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IDENTIFICATION OF TWO-PLAYER SITUATIONS WHERE COOPERATION IS PREFERABLE TO USE OF PERCENTILE GAME THEORY

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ABSTRACT

Considered is discrete two-person game theory where the players choose their strategies separately and independently. A generally applicable form of percentile game theory, using mixed strategies, has been developed where player i can select a $100\alpha_i$ percentile criterion and determine a solution that is optimum to him for this criterion ($i=1,2$). For example, median game theory occurs when the players decide to select $\alpha_1=\alpha_2=1/2$. The only requirement for usability is that, separately, each player can rank the outcomes for the game (pairs of payoffs, one to each player) according to their desirability to him. When cooperation can occur, however, cooperative choice of strategies can have

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advantages compared to a specified use, or class of uses, of percentile game theory (a use is defined by the values of the two percentiles). This paper identifies situations where cooperation is definitely preferable, for two types of cooperation. No side payments are made for one type of cooperation. This type can occur for any situation where percentile game theory is applicable. Side payments can be made for the other type of cooperation. This type occurs for situations where all payoffs can be expressed in a common unit and satisfy arithmetical operations. Rules are given for deciding when cooperation is definitely advantageous. Cooperation is always definitely preferable when $\alpha_1+\alpha_2>1$.

INTRODUCTION AND DISCUSSION

A generally applicable form of discrete two-person game theory based on percentile considerations has been developed (ref. 1) for situations where the players select their strategies separately and independently. The payoffs can be of almost any nature and some of them may not even be numerical. The pairs of payoffs, one to each player, that occur for the various combinations of strategy choices are the possible outcomes for the game. These outcomes, and the preferences of the players, are such that separately each player is able to rank the outcomes according to their relative desirability to him (including equal desirability as a possibility). The two rankings do not necessarily bear any relationship to each other.

For percentile game theory, player i selects a percentile $100\alpha_i$ and wants to play the game optimally with respect to this percentile ($i=1, 2$). A largest level of desirability (which corresponds to one or more outcomes O_i) occurs for the i-th player such that he can assure, with probability at least α_i, that an outcome with at least this desirability is obtained. This can be done simultaneously by both players. A method for determining O_i and an optimum (mixed) strategy for each player is given in ref. 1. The Appendix contains an outline of this method. Incidentally, median game theory occurs for the special case where $\alpha_1=$

$\alpha_2=1/2$.

It is to be noted that a ranking of the outcomes by a player not only considers the payoffs to him but also the corresponding payoffs to the other player. Thus, to the player doing the ranking, his ranking shows the relative desirability of what can occur for the game, including what occurs for the other player. Hence, for separate and independent choice of strategies, and given values for α_1 and α_2, the optimum solutions developed in ref. 1 are as good as can be obtained on this basis.

The usefulness of the percentile approach (for specified α_1 and α_2) is not so evident when the players can cooperate in selection of strategies. With cooperation, they may be able to obtain a game outcome that, to both, is preferable to use of solutions that are optimum when there is no cooperation. This paper is concerned with identification of cases where suitable use of cooperation is definitely preferable to use of percentile game theory. In making this identification, the preference rankings of the game outcomes are considered to be known for both players. Of course, the values of α_1 and α_2 can affect this preference. In fact, cooperation is found to always be definitely advantageous when $\alpha_1+\alpha_2>1$. Some results on cooperation have already been developed for $\alpha_1=\alpha_2=1/2$ and are given in ref. 2.

Two types of cooperation are considered. For the first type, no payments from one player to the other are made (no side payments). Cooperation of the first type can occur for virtually all situations (all situations where percentile game theory is applicable).

Cooperation of the second type can involve side payments but imposes a condition on the payoffs. That is, for side payments to be meaningful, the totality of payoffs should be expressible in a common unit and satisfy arithmetical operations (the operations of addition and subtraction, at the least).

For both types of cooperation, a restriction is sometimes imposed on the allowable freedom in ranking of the outcomes. This restriction is oriented toward situations where the players behave in a competitive manner. Specifically, for a given player, relative desirability is required to be a nondecreasing function of the desirability level of his payoff when the desirability level of the payoff for the other player (to the other player) is nonincreasing. This restriction would seem acceptable for any situation of practical interest where the players behave competitively and, separately, each player is able to rank his payoffs according to increasing desirability level (to him).

