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ON QUEUE DISCIPLINES

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Abstract

In a single-server queue we consider, at first, the characteristics of the well-known queue disciplines on a sample path of the queueing process and compare those disciplines in the average waiting time and the queue-length on a sample path. Secondly, we show that the expected waiting time and the moments of the queue-length in the steady state are invariant in a certain class of queue disciplines.

1. Introduction

Invariants in a certain class of queue disciplines were found by several authors. Welch [14] proved the fact that the distribution of the length of the busy period is independent of the discipline. Little [5] presented a remarkable relation $L=\lambda W$ and stated the relation being free from queue disciplines. Kleinrock [4] and Schrage [10], further, established the conservation law named by the former.

For the queueing system $M/G/1$, Takács [11] compared the variances of the equilibrium waiting time distribution for three disciplines; first-come first-served, random service and last-come first-served. Also, Schrage [9] proved that with the shortest remaining processing time discipline the queue-length at any point in time is less than or equal to the queue-length for any other discipline in a defined class simultaneously acting on the same sequence of arrivals and service times.

Motivation of our study in this paper was brought about by the above consideration. We consider in the next section each characteristic of some particular disciplines acting on the same sequence of arrivals and service times for a single-server queueing system. And also we will compare the average waiting times and the queue-lengths at any point in time for disciplines in a certain class. The results obtained in this section do not depend on any assumptions about the distribution of either the inter-arrival times or service times, but in the last section it is only assumed that the service times are independent of each other.

In the last section, we show that the expected waiting time and the moments of the queue-length in the steady state are invariant in a certain class of disciplines.

Here, we will define some particular disciplines being treated with this paper. "*First-come, first-served*" discipline means that the order of service is the same with that of arrival. "*Last-come, first-served*" discipline means that the order of service is reverse order of arrival. "*Random service*" discipline means that the customer being served is selected at random from queue. "*Shortest service time*" discipline means that the customer having the shortest service time is first served. "*Largest service time*" discipline means that the customer having the largest service time is first served. "*Shortest remaining service time*" discipline means that priority is assigned to customers according to the length of service remaining time, with highest priority going to the customer with least processing time left. "*Largest remaining service time*" discipline means that priority is assigned to customers according to the length of service remaining time, with highest priority going to the customer with longest processing time left.

2. Characteristics of disciplines

Throughout the paper we will deal with the disciplines which do not affect the sequence of arrival times.

Let C_0 be a class of disciplines satisfying the following conditions:

1. All customers remain in the system until completely served;
2. The server is never idle if there is a customer for service;
3. There is no preemption;
4. Service times of any customers remaining in the system are unknown by the server. And let C_1 be a class of disciplines satisfying the above conditions except but the fourth condition. Also let C_2 be a wide class of disciplines satisfying only the first two conditions. For these classes it holds the inclusion relation as follows:

$$C_0 \subset C_1 \subset C_2.$$

The well-known disciplines such as "*first-come, first-served* (*FCFS*)", "*random service* (*RS*)" and "*last-come, first-served* (*LCFS*)" are elements of the class C_0. But both of the shortest service time (SST) discipline and the largest service time (LST) discipline are elements of the class C_1 but not contained in the class C_0. The shortest remaining service time ($SRST$) and the largest remaining service time ($LRST$) discipline are in the class C_2 but not in the class C_1.

In a given system we define the stochastic process $X(t)$, $t \in (-\infty, \infty)$ as follows: For each ω,

$$X(t, \omega) = \begin{cases} 1, & \text{if there are customers in the system at time } t, \\ 0, & \text{otherwise.} \end{cases}$$

Then, Welch [14] proved the following fact.

Proposition 2.1. For any fixed ω, $X(t, \omega)$ is independent of disciplines for the class C_2, that is, the sample path of $X(t)$ is invariant for all disciplines of the class C_2.

