q'}}{\partial y_1}\right]^{⑧}_{y_1=\bar{y}_1(q)}=c'-p+(h+pq')\Phi_1*\Phi_2(\bar{y}_1(q)+y_2)+p(1-q')\Phi_1(\bar{y}_1(q))$$

On the other hand, since $\bar{y}_1(q)$ and $\bar{y}_2(q)$ satisfy Eq. (12),

⑧; we write $\frac{\partial G^k(\boldsymbol{y})}{\partial y_1}$ in Eq. (5) by $\frac{\partial G^k q'}{\partial y_1}$ when $q=q'$ in order to make a parameter q clear.

$$c'-p=-(h+pq)\Phi_{1}*\Phi_{2}(\bar{y}_1(q)+\bar{y}_2(q))-p(1-q)\Phi_1(\bar{y}_1(q)) \tag{25}$$

Substituting Eq. (25) into (24), we get

$$\left[\frac{\partial G^1{}_{q'}}{\partial y_1}\right]_{y_1=\bar{y}_1(q)}=h\{\Phi_{1}*\Phi_{2}(\bar{y}_1(q)+y_2)-\Phi_{1}*\Phi_{2}(\bar{y}_1(q)+\bar{y}_2(q))\} \tag{26}$$

$$+p(q-q')\Phi_1(\bar{y}_1(q))-pq\Phi_{1}*\Phi_{2}(\bar{y}_1(q)+\bar{y}_2(q))$$

$$+pq'\Phi_{1}*\Phi_{2}(\bar{y}_1(q)+y_2)$$

And, Theorem 3, for $\boldsymbol{\Delta}>\hat{\boldsymbol{y}}$, implies that,

$$\Phi_1(\bar{y}_1(q))>\Phi_1(\hat{y}_1)=\frac{p-c'}{h+p}$$

and

$$\Phi_{1}*\Phi_{2}(\bar{y}_1(q)+\bar{y}_2(q))=\frac{1}{h+pq}\{p-c'-p(1-q)\Phi_1(\bar{y}_1(q))\}$$

$$<\frac{1}{h+pq}\{(h+p)\Phi_1(\bar{y}_1(q))-p(1-q)\Phi_1(\bar{y}_1(q))\}=\Phi_1(\bar{y}_1(q)) \tag{27}$$

Thus, if $y_2>\bar{y}_2(q)$, by Eqs. (26), and (27), we can see that

$$\left[\frac{\partial G^1{}_{q'}}{\partial y_1}\right]_{y_1=\bar{y}_1(q)}>p(q-q')\{\Phi_1(\bar{y}_1(q))-\Phi_{1}*\Phi_{2}(\bar{y}_1(q)+\bar{y}_2(q))\}>0 \tag{28}$$

From (28) and using the convexity of $G^1(\boldsymbol{y})$, we get

$$G^1{}_{q'}(\bar{y}_1(q), y_2)<G^1{}_{q'}(y_1, y_2)\ , \quad \text{when} \quad y_2>\bar{y}_2(q) \quad \text{and} \quad y_1>\bar{y}_1(q)\ .$$

Similarly,

$$G^1{}_{q'}(y_1, \bar{y}_2(q))<G^1{}_{q'}(y_1, y_2)\ , \quad \text{when} \quad y_2>\bar{y}_2(q) \quad \text{and} \quad y_1>\bar{y}_1(q)\ .$$

Then, $\bar{y}_1(q')>\bar{y}_1(q)$ and $\bar{y}_2(q')>\bar{y}_2(q)$ does not occur. So we get $\bar{\boldsymbol{y}}(q)>\bar{\boldsymbol{y}}(q')$, because $Y=\{\bar{\boldsymbol{y}}(q)\,;\ 0\leqq q\leqq 1\}$ has Property A.