Let (p_1, p_2) denote an overall "outcome" for the game where, when side payments can be made, p_i is the overall amount received by player i, $(i=1, 2)$. Thus, the values of p_1 and p_2 are influenced by a payment made from one player to the other. Also, let $(p_1^{(g)}, p_2^{(g)})$ denote an actual outcome of the game, as determined by the payoff matrices. Only the $(p_1^{(g)}, p_2^{(g)})$ are compared for the first type of cooperation. All (p_1, p_2) such that p_1+p_2 equals $p_1^{(g)}+p_2^{(g)}$, for some game outcome, are compared (and can possibly occur) for the second type of cooperation. The $(p_1^{(g)}, p_2^{(g)})$ are, of course, included in the totality of the (p_1, p_2).

Cooperation is considered to be definitely advantageous, compared to optimum use of percentile game theory (with given α_1, α_2), when both players can gain by agreeing on an achievable (p_1, p_2). That is, for the first type of cooperation, they agree to select strategies so that a determined game outcome is obtained. For the second type of cooperation, the agree to choose strategies and make side payments so that a determined overall outcome occurs.

A rule for deciding when cooperation is definitely advantageous is given in the next section. Some implications of this rule, for competitive games and for games in general, are considered in the final section.

GENERAL RULE

Let the totality of game outcomes $(p_1^{(g)}, p_2^{(g)})$ be ranked according to increasing desirability separately by each player. There is a smallest subset $S_i(\alpha_i)$ such that all other game outcomes are less desirable to player i than those of this subset and also an outcome of this subset can be assured with probability at least α_i by player i, (i=1, 2). Statement of a method that can be used to determine $S_i(\alpha_i)$ is given in ref. 1. An outline of this method occurs in the Appendix. It is to be noted that $S_1(1/2)$ and $S_2(1/2)$ can differ slightly from the subsets S_{I} and S_{II} of ref. 2.

The subsets $S_1(\alpha_1)$ and $S_2(\alpha_2)$ provide the basis for a general rule that can be used to decide when cooperation is definitely preferable to percentile game theory (with α_1 and α_2 specified).

General Rule: Suitable use of cooperation is definitely preferable for both players when an achievable outcome (p_1, p_2) occurs that is at least as desirable to player 1 as one or more of the game outcomes of $S_1(\alpha_1)$ and also is at least as desirable to player 2 as one or more of the game outcomes of $S_2(\alpha_2)$.

This rule follows from the consideration that an achievable (overall) outcome with at least the minimum level of desirability for both $S_1(\alpha_1)$ and $S_2(\alpha_2)$ can be obtained with certainty. For player i, optimum use of the percentile approach only assures that an outcome with desirability level at least the minimum for $S_i(\alpha_i)$ is obtained with probability at least α_i, (i=1, 2).

The achievable (p_1, p_2) for the first type of cooperation are the ($p_1^{(g)}$, $p_2^{(g)}$). All (p_1, p_2) that satisfy $p_1+p_2=p_1^{(g)}+p_2^{(g)}$, for some game outcome, are achievable for the second type of cooperation.

Suitable use of cooperation is definitely advantages, for both types of cooperation, when the same game outcome occurs in both $S_1(\alpha_1)$ and $S_2(\alpha_2)$. This always happens, as shown in the next section, when $\alpha_1+\alpha_2>1$. However, $S_1(\alpha_1)$ and $S_2(\alpha_2)$ need not have any game outcomes in common when $\alpha_1+\alpha_2\leq 1$. Of course, the number of outcomes in $S_i(\alpha_i)$ is a nondecreasing function of α_i, so situations can occur where cooperation is not definitely preferable for given α_1, α_2 but is definitely preferable for sufficient increase in α_1 and/or α_2 (even though $\alpha_1+\alpha_2\leq 1$ in all cases). Also, the minimum level of desirability for outcomes of $S_i(\alpha_i)$ is a nonincreasing function of α_i and situations can occur where the players do not want to use α_1 and α_2 values which result in $\alpha_1+\alpha_2>1$.

A game outcome that is common to both $S_1(\alpha_1)$ and $S_2(\alpha_2)$ must occur if the first type of cooperation is to be definitely preferable. There is, however, a broad class of situations where the second type of cooperation is preferable, even though $\alpha_1+\alpha_2\leq 1$. Some consideration of situations where the second type is preferable occurs in the next section.

