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# ON AN EXTENSION OF THE MAXIMUM-FLOW MINIMUM-CUT THEOREM TO MULTICOMMODITY FLOWS

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## Introduction

The extension of the maximum-flow minimum-cut theorem for single-commodity network flows to the case of multicommodity flows has been discussed by many researchers (see [1] for the details). Recently, K. Onaga established a necessary and sufficient condition for the existence of a feasible multicommodity flow configuration on a capacity-constrained undirected network when the locations of source and sink as well as the total flow value are given for each commodity [2]. His condition seems to be of fundamental significance in multicommodity network-flow theory although it is not combined with a practical computational algorithm. However, the proof developed in [2] is purely graphical and fairly complicated. In the present paper, a simple proof to his condition will be given based on the duality theorem in linear programming, where some improvement will be made also on the statement of the condition itself.

## 1. Description of the Problem

Let $N=(V, A)$ be a connected network with the set of vertices $V$ and the set of arcs $A(=V\times V)$, where we assume that $N$ is symmetric, *i.e.* that if $(\alpha, \beta)\in A$ then $(\beta, \alpha)\in A$. We shall denote by $((\alpha, \beta))$ an unordered pair of vertices $(\alpha, \beta\in V)$, and by $\tilde{A}$ the set of unordered pairs $((\alpha, \beta))$ such that $(\alpha, \beta)\in A$. The "flow" in arc $(\alpha, \beta)$ of commodity $i$ is denoted by $x_{i,(\alpha,\beta)}$, where the number of commodities is assumed to be $Q$, *i.e.* $i$ ranges from 1 to $Q$. Furthermore, for each commodity $i$, a source vertex $s_i$ and a sink vertex $t_i$ are prescribed. The characteristics of the network are specified by defining the "capacity" $c_{((\alpha,\beta))}$ for each unordered pair $((\alpha, \beta))$ in $\tilde{A}$.

By the "capacity constraints" we mean the inequalities to be satisfied by flows $x_{i,(\alpha,\beta)}$:

$$\sum_{i=1}^{Q}(x_{i,(\alpha,\beta)}+x_{i,(\beta,\alpha)})\leqq c_{((\alpha,\beta))} \quad \text{for} \quad ((\alpha, \beta))\in\tilde{A} . \tag{1.1}$$

Each commodity $i$ is required to be conveyed at least $r_i$ units ($r_i$ being given nonnegative values) from $s_i$ to $t_i$ through $N$ and the flows must satisfy the continuity conditions, *i.e.* the following conditions must hold for every $i(=1, \cdots, Q)$ and every $\alpha\in V$:

$$\begin{aligned} -\sum_{\beta\in V}x_{i,(\alpha,\beta)}+\sum_{\beta\in V}x_{i,(\beta,\alpha)} &= 0 \qquad (\alpha\neq s_i \text{ or } t_i),\\ &\leqq -r_i \quad (\alpha=s_i),\\ &\geqq r_i \qquad (\alpha=t_i), \end{aligned} \tag{1.2}$$

where the nonnegativity of $x_{i,(\alpha,\beta)}$'s is understood:

$$x_{i,(\alpha,\beta)}\geqq 0 \qquad (i=1, \cdots, Q\,;\ (\alpha, \beta)\in A). \tag{1.3}$$

The problem is to find a necessary and sufficient condition for the existence of a set of values $x_{i,(\alpha,\beta)}$'s which satisfy (1.1), (1.2) and (1.3) when $N=(V, A)$, $c_{((\alpha,\beta))}$'s, $s_i$'s, $t_i$'s and $r_i$'s are given.

## 2. Formulation of the Dual Problem

According to the well-known duality arguments, let us consider the dual problem of the problem posed in §1 (see, *e.g.*, [3]). We introduce the dual variables (or Lagrange multipliers) $y_{((\alpha,\beta))}$'s (one for each unordered pair $((\alpha, \beta))\in \tilde{A}$) corresponding to the conditions (1.1) and $z_{i,\alpha}$'s corresponding to the conditions (1.2), and we consider the problem of minimizing the function:

$$g(y, z)\equiv \sum_{((\alpha,\beta))\in \tilde{A}} c_{((\alpha,\beta))}\cdot y_{((\alpha,\beta))}-\sum_{i=1}^{Q} r_i\cdot(z_{i,s_i}-z_{i,t_i}) \tag{2.1}$$

under the conditions:

$$\left.\begin{aligned} &y_{((\alpha,\beta))}-z_{i,\alpha}+z_{i,\beta}\geqq 0\ ,\\ &y_{((\alpha,\beta))}-z_{i,\beta}+z_{i,\alpha}\geqq 0 \end{aligned}\right\}\ ((\alpha, \beta))\in \tilde{A}\ ;\quad i=1, \cdots, Q\ , \tag{2.2}$$

and

$$\left.\begin{aligned} &z_{i,s_i}\geqq 0\ ,\quad z_{i,t_i}\leqq 0\quad (i=1, \cdots, Q)\ ;\\ &y_{((\alpha,\beta))}\geqq 0\qquad ((\alpha, \beta))\in \tilde{A})\ . \end{aligned}\right\} \tag{2.3}$$