Corollary

Let $\hat{y}_i{}^\beta$ be y such that $\Phi_i(y)=\beta$ for $0<\beta<1$, $\hat{y}_i{}^\beta=0$ for $\beta\leqq 0$, and $\hat{y}_i{}^\beta=\infty$ for $\beta\geqq 1$. Then, for all $0<q<1$,

(29) $$\max(\hat{y}^{[p-c'-(h+pq)]/p(1-q)},\ \bar{y}(1)) < \bar{y}(q) < \hat{y}$$

or

(30) $$\hat{y} < \bar{y}(q) < \min(\hat{y}^{[p-c']/p(1-q)},\ \bar{y}(1))$$

Proof. From Eq. (5)

$$\Phi_1(\bar{y}_1(q)) = \frac{p-c'}{p(1-q)} - \frac{(h+pq)}{p(1-q)}\ \Phi_1 * \Phi_2(\bar{y}_1(q)+\bar{y}_2(q))$$

so

$$\frac{p-c'-(h+pq)}{p(1-q)} < \Phi_1(\bar{y}_1(q)) < \frac{p-c'}{p(1-q)}$$

And trivially,

$$\frac{p-c'}{p(1-q)} > \frac{p-c'}{h+p} > \frac{p-c'-(h+pq)}{p(1-q)}$$

Then, these relations and Theorem 5 imply the Corollary.

Example 4.1.4. Let $\varphi_i(\xi)$ be Normal density with mean m_i and common variance σ^2, for $i=1,2$. Then, from Example 4.1.1 and Corollary, we get for all $q\,(0<q<1)$,

$$m_i+\sigma\max(Z_{\gamma'},\ \frac{1}{\sqrt{2}}Z_\gamma) < \bar{y}_i(q) < m_i+\sigma Z_\gamma \qquad \text{for } i=1,2\ ,$$

where $\gamma' = \dfrac{p-c'-h-pq}{p(1-q)}$ and $\gamma = \dfrac{p-c'}{h+p}$

We put $p=4h$ and $c'=0$, then $Z_\gamma \doteqdot 0.84$
And moreover put $q=0$, then $Z_{\gamma'} \doteqdot 0.67$, so

$$m_i+0.67\sigma < \bar{y}_i(o) < m_i+0.84\sigma$$

In the same case, when $q>0.1$, we have

$$m_i+0.59\sigma < \bar{y}_i(q) < m_i+0.84\sigma \qquad \text{for } i=1,2\ .$$

4.2. The case of $k_1=k_2\neq 1$

Next, we consider the case of $k_1=k_2=k\neq 1$. In order to stress the parameter k, we denote $\bar{y}(q)$ by $\bar{y}^k(q)$ when the substitute rate is k.

Theorem 6. $\bar{y}_i^k(q)$ is a continuous function of k, $(0<k\leqq 1)$ for any fixed q, and $i=1, 2$.

We can prove it in the similar argument as Theorem 4, so omitted. Next, we consider bounds of $\bar{y}^k(q)$.

Proposition 3. For any $0<k\leqq 1$ and $0\leqq q<1$,

$$\hat{y}^{1/p(1-q)\cdot\{p-c'-(h+pq)(1+k)\}}\leqq\bar{y}^k(q)\leqq\hat{y}^{[p-c']/p(1-q)} \tag{31}$$

Proof. Putting Eqs. (5) and (6) be zero, and using $0\leqq\Psi_i$,[9] $\Psi_i'\leqq 1$, for $i=1, 2$, we obtain (31).

Proposition 4. Assume that $\Phi_1(\xi)=\Phi_2(\xi)\equiv\Phi(\xi)$, then $\bar{y}^k{}_1(q)=\bar{y}^k{}_2(q)\equiv\bar{y}^k(q)$ and $\bar{y}^k(q)$ is in the following region.