Now we will introduce the following notations, where the subscript d refers to the discipline d being used in the system;

W_n^d = elapsed time in the system for the n-th arrival customer,

w_n^d = total waiting time in the queue for the n-th arrival customer,

$N^d(t)$ = number of customers in the system at time t,

$n^d(t)$ = number of customers in the queue at time t,

t_n = arrival time of the n-th customer,

c_n^d = departure time of the n-th arrival customer,

τ_n^d = departure time of the n-th departing customer,

S_n = service time of the n-th arrival customer,

S_n^d = n-th processing time,

$T_n = t_{n+1} - t_n$ = n-th inter-arrival time.

For a fixed ω, we have the following relations on a specified busy period, where the starting point of the busy period is taken as time zero.

Proposition 2.2.

(2.1) $$\sum_n W_n^{d_1}(\omega) = \sum_n W_n^{d_2}(\omega) + \sum_n (\tau_n^{d_1}(\omega) - \tau_n^{d_2}(\omega)) \quad \text{for any } d_1, d_2 \in C_2.$$

(2.2) $$\sum_n W_n^d(\omega) = \sum_n w_n^d(\omega) + \sum_m S_m^d(\omega) \qquad \text{for any } d \in C_2.$$

(2.3) If $\tau_n^{d_1}(\omega) \leqq \tau_n^{d_2}(\omega)$ for all n and for any d_1, $d_2 \in C_2$, then it follows $N^{d_1}(t, \omega) \leqq N^{d_2}(t, \omega)$ for any t, vise versa.

In the case with a constant service we have from (2.1) and (2.2)

$$\sum_n W_n^{d_1}(\omega) = \sum_n W_n^{d_2}(\omega) \qquad \text{for any } d_1,\ d_2 \in C_1$$

and

$$\sum_n w_n^{d_1}(\omega) = \sum_n w_n^{d_2}(\omega) \qquad \text{for any } d_1,\ d_2 \in C_1.$$

Proof of (2.1). From the relations; $W_n^d(\omega) = c_n^d(\omega) - t_n(\omega)$ and $\sum_n c_n^d(\omega) =$

$\sum_n \tau_n^d(\omega)$, we have $\sum_n W_n^{d_1}(\omega)=\sum_n(\tau_n^{d_1}(\omega)-t_n(\omega))$ as well as $\sum_n W_n^{d_2}(\omega)=\sum_n(\tau_n^{d_2}(\omega)-t_n(\omega))$. That is,

$$\sum_n W_n^{d_1}(\omega)-\sum_n W_n^{d_2}(\omega)=\sum_n \tau_n^{d_1}(\omega)-\sum_n \tau_n^{d_2}(\omega).$$

(2.2) is clearly derived from the relation; $W_n^d(\omega)=w_n^d(\omega)+\sum_m{}^* S_m(\omega)$, where $\sum_m{}^*$ is taken over all processing times of the n-th arrival customer.

Proof of (2.3). We define

$$I(t,\omega)=\max\ \{n \mid t_n(\omega)\leqq t\},$$

and

$$O^d(t,\omega)=\begin{cases} 0, & \text{if } \tau_1^d(\omega)\geqq t,\\ \max\ \{n \mid \tau_n^d(\omega)\leqq t\}, & \text{otherwise.}\end{cases}$$

Then $N^d(t,\omega)=I(t,\omega)-O^d(t,\omega)$. Therefore it will be sufficient to prove the relation: $\tau_n^{d_1}(\omega)\leqq\tau_n^{d_2}(\omega)$ for all $n \rightleftarrows O^{d_1}(t,\omega)\geqq O^{d_2}(t,\omega)$ for all t. But, this relation is easily derived by the definition of $O^d(t,\omega)$.

Maxwell [6] established the following formula.

Proposition 2.3. On a busy period with its length T,

$$L^d(\omega)=\frac{1}{T(\omega)}\int_0^{T(\omega)} N^d(t,\omega)\,dt=\frac{\lambda(\omega)}{\gamma(\omega)}\sum_{n=1}^{\gamma} W_n^d(\omega)=\lambda(\omega)\overline{W}^d(\omega)$$

for any $d\in C_2$, where $\lambda(\omega)=\gamma(\omega)/T(\omega)$, $\overline{W}^d(\omega)=\frac{1}{\gamma(\omega)}\sum_{n=1}^{\gamma} W_n^d(\omega)$ and $\gamma(\omega)$ is the number of arrivals during the busy period.