Then, as is well known,

> *the* $x_{i,(\alpha,\beta)}$*'s satisfying* (1.1)~(1.3) *exist if and only if* $g(y, z)$ *is bounded downwards for such* $y_{((\alpha,\beta))}$*'s and* $z_{i,\alpha}$*'s as satisfy* (2.2) *and* (2.3)[1].

Since, by virtue of the homogeneity of (2.1)~(2.3) in $y$ and $z$, we may adopt "0" for the lower bound, if it exists, of $g(y, z)$, we can restate the necessary and sufficient condition for the existence of $x_{i,(\alpha,\beta)}$'s satisfying (1.1)~(1.3) as follows:

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| $g(y, z)\leqq 0$<br>for any $y_{((\alpha,\beta))}$ and $z_{i,\alpha}$ satisfying (2.2) and (2.3). | (2.4) |

1) Here it should be noted that the condition (2.2) and (2.3) are trivially satisfied by null $y_{((\alpha,\beta))}$'s and null $z_{i,\alpha}$'s.

## 3. A Simplification by Graphical Consideration

Since the dual variables $z_{i,\alpha}$'s appear in (2.1) and (2.2) as the difference of two $z_{i,\alpha}$'s with the same $i$, we may put, with no loss in generality,

$$z_{i,t_i}=0 \quad \text{for every} \quad i\ (=1, \cdots, Q). \tag{3.1}$$

Furthermore, (2.2) may be rewritten as

$$y_{((\alpha,\beta))} \geqq |z_{i,\alpha}-z_{i,\beta}| \quad \text{for every} \quad ((\alpha,\beta))=\tilde{A} \quad \text{and every} \quad i\ (=1, \cdots, Q). \tag{3.2}$$

Let us fix a set of values $y_{((\alpha,\beta))}$'s arbitrarily, and consider to minimize $g(y, z)$ by varying the values of $z_{i,\alpha}$'s under the conditions (2.3), (3.1) and (3.2). As is evident from the expression of (2.1), $g(y, z)$ is the smaller the larger the $z_{i,\alpha}$'s are. Owing to (3.1) and (3.2), the maximum possible value of $z_{i,s_i}$, for each $i$, is equal to the length (which we shall denote by $R_i(y)$) of the shortest route from vertex $s_i$ to vertex $t_i$, where each $y_{((\alpha,\beta))}$ is regarded as the length of the corresponding arcs $(\alpha, \beta)$ and $(\beta, \alpha)$[2]. In other words, we have

$$\min_{z_i,\alpha} g(y, z)=\sum_{((\alpha,\beta))\in\tilde{A}} c_{((\alpha,\beta))}\cdot y_{((\alpha,\beta))}-\sum_{i=1}^{Q} r_i\cdot R_i(y) . \tag{3.3}$$

Thus we have been led to the following form of the required condition:

$$\boxed{\sum_{((\alpha,\beta))\in\tilde{A}} c_{((\alpha,\beta))}\cdot y_{((\alpha,\beta))} \geqq \sum_{i=1}^{Q} r_i\cdot R_i(y) \quad \text{for any nonnegative } y_{((\alpha,\beta))}\text{'s } ((\alpha,\beta))\in\tilde{A},} \tag{3.4}$$

which is essentially the same form as was given in [2].

2) This is the "dual" theorem of the maximum-flow minimum-cut theorem, which is called in [4] the "maximum-separation minimum-route theorem".