IMPLICATIONS OF RULE

First, consider verification of the statement that cooperation is always definitely preferable when $\alpha_1+\alpha_2>1$. This follows from

Theorem 1. *If $S_1(\alpha_1)$ and $S_2(\alpha_2)$ have no game outcomes in common, then $\alpha_1+\alpha_2\leq 1$.*

Proof: An outcome of $S_i(\alpha_i)$ can be assured with probability at least α_i by player i, $(i=1, 2)$. This implies that the complement of $S_1(\alpha_1)$ in the totality of game outcomes, denoted by $\bar{S}_1(\alpha_1)$, can be assured by player 2 with probability at most $1-\alpha_1$. However, $\bar{S}_1(\alpha_1)$ contains $S_2(\alpha_2)$, which implies that $1-\alpha_1\geq\alpha_2$, or $\alpha_1+\alpha_2\leq 1$.

This theorem holds, in particular, when α_1 and α_2 are attainable values (ref. 1). A value $\alpha_i^{(o)}$ is attainable for α_i if player i cannot assure an outcome of $S_i(\alpha_i^{(o)})$ with probability greater than $\alpha_i^{(o)}$. When a specified value for α_i is not attainable, the value actually used for α_i is the smallest of the attainable values that exceeds the specified value. It is advisable to use attainable values for α_1 and α_2. Then, a true situation of $\alpha_1^{(o)}+\alpha_2^{(o)}>1$ does not occur when $\alpha_1+\alpha_2\leq 1$ for the specified values.

Next, consider the advantages of cooperation for games that are "competitive." Here, a game is considered to be competitive if and only if the totality of game outcomes can be arranged in a sequence so that the payoffs for player 1 have nondecreasing desirability (to him) and also the payoffs for player 2 have nonincreasing desirability level (to him). This generalizes the concept of competitive games given in ref. 3 for the case of payoffs that are numbers.

For a competitive game and the restriction imposed for competitive behavior of the players, $S_1(\alpha_1)$ and $S_2(\alpha_2)$ have no outcomes in common when $\alpha_1+\alpha_2<1$ for attainable values. This follows from

Theorem 2. *Let $\alpha_1^{(o)}$ and $\alpha_2^{(o)}$ be attainable values for α_1 and α_2, respectively. When the restriction for competitive behavior holds, the game is competitive, and $\alpha_1^{(o)}+\alpha_2^{(o)}<1$, then the subsets $S_1(\alpha_1^{(o)})$ and $S_2(\alpha_2^{(o)})$ are mutually exclusive and there is at least one game outcome that does not belong to either of them.*

Proof: If this were not true, the restriction on ranking outcomes due to the competitive behavior would imply that the union of $S_1(\alpha_1^{(o)})$ and $S_2(\alpha_2^{(o)})$ is the totality of the game outcomes. Then, $S_2(\alpha_2^{(o)})$ contains $\bar{S}_1(\alpha_1^{(o)})$. Also, $1-\alpha_1^{(o)}$ is an attainable value for α_2, since player 2 can assure at outcome of $\bar{S}_1(\alpha_1^{(o)})$ with probability at least $1-\alpha_1^{(o)}$ but not in excess of $1-\alpha_1^{(o)}$ (because player 1 can assure an outcome of $S_1(\alpha_1^{(o)})$ with probability at least $\alpha_1^{(o)}$ but not in excess of $\alpha_1^{(o)}$). This, combined with the relationship between $S_2(\alpha_2^{(o)}$ and $\bar{S}_1(\alpha_1^{(o)})$, implies that $\alpha_2^{(o)}\geq 1-\alpha_1^{(o)}$, or $\alpha_1^{(o)}+\alpha_2^{(o)}\geq 1$, a contradiction.

Thus, for competitive behavior, a competitive game, and use of attainable α_i, the subsets $S_1(\alpha_1)$ and $S_2(\alpha_2)$ necessarily have at least one outcome in common when $\alpha_1+\alpha_2>1$ and necessarily are mutually exclusive when $\alpha_1+\alpha_2<1$. There may, or may not, be any outcomes in common when $\alpha_1+\alpha_2=1$.