We will now state the characteristics of the well-known disciplines: *FCFS*, *LST*, *SRST* and *LRST* discipline. On a busy period the following relations from (2.4) to (2.15) are true.

Proposition 2.4. For *FCFS* discipline,

(2.4) $$\min_{d\in C_2}\ \{\max_n\ W_n^d(\omega)\}=\max_n\ W_n^{FCFS}(\omega).$$

Proof. We will assume that k customers are served in a specified busy period. For *FCFS* discipline, let the i-th arrival customer have the most largest elapsed time in the system, that is,

$$W_i^{FCFS}(\omega)=\max_{1\leqq n\leqq k} W_n^{FCFS}(\omega)$$

for a fixed ω. For any discipline $d\in C_2$ we consider two cases; $c_i^d\geqq c_i^{FCFS}$ or $c_i^d<c_i^{FCFS}$. In the former case,

$$\max_{1\leqq n\leqq k} W_n^d(\omega)\geqq W_i^d(\omega)\geqq W_i^{FCFS}(\omega)=\max_{1\leqq n\leqq k} W_n^{FCFS}(\omega).$$

In the latter case, there is at least one customer (say the j-th) satisfying the conditions; $t_j<t_i$ and $c_j^d\geqq c_i^{FCFS}$. Then,

$$\max_{1\leqq n\leqq k} W_n^d(\omega)\geqq W_j^d(\omega)\geqq W_i^{FCFS}(\omega)=\max_{1\leqq n\leqq k} W_n^{FCFS}(\omega).$$

Thus, (2.4) is proved in either case.

Proposition 2.5. For *LST* discipline,

(2.5) $$\tau_n^{LST}(\omega)=\max_{d\in C_1} \tau_n^d(\omega) \qquad \text{for all } n,$$

(2.6) $$\sum_n W_n^{LST}(\omega)=\max_{d\in C_1} \sum_n W_n^d(\omega)$$

and

(2.7) $$N^{LST}(t,\omega)=\max_{d\in C_1} N^d(t,\omega) \qquad \text{for all } t.$$

Proof of (2.5). Let $t_1=0$ and let k customers be served in a specified busy period. And let H_m^d be the set of service times of queueing customers at the m-th departure time. That is,

$$H_m^d=\{S_i|t_i<\tau_m^d\}-\{S_i^d|1\leqq i\leqq m\}.$$

Of course, H_k^d is empty and $S_i^d\in H^d_{i-1}$. From the definition of *LST* discipline $S_i^{LST}=\max\{S_j|S_j\in H_{i-1}^{LST}\}$. Now let us prove (2.5). At first, $H_1^{LST}=H_1^d$ since $\tau_1^{LST}=\tau_1^d=S_1$. Next $\tau_2^{LST}=S_1+S_2^{LST}=S_1+\max\{S_i|S_i\in H_1^{LST}\}\geqq S_1+S_2^d=\tau_2^d$ and then $\bigcup_{m=1}^{2} H_m^{LST}\supset\bigcup_{m=1}^{2} H_m^d$.

$$\tau_3{}^{LST} = S_1 + S_2{}^{LST} + S_3{}^{LST} = S_1 + \max\{S_i | S_i \in H_1\}$$
$$+ \max\{S_i | S_i \in H_2{}^{LST}\}$$
$$= S_1 + \max\{S_2{}^{LST} + S_i | S_i \in \{\bigcup_{m=1}^{2} H_m{}^{LST} - \{S_2{}^{LST}\}\}\}$$
$$\geqq S_1 + \max\{S_2{}^{LST} + S_i | S_i \in \{\bigcup_{m=1}^{2} H_m{}^{d} - \{S_2{}^{LST}\}\}\}$$
$$\geqq S_1 + \max\{S_2{}^{d} + S_i | S_i \in \{\bigcup_{m=1}^{2} H_m{}^{d} - \{S_2{}^{d}\}\}\}$$
$$\geqq S_1 + S_2 + S_3 = \tau_3{}^{d},$$

then $\bigcup_{m=1}^{3} H_m{}^{LST} \supset \bigcup_{m=1}^{3} H_m{}^{d}$ and so on. Thus we have (2.5).