$$\max(\bar{y}'', \hat{y}^{\gamma''})\leqq\bar{y}^k(q)\leqq\min(\bar{y}', \hat{y}''') , \tag{32}$$

where

$$\gamma''=\frac{-p(1-q)+\sqrt{p^2(1-q)^2+4(p-c')(h+pq)-4(1+k)(h+pq)^2}}{2(h+pq)}$$

$$\gamma'''=\frac{-p(1-q)+\sqrt{p^2(1-q)^2+4(p-c')(h+pq)}}{2(h+pq)}$$ (but it has meaning only when γ'' is a real number.), and $\bar{y}', \bar{y}''$ are the solution of the following equations

$$c'-p+(h+pq)k\Phi*\Phi^{[1/k]}\left(\left(1+\frac{1}{k}\right)\bar{y}'\right)+p(1-q)\Phi(\bar{y}')=0 \tag{33}$$

$$c'-p+(h+pq)\Phi*\Phi\left(\left(1+\frac{1}{k}\right)\bar{y}''\right)+p(1-q)\Phi(\bar{y}'')=0 \tag{34}$$

---

⑨; $\Psi_1=\iint_{R1+R2}\varphi(\xi)d\xi$, $\Psi_1'=\iint_{R4}\varphi(\xi)d\xi$,

$\Psi_2=\iint_{R1+R4}\varphi(\xi)d\xi$, $\Psi_2'=\iint_{R2}\varphi(\xi)d\xi$.

respectively, where $\Phi^{[1/k]}$ denote the probability distribution of $\frac{\xi}{k}$.
In the special case of $q=1$, the relation (32) becomes as follows

$$\bar{y}'' \leqq \bar{y}^k(q) \leqq \min(\bar{y}', \hat{y}^{\sqrt{p-c'/h+p}}) , \tag{35}$$

where $\bar{y}'$ and $\bar{y}''$ are obtained from the following equations

$$\Phi * \Phi^{[1/k]}\left(\left(1+\frac{1}{k}\right)\bar{y}'\right)=\frac{p-c'}{k(h+p)} \tag{36}$$

$$\Phi * \Phi \left(\left(1+\frac{1}{k}\right)\bar{y}''\right)=\frac{p-c'}{h+p} \tag{37}$$

Proof. Putting Eq. (5) be zero, and from $0 \leqq \Psi$ ⑩ $\leqq 1$, we obtain the right hand side of the both inequalities (32). Next, from the inequality of

$$\Phi * \Phi \left(\left(1+\frac{1}{k}\right)y\right)=\iint_{\xi_1+\xi_2 \leqq (1+1/k)y} \varphi(\boldsymbol{\xi})\, d\boldsymbol{\xi} \geqq (1+k)\Psi+(\Phi(\bar{y}^k(q)))^2$$

$$\geqq k\iint_{\xi_1+\xi_2/k \leqq (1+1/k)y} \varphi(\boldsymbol{\xi})\, d\boldsymbol{\xi} \equiv k\Phi * \Phi^{[1/k]}\left(\left(1+\frac{1}{k}\right)y\right)$$

and let

$$g_1(y)=c'-p+(h+pq)k\Phi * \Phi^{[1/k]}\left(\left(1+\frac{1}{k}\right)y\right)+p(1-q)\Phi(y)$$

$$g_2(y)=c'-p+(h+pq)\Phi * \Phi \left(\left(1+\frac{1}{k}\right)y\right)+p(1-q)\Phi(y) ,$$

then

$$g_1(y) \leqq \frac{\partial G^k(y)}{\partial y} \quad \text{and} \quad g_2(y) \geqq \frac{\partial G^k(y)}{\partial y}$$

for all y. Since $\frac{\partial G^k(y)}{\partial y}$, $g_1(y)$, and $g_2(y)$ are monotone functions of y,

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⑩; $\Psi=\iint_{R2} \varphi(\xi)\, d\xi$.

we have $\bar{y}'' \leqq \bar{y}^k(q) \leqq \bar{y}'$, where $\bar{y}^k(q)$, $\bar{y}'$ and $\bar{y}''$ satisfy $\left[\frac{\partial G^k(y)}{\partial y}\right]_{y=\bar{y}^k(q)}=0$, $g_1(\bar{y}')=0$, and $g_2(\bar{y}'')=0$ respectively.

Example 4.2.1. Let $\varphi(\xi)$ be $N(m, \sigma^2)$.