Occurrence of a common outcome for $S_1(\alpha_1)$ and $S_2(\alpha_2)$ is not necessary for definite preference of the second type of cooperation. For example, cooperation often happens to be preferable in the somewhat common situation where an increase of zero or more in the payoff received by a player, combined with addition of at most this amount (negative amounts possible) to the payoff for the other player, yields another outcome that is at least as desirable to him. Here, the payoffs are expressed in the same unit and justification for the preceding assertion follows from

Theorem 3. *Suppose that, for any outcome $(p_1^{(g)}, p_2^{(g)})$, a modifi-*

cation $(p_1^{(g)}+h_1,\ p_2^{(g)}+h_2)$, *to some achievable outcome, is as least as desirable to player i as* $(p_1^{(g)},\ p_2^{(g)})$ *when* $h_i\geq 0$ *and the other h is at most equal to* h_i, $(i=1,2)$. *Then, if there exists an outcome* $(p_1',\ p_2')$ *of* $S_1(\alpha_1)$, *an outcome* $(p_1'',\ p_2'')$ *of* $S_2(\alpha_2)$, *and a game outcome* $(p_1''',\ p_2''')$ *such that* $p_2'-p_2''\geq p_1'-p_1''$ *and* $p_1'''+p_2'''\geq \max(p_1'+p_2',\ p_1''+p_2'')$, *the second type of cooperation is definitely advantageous.*

Proof: Let p_1''' and p_2''' be expressed in the form

$$p_1'''=p_1'+a=p_1''+b,\ p_2'''=p_2'+A=p_2''+B\ .$$

The conditions of the theorem imply that

$$a+b\geq 0\ ,\quad A+B\geq 0\ ,\quad B-A+a-b\geq 0\ .$$

Suppose that the players use $(p_1''',\ p_2''')$ and that player 2 makes a side payment of $C=(b+B-a-A)/4$ to player 1. The result is

$$(p_1'''+C,\ p_2'''-C)=[p_1'+(3a+b+B-A)/4,\ p_2'+(3b-B+a+A)/4]\ ,$$

with a nonnegative amount added to p_1' and at most this amount added to p_2', since

$$3a+b+B-A\geq 3a+b+b-a=2(a+b)\geq 0\ ,$$
$$(3a+b+B-A)-(3b-B+a+A)=2(B-A+a-b)\geq 0\ .$$

Likewise,

$$(p_1'''+C,\ p_2'''-C)=[p_1''+(3A+b+B-a)/4,\ p_2''+(3B-b+a+A)/4]\ ,$$

where a nonnegative amount is added to p_2'' and at most this amount is added to p_1''. Thus, from the supposition of the theorem, the achievable outcome $(p_1'''+C,\ p_2'''-C)$ is at least as desirable to player 1 as $(p_1',\ p_2')$ of $S_1(\alpha_1)$ and as least as desirable to player 2 as $(p_1'',\ p_2'')$ of $S_2(\alpha_2)$.

APPENDIX

The same results are applicable to each player and are stated for player *i*. Here the players select strategies separately and independently, and the only outcomes considered are game outcomes. A way of determining the largest level of desirability (corresponds to one or more outcomes O_i) that player i can assure with probability at least α_i is outlined. Then, a method for determining an optimum strategy for player i is described. These procedures are based on a making of the payoff positions in the matrix for player i. A more detailed statement of these methods is given in ref. 1.

First, mark the position(s) in the payoff matrix for player i of the outcome(s) with the highest level of desirability to him. Next, also mark the position(s) of the outcome(s) with the next to highest level of desirability. Continue this marking, according to decreasing level of desirability, until the first time that player i can assure the occurrence of an outcome in the marked set with probability at least α_i. The last (and lowest) level desirability for which a marking ultimately occurred is the highest level that can be assured by player i with probability at least α_i, with O_i being the outcome(s) marked last. The set $S_i(\alpha_i)$ consists of the outcomes whose desirability level, to player i, is at least as great as the level for O_i.

The method for determining whether a set of marked outcomes can be assured with probability at least α_i consists in first replacing all of their positions (marked) in the matrix for player i by unity and all other positions by zero. The resulting matrix of ones and zeroes is considered to be for player i in a zero-sum game with an expected-value basis. Player i can assure an outcome of the marked set with probability at least α_i if and only if the value of this game, for him, is at least α_i.

Now, consider determination of an optimum strategy for player i. Use the matrix marking that, ultimately, resulted in the smallest marked set (according to the marking procedure) such that an outcome of this set can be assured by player i with probability at least α_i. Replace the marked positions by unity and the others by zero. Treat the resulting matrix as that for player i in a zero-sum game with an expected-value basis. An optimum strategy for player i in this game is α_i-optimum for him.

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