(2.6) is easily induced from (2.1) and (2.5). (2.7) is also induced from (2.3) and (2.5).

Proposition 2.6. For *SRST* discipline,

(2.8) $$\tau_n{}^{SRST}(\omega) = \min_{d \in C_2} \tau_n{}^{d}(\omega) \qquad \text{for all } n,$$

(2.9) $$\sum_n W_n{}^{SRST}(\omega) = \min_{d \in C_2} \sum_n W_n{}^{d}(\omega)$$

and

(2.10) $$N^{SRST}(t, \omega) = \min_{d \in C_2} N^{d}(t, \omega) \qquad \text{for all } t.$$

Proof of (2.8) is given by Schrage [9]. (2.9) is induced from (2.1) and (2.8), (2.10) is also obtained from (2.3) and (2.8).

Proposition 2.7. In the system with *LRST* discipline in which both service time and inter-arrival time may be taken only multiples of a unit as their values,

(2.11) $$\tau_n{}^{LRST}(\omega) = \max_{d \in C_2} \tau_n{}^{d}(\omega) \qquad \text{for all } n,$$

(2.12) $$\sum_n W_n{}^{LRST}(\omega) = \max_{d \in C_2} \sum_n W_n{}^{d}(\omega),$$

(2.13) $$N^{LRST}(t, \omega) = \max_{d \in C_2} N^{d}(t, \omega) \qquad \text{for all } t.$$

Proof. The equation (2.11) is proved by a similar way as being used in (2.8), which is a slight modification of Schrage's method [9]. (2.12) and (2.13) are derived from (2.1), (2.3) and (2.11).

Now, on a busy period we will compare the average waiting times and the queue-lengths at any time for several disciplines. Let $\overline{W}^d(\omega)$ be the average waiting time: $\overline{W}^d=\frac{1}{k}\sum_{n=1}^{k} W_n^d(\omega)$, where k is the number of arrival customers in the busy period. "$d_1 \rightarrow d_2$" means that $\overline{W}^{d_1}(\omega) \geqq \overline{W}^{d_2}(\omega)$ for all ω.

Proposition 2.8.

(2.14) $\quad FCFS \rightarrow SST$

Proof. Let $t_1=0$. H_m^d is the set as defined in the proof of (2.5). From (2.1) it is sufficient to prove that $\tau_n^{FCFS}(\omega) \geqq \tau_n^{SST}(\omega)$ for all n and ω. At first, $H_1^{FCFS}=H_1^{SST}$ since $\tau_1^{FCFS}=\tau_1^{SST}=S_1$. Next, $\tau_2^{FCFS}=S_1+S_2 \geqq S_1+\min\{S_i|S_i \in H_1^{SST}\}=S_1+S_2^{SST}=\tau_2^{SST}$. And also $\{S_i|2\leqq i \leqq j+1\} \subset \bigcup_{m=1}^{j} H_m^{SST}$ $(1 \leqq j \leqq k-1)$, since $\tau_i^{SST}>t_{i+1}$.

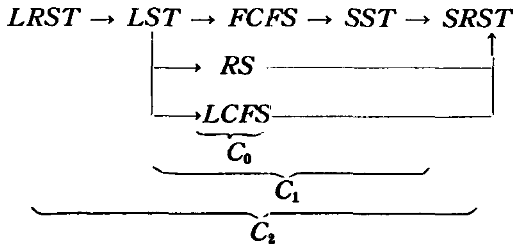
$$
\begin{aligned}
\tau_3^{SST} &= S_1+S_2^{SST}+S_3^{SST} \\
&= S_1+\min\{S_i|S_i \in H_1^{SST}\}+\min\{S_i|S_i \in H_2^{SST}\} \\
&= S_1+\min\{S_2^{SST}+S_i|S_i \in \{\bigcup_{m=1}^{2} H_m^{SST}-\{S_2^{SST}\}\}\} \\
&\leqq S_1+\min\{S_2^{SST}+S_i|S_i \in \{\{S_i|2\leqq i\leqq 3\}-\{S_2^{SST}\}\}\} \\
&\leqq S_1+\min\{S_2+S_i|S_i \in \{\{S_i|2\leqq i\leqq 3\}-\{S_2\}\}\} \\
&= S_1+S_2+S_3=\tau_3^{FCFS}
\end{aligned}
$$