When $q=1$, from Eqs. (36) and (37),

$$\bar{y}' = m + \frac{\sqrt{1+k^2}}{1+k}\sigma Z_{\gamma/k}\ , \qquad \bar{y}'' = \frac{2k}{1+k}m + \frac{\sqrt{2k}}{1+k}\sigma Z_\gamma\ ,$$

where Z_ν is ν-th fractile of $N(0,1)$.

Then, when $k>\gamma$.

$$\frac{2k}{1+k}m + \frac{\sqrt{2k}}{1+k}\sigma Z_\gamma \leqq \bar{y}^k(1) \leqq m + \sigma \min\left(\frac{\sqrt{1+k^2}}{1+k}Z_{\gamma/k}\ ,\ Z_{\sqrt{\gamma}}\right)$$

Theorem 7. When both products have same probability distribution $\Phi(\xi)$, then $(\bar{y}^k{}_1(q)=\bar{y}^k{}_2(q)\equiv\bar{y}^k(q))$

(38) if $\Delta^k \gtrless \hat{y}$, then $\bar{y}^k(q) \gtrless \hat{y}$

and

(39) if $\Delta^{k'} \gtrless \hat{y}$, then $\Delta^{k'} \gtrless \bar{y}^k(q) \gtrless \hat{y}$

for all $q\,(0\leqq q\leqq 1)$ and any fixed $k\,(0<k\leqq 1)$, where $\Delta^k(>0)$ and $\Delta^{k'}(>0)$ satisfy the following equation

$$(40)\qquad \int_0^{\Delta^k} \Phi\left(\left(1+\frac{1}{k}\right)\Delta^k - \frac{\xi}{k}\right)\varphi(\xi)\,d\xi = \frac{\gamma(1+k\gamma)}{1+k}$$

$$(41)\qquad (1+k)\int_0^{\Delta^{k'}} \Phi\left(\left(1+\frac{1}{k}\right)\Delta^{k'} - \frac{\xi}{k}\right)\varphi(\xi)\,d\xi - k[\Phi(\Delta^{k'})]^2 = \gamma\ ,$$

respectively. (Notice that $\Delta^{k'}=\bar{y}^k(1)$)

Proof. In an analogous method as the proof of Theorem 3, evaluating the sign of $\frac{\partial G^k{}_q}{\partial y}$ at $y=\hat{y}$, we obtain the results.

Theorem 8. When both products have same probability distribution $\Phi(\xi)$, $\bar{y}^k(q)\ (\equiv \bar{y}^k{}_1(q)=\bar{y}^k{}_2(q))$ is a monotone function of $q\,(0\leqq q\leqq 1)$ for any fixed k. More precisely.

$$\text{if } \Delta^{k\prime}>\hat{y}, \quad \text{then } \bar{y}^k(1)>\bar{y}^k(q)>\bar{y}^k(q')>\hat{y}$$
$$\text{if } \Delta^{k\prime}<\hat{y}, \quad \text{then } \bar{y}^k(1)<\bar{y}^k(q)<\bar{y}^k(q')<\hat{y} ,$$

when $q>q'$.

Proof. From Eq. (5),

$$\left[\frac{\partial G^k{}_{q'}}{\partial y}\right]_{y=\bar{y}^k(q)} = c'-p+(1+k)(h+pq')\int_0^{y^k(q)} \Phi\left(\left(1+\frac{1}{k}\right)\bar{y}^k(q) - \frac{\xi}{k}\right)\varphi(\xi)\,d\xi - k(h+pq')[\Phi(\bar{y}^k(q))]^2 + p(1-q')\Phi(\bar{y}^k(q))$$
$$= \frac{(h+p)p(q-q')}{h+pq}\left\{\Phi(\bar{y}^k(q)) - \frac{p-c'}{h+p}\right\}$$