## 4. Discussions

(a) From the basis theorem of linear programming and the homogeneity of (2.2) and (2.3), it follows that only nonnegative integers are sufficient to consider as the values of $y_{((\alpha,\beta))}$'s in examining the validity of the condition (3.4). Furthermore, we need not deal with "all" nonnegative integers but only with those integers which are equal to a minor determinant of the coefficient matrix in the standard-from expression of the condition (2.2). Therefore, if the maximum absolute value of the minor determinants is $M$, then we have only to deal with those nonnegative integers which do not exceed $M$. This theoretical bound for $y$ is not noted in [2]. However, the value of $M$ is, practically, difficult to determine.[3)]

(b) When nonnegative integers $y_{((\alpha,\beta))}$'s are given, we can consider a set of arc-pairs (*i.e.* $(\alpha, \beta)$ and $(\beta, \alpha)$ for an $((\alpha, \beta))\in\tilde{A}$), counted $y_{((\alpha,\beta))}$ times, for every $((\alpha, \beta))\in\tilde{A}$, and interpret the left-hand side of the inequality in (3.4) as the "value" associated with the capacities of such a set. This may be regarded as an extension of the concept of the value of a cut in the single-commodity case. In the single-commodity case ($Q=1$), if $y_{((\alpha,\beta))}=1$ for the $((\alpha, \beta))$'s in a cut separating $s_1$ from $t_1$ and $y_{((\alpha,\beta))}=0$ for the other $((\alpha, \beta))$'s then we obviously have $R_1(y)=1$, so that the right-hand side of (3.4) is equal to $r_1$, the amount of total flow required to be conveyed from the source to the sink. It is not difficult to see that, in case $Q=1$, (3.4) is satisfied for arbitary nonnegative $y$'s if it is satisfied for the $y$'s of this kind. Hence, in this case, the condition (3.4) is equivalent to saying that the amount of total flow cannot exceed the value of any cut. Thus, the maximum-flow minimum-cut

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3) It is one of challenging problems in network-flow theory to clarify the dependence of $M$ on the number of commodities as well as on the underlying network structure. A trivial bound for $M$ is $M\leqq Q^{|\tilde{A}|}$, where $|\tilde{A}|$ is the number of elements of $\tilde{A}$ (*cf.* [5]).

theorem has theoretically been extended to the multicommodity case. However, no practical algorithm for discerning whether the condition (3.4) is satisfied by a given network or not has been known in the multicommodity case unlike the single-commodity case.

(c) The foregoing argument for "undirected" networks can easily be modified to directed networks. In a directed network, each arc $(\alpha, \beta)\in A$ is provided with a capacity $c_{(\alpha,\beta)}$, and the capacity constraints are

$$\sum_{i=1}^{Q} x_{i,(\alpha,\beta)} \leqq c_{(\alpha,\beta)} \qquad \text{for every} \quad (\alpha, \beta)\in A, \tag{4.1}$$

in place of (1.1). The conditions (1.2) and (1.3) on $x_{i,(\alpha,\beta)}$'s remain the same. In addition to $z_{i,\alpha}$'s, the dual variables $y_{(\alpha,\beta)}$'s (in place of $y_{((\alpha,\beta))}$'s) are to be adopted, one for each arc. (2.1) is replaced by

$$g(y, z)=\sum_{(\alpha,\beta)\in A} c_{(\alpha,\beta)}\cdot y_{(\alpha,\beta)}-\sum^{Q} r_i\cdot(z_{i,s_i}-z_{i,t_i}) \ , \tag{4.2}$$

(2.2) by

$$y_{(\alpha,\beta)}-z_{i,\alpha}+z_{i,\beta}\geqq 0 \qquad ((\alpha, \beta)\in A\ ; \quad i=1, \cdots, Q) \ , \tag{4.3}$$

and (2.3) by

$$\left.\begin{array}{l} z_{i,s_i}\geqq 0 \ , \quad z_{i,t_i}\leqq 0 \qquad (i=1, \cdots, Q)\ ; \\ y_{(\alpha,\beta)}\geqq 0 \qquad ((\alpha, \beta)\in A) \ , \end{array}\right\} \tag{4.4}$$

respectively. Then, the necessary and sufficient condition for the existence of a feasible solution for (4.1), (1.2) and (1.3) is written as

$$\boxed{\begin{array}{l} \displaystyle\sum_{(\alpha,\beta)\in A} c_{(\alpha,\beta)}\cdot y_{(\alpha,\beta)}\geqq \sum_{i=1}^{Q} r_i\cdot R_i(y) \\ \qquad \text{for any nonnegative} \quad y_{(\alpha,\beta)}\text{'s} \quad ((\alpha, \beta)\in A) \ , \end{array}} \tag{4.5}$$

where, for each $i$, $R_i(y)$ is the shortest distance from vertex $s_i$ to vertex $t_i$ with $y_{(\alpha,\beta)}$ as the length of arc $(\alpha, \beta)$ in the positive direction (the length of an arc in the negative direction is regarded as infinity).

(d) The idea of considering the shortest route problem with the Lagrange multipliers for capacity constraints as the lengths of arcs goes back to Ford and Fulkerson [6].

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