and so on. Thus (2.14) is proved inductively.

Summarizing the above results, we have following relation.

Proposition 2.9.

(2.15)



If we take $N^d(t,\omega)$ as another measure, we have the same relation as (2.15), where "$d_1 \to d_2$" means that $N^{d_1}(t,\omega) \geqq N^{d_2}(t,\omega)$ for all t and ω.

Further, we will give some examples for the relations; $FCFS \not\rightleftarrows RS$, $RS \not\rightleftarrows LCFS$, $RS \not\rightleftarrows SST$ and etc.

Example 1. $SST \nrightarrow RS(=FCFS)$, $SST \nrightarrow LCFS$, $RS(=SST) \nrightarrow FCFS$, $RS(SST) \nrightarrow LCFS$ and $FCFS \nrightarrow LCFS$. There are 8 customers. Their service times and arrival times are given as follows.

n	1	2	3	4	5	6	7	8
t_n	0	2	5	8	9	13	17	19
S_n	4	6	2	3	1	4	2	4

Then,

d	$LRST$	LST	$LCFS$	$FCFS$	SST	$SRST$
ΣW_n	107	60	54	52	49	43

Example 2. $RS(=LCFS) \nrightarrow SST$, $LCFS \nrightarrow SST$, $FCFS \nrightarrow RS(=LST)$, $LCFS \nrightarrow RS(=FCFS)$ and $LCFS \nrightarrow FCFS$. There are 5 customers. Their service times and arrival times are given as follows.

n	1	2	3	4	5
t_n	0	2	5	11	12
S_n	6	4	8	2	1

Then,

d	*LRST*	*LST*	*FCFS*	*SST*	*LCFS*	*SRST*
ΣW_n	65	49	45	44	43	34

Remark 2.1. All relations (2.1)~(2.15) hold on a typical n cycles, where one cycle is composed of a busy period and successive idle time.

Remark 2.2. In the system with constant service, there is a following relation among the sample variances, $\hat{V}(W^d(\omega))=\frac{1}{k}\sum_{n=1}^{k}(W_n^d(\omega)-\overline{W}^d(\omega))^2$ for $d\in C_1$.

$$\hat{V}(W^{FCFS}(\omega))\leqq\hat{V}(W^d(\omega))+\hat{V}(W^{LCFS}(\omega))$$

for all ω and $d\in C_1$.

The inequality of the left-hand side is given by Kingman [3] and that of the right-hand side is given by Tambouratzis [13].

3. A Conservation Law

We will deal with stationary queues in this section. As stated in the first part of the preceeding section, we can see that the stationary distribution of the length of a busy period and that of an idle time for each discipline $d\in C_2$ are the same with those for a certain discipline $d\in C_2$.

Throughout the section we assume that the necessary moments exist and we drop the subscripts when it is not necessary. In the system $G/G/1$ with $\rho<1$, where $\rho=\frac{E(S)}{E(T)}$, the expectation of the length of a busy period (say $E(B)$) is given by

$$E(B)=\frac{\rho E(I)}{I-\rho},$$

where $E(I)$ is the expectation of the length of an idle time. This quantity is invariant for any discipline $d\in C_2$. Rice [8] has given the above equation without defining the class of using discipline precisely. It would be difficult to find $E(I)$ in general. In the system $GI/G/1$ with $\rho<1$, we have

$$E(T)-E(S)\leqq E(I)\leqq E(T) \qquad \text{for any discipline } d\in C_2.$$

In the system $G/G/1$ with $\rho<1$, the probability P_0 that the server is idle at an instant selected at random is given by

$$P_0=1-\rho \qquad \text{for any discipline } d\in C_2.$$

Rice has given this formula in his paper [8].