Thus, from Theorem 7, if $\Delta^{k\prime} \gtrless \hat{y}$, then

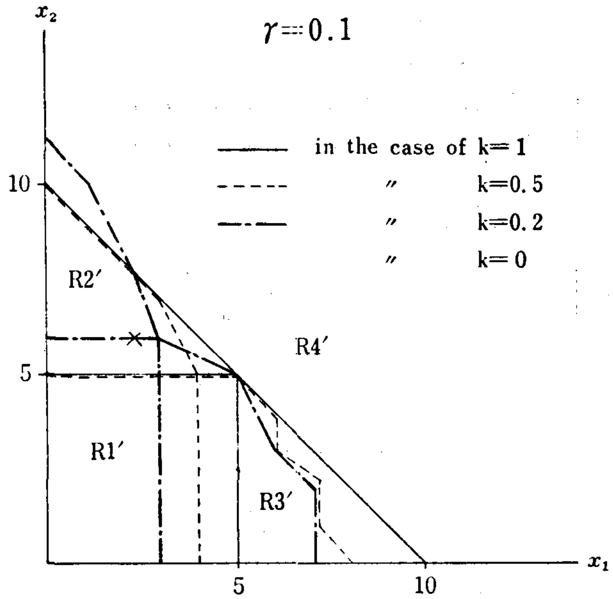
$$\Phi(\bar{y}^k(q)) \gtrless \frac{p-c'}{h+p} \quad \text{and} \quad \left[\frac{\partial G^k{}_{q'}}{\partial y}\right]_{y=\bar{y}^k(q)} \gtrless 0 ,$$

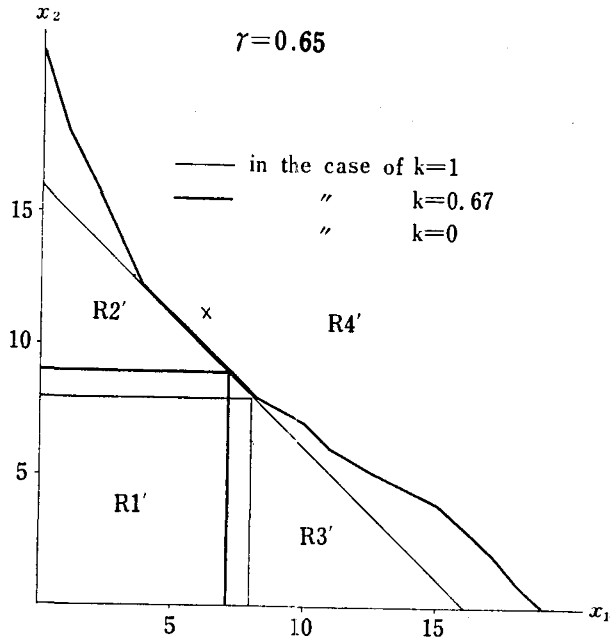
so we obtain $\bar{y}^k(q') \lesseqgtr \bar{y}^k(q)$, respectively. Now, when $\varphi_1(\cdot) \neq \varphi_2(\cdot)$, we only obtain that if $\hat{y}<\bar{y}^k(o)$ then $[\bar{y}^k{}_1(q')>\bar{y}^k{}_1(q)$ and $\bar{y}^k{}_2(q')>\bar{y}^k{}_2(q)]$ does not hold for $q>q'$. But the set $\{\bar{y}^k(q), 0<q<1\}$ has not, in general, Property A, then Theorem 8 does not hold for $\varphi_1 \neq \varphi_2$.

5. Numerical Examples

When the cost functions are symmetric and $q_1=q_2=1$, we calculate the optimal levels varying the values of k and $\gamma=\dfrac{p-c'}{h+p}$.

Example 5.1. When demand distributions are Poisson distribution with mean 5 (product 1) and 10 (product 2), respectively, the respective optimal ranges are as follows.


x_2
$\gamma=0.1$
in the case of k=1
〃 k=0.5
〃 k=0.2
〃 k=0
10
5
R2′
R4′
R1′
R3′
5
10
x_1


x_2
$\gamma=0.65$
in the case of k=1
〃 k=0.67
〃 k=0
15
10
5
R2′
R4′
R1′
R3′
5
10
15
x_1

When both demand distributions are the same Poisson distribution with $\lambda=5$ or $\lambda=10$, optimal levels are as follows.