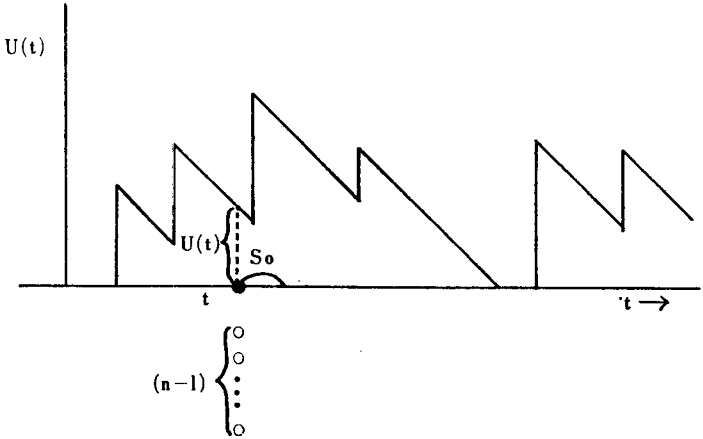
Let us define $U(t)$ as the total remaining processing time present in the system at time t. Beneš [1] and Tahács [12] define a function $W(t)$ similar to $U(t)$, which they call the virtual waiting time, which is the time a customer would have to wait for service if he arrived at time t under *FCFS* discipline. But $U(t)$ is different from $W(t)$ in that it does not, in general, represent a customer's waiting time under the other discipline.

Theorem 3.1. In the system $G/G/1$ with $\rho<1$, where the service times S_n $(n=1,2,\cdots)$ are independent of each other, then

$$E(U)=\rho\frac{E(S^2)}{2E(S)}+E(S)L_q^d \qquad \text{for all } d\in C_0, \tag{3.1}$$

where L_q^d is the expected queue size for a discipline d.

Proof. Let us assume that there are N customers in the system at time t and let $P_n=P(N=n)$.



$$U(t)=\begin{cases} 0 & \text{when } N=0 \\ S_0+S_1+\cdots+S_{n-1} & \text{when } N=n\geqq 1. \end{cases}$$

Taking expectations of both sides,

$$\begin{aligned} E(U)&=0\cdot P_0+E(S_0|N=1)P_1+E(S_0+S_1|N=2)P_2+\cdots \\ &=\{E(S_0|N=1)P_1+E(S_0|N=2)P_2+\cdots\} \\ &\quad +E(S)\sum_{n=1}^{\infty}(n-1)P_n \\ &=E(S_0|N>0)(1-P_0)+E(S)\sum_{n=1}^{\infty}(n-1)P_n \\ &=\rho\frac{E(S^2)}{2E(S)}+E(S)L_q, \end{aligned}$$

where the equation $E(S_0|N>0)=\dfrac{E(S^2)}{2E(S)}$ is derived by using renewal theory.

Remark 3.1. Kleinrock [4] proved the theorem for the system $M/G/1$, but his proof was limited to the case $U(t)>0$. Recently, Schrage [10] proved the theorem in the case where service times might be taken multiples of a unit as their values.

Corollary 3.1. Under the same assumption as which is stated in the theorem,

$$(3.2)\qquad E(W^d)=\frac{1}{\rho}E(U)-\frac{E(S^2)}{2E(S)}\qquad \text{for all } d\in C_0$$

and

$$(3.3)\qquad E(W^d)\geqq E(W^{SST})\qquad \text{for all } d\in C_0.$$

Proof. At first, we note that $E(U)$ is invariant for all $d\in C_0$. Then we see from (3.1) that $L_q{}^d$ is the same value for all $d\in C_0$. Further, by using Little's result [5], that is,

$$E(W^d)=E(T)L_q{}^d$$

we can obtain (3.2) at once. From (2.14) we can see that

$$N^{FCFS}(t,\omega)\geqq N^{SST}(t,\omega)\qquad \text{for all } t \text{ and } \omega.$$

Then $L_q{}^{FCFS}\geqq L_q{}^{SST}$, that is to say, $L_q{}^d\geqq L_q{}^{SST}$ for all $d\in C_0$. From Little's result we obtain (3.3) immediately.

Remark 3.2. Phips [7] proved the inequality (3.3) in the system $M/G/1$ for only two disciplines: *FCFS* and *SST*.

Corollary 3.2. In the system $M/G/1$.

$$(3.4)\qquad E(W^d)=\frac{\rho^2+\lambda^2 Var(S)}{2(1-\rho)\lambda}\qquad \text{for all } d\in C_0,$$

where $\lambda=1/E(T)$.

Proof. From (3.2), the value of $E(W^d)$ is the same for any $d\in C_0$. Then it is sufficient to prove (3.4) for *FCFS* discipline. $U(t)$ and the virtual waiting time $W(t)$ introduced by Takács are identical for *FCFS* discipline. In the steady state of the system $M/G/1$, Takács proved that $E(W(t))=E(W^{FCFS})$. Then, $E(W^{FCFS})=E(U(t))$. Substituting $E(W^{FCFS})$ into the right-hand side of (3.2) we have (3.4) easily.

Remark 3.3. The formula (3.4) was obtained, at first, by Kendall [2].

Finally we will deal with the n-th moment $E\{(L^d)^n\}$ of queue size L^d in the system $G/G/1$, where the service times are independent of each other.

Proposition 3.1. For the second moment $E\{(L^d)^2\}$,

$$E(U^2)=\rho\frac{E(S^3)}{3E(S)}+E(S^2)L_q+Var(S)L_q \qquad (3.5)$$

$$+E^2(S)E\{(L^d)^2\} \qquad \text{for all } d\in C_0,$$

where $L_q=E(L^d)$.

Proof. We will use the same notations as which is used in the theorem, then

$$U^2(t)=\begin{cases}0 & \text{when } N=0\\ (S_0+S_1+\cdots+S_{n-1})^2 & \text{when } N=n\geqq 1.\end{cases}$$

Taking expectations of both sides, we have

$$\begin{aligned}E(U^2)&=\rho E(S_0^2)+2E(S_0)E(S)\sum_{n=1}^{\infty}(n-1)P_n\\&\quad+E^2(S)\sum_{n=1}^{\infty}(n-1)(n-2)P_n+E(S^2)\sum_{n=1}^{\infty}(n-1)P_n\\&=\rho E(S_0^2)+2E(S_0)E(S)L_q+E(S^2)L_q\\&\quad+E^2(S)\{E\{(L^d)^2\}-L_q\}\\&=\rho\frac{E(S^3)}{3E(S)}+E(S^2)L_q+Var(S)L_q+E^2(S)E\{(L^d)^2\},\end{aligned}$$

where the equation $E(S_0^2)=\dfrac{E(S^3)}{3E(S)}$ is obtained by using renewal theory. From this fact, the second moment of queue size is invariant for all $d\in C_0$.

Similary, we have the following third moment of queue size.

$$E(U^3)=\rho E(S_0^3)+L_q\{3E(S^2)E(S)+3E(S_0)E(S^2)+E(S^3)\}$$
$$+3\{E(L^2)-L_q\}\{E(S_0)E^2(S)+E(S)E(S^2)\}$$
$$+E^3(S)\{E\{(L^d)^3\}-3E(L^2)+2L_q\}$$

for all $d\in C_0$. From this equation the third moment of queue size is also invariant for all $d\in C_0$.

Repeating such a way, the expression of the n-th moment of queue size is presented. From the expressions, the n-th moment of queue size is independent of queue discipline $d\in C_0$.

Proposition 3.2. If $E\{(L^{d_0})^n\}<\infty$ for a discipline $d_0\in C_0$, then

$$E\{(L^d)^n\}=E\{(L^{d_0})^n\} \qquad \text{for all } d\in C_0.$$

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