Optimal levels $(\bar{y}_1{}^k(1)=\bar{y}_2{}^k(1))$

$\lambda=5$ ($\lambda=10$, in the parenthesis)

k \ γ	0.01	0.1	0.3	0.5	0.6	0.7	0.8	0.9	0.95
0	1 (4)	2 (6)	4 (8)	5 (10)	6 (11)	6 (12)	7 (13)	9 (14)	10 (15)
0.1	2 (4)	3 (7)	4 (9)	5 (10)	6 (11)	6 (12)	7 (13)	8 (14)	9 (15)
0.2	2 (5)	3 (7)	4 (9)	5 (10)	6 (11)	6 (11)	7 (12)	8 (14)	9 (15)
0.25	2 (5)	3 (7)	4 (9)	5 (10)	5 (11)	6 (11)	7 (12)	8 (14)	8 (15)
0.5	2 (5)	3 (7)	4 (9)	5 (10)	5 (10)	6 (11)	6 (12)	7 (13)	8 (14)
0.6	2 (5)	3 (7)	4 (9)	5 (10)	5 (10)	6 (11)	6 (12)	7 (13)	8 (14)
0.8	2 (5)	3 (7)	4 (9)	5 (10)	5 (10)	6 (11)	6 (12)	7 (13)	7 (13)

Example 5.2.

$\varphi_1(\xi)=\varphi_2(\xi)=\lambda e^{-\lambda\xi}(\xi>0)$, then optimal levels $\lambda\bar{y}^k(1)$ are as follows. It shows that optimal levels are a monotone increasing function of k for the values of γ between about near 0 and 0.5, a monotone decreasing of k for γ between about 0.8 and near 1, and not a monotone function of k when $\gamma=0.6$ and 0.7.

values of $\lambda\bar{y}^k(1)$

k \ γ	0.01	0.1	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
0	.010	.105	.357	.511	.693	.916	1.204	1.609	2.303	2.996
0.1	.043	.168	.402	.539	.703	.906	1.172	1.555	2.224	2.904
0.167	.051	.191	.430	.563	.718	.910	1.161	1.529	2.173	2.841
0.2	.053	.200	.442	.574	.727	.914	1.159	1.515	2.150	2.810
0.25	.057	.210	.457	.588	.739	.921	1.158	1.501	2.117	2.763
0.33	.061	.224	.477	.609	.758	.935	1.162	1.487	2.069	2.687
0.5	.067	.241	.506	.640	.788	.961	1.177	1.479	2.005	2.559
0.6	.069	.248	.518	.653	.802	.974	1.187	1.481	1.981	2.501
0.8	.072	.259	.536	.674	.823	.995	1.205	1.488	1.956	2.419

Appendix

Proof of Theorem 4.

This theorem holds in the case of $k \neq 1$. So we prove the continuity of $\bar{y}^k(q)$ in q for any fixed $k(0<k\leqq 1)$.

In Eqs. (5) and (6), let $\frac{\partial G^k}{\partial y_i}$ denote by $f_i(q, y_1, y_2)$, for $i=1, 2$.

For any q, there exists one minimizing point $\bar{y}(q)=(\bar{y}_1(q), \bar{y}_2(q))$ which satisfies $f_i(q, \bar{y}_1(q), \bar{y}_2(q))=0$ $(i=1, 2)$.

Now, f_1, f_2 and their 1st order derivatives are continuous in this neighborhood. And moreover, from Eqs. (7), (9) and (10), we have the Jacobian

$$J=\frac{\partial(f_1, f_2)}{\partial(y_1, y_2)}>0 .$$

Thus by the well known theorem of the implicit function, we conclude that there exist continuous functions $\bar{y}_1(q)$ and $\bar{y}_2(q)$ which satisfy $f_i(q, \bar{y}_1(q), \bar{y}_2(q))=0$ $(i=1, 2)$.

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