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A NOTE ON A PROBABILITY PROBLEM ARISING IN RELIABILITY AND TRAFFIC STUDIES

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Summary and Introduction

In the reliability context of reliability, traffic, and other studies, a problem arises in evaluating a performance of system which is used intermittently with failure and repair. Gaver [1] defines a measure of the system reliability called 'disappointment' time, the time to system failure during a usage period, or to occurrence of a demand during a system inoperative period, whichever occurs first. He obtained the Laplace-Stieltjes (LS) transform of the time distribution to the 'disappointment' time, measured from an instant at which the system became operative. From the LS transform the mean time was easily obtained. His assumptions are that the types of the failure and the repair time distributions are both arbitrary, and the occurrence time distribution and the holding time distribution of a need are exponential.

In this note we obtain an LS transform of the time distribution to 'disappointment' time under a generalized assumption that the type of the holding time distribution of a need is also arbitrary in Gaver [1]. Thus our results include those of Gaver [1] as a special case. In the analysis we simply apply the signal flow graph method [2].

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Assumptions

The following assumtions are made:

(i) The type of the failure time distribution $\alpha(t)$ $(t \geq 0)$ of the system is arbitrary. The system performs its function perfectly until the occurrence of the failure.

(ii) The type of the repair time distribution $\beta(t)$ $(t \geq 0)$ of the system is arbitrary. After the repair the system recovers its function perfectly. Each switchover time is neglected.

(iii) The occurrence time distribution of a need is exponential, *i.e.*, $F(t)=1-\exp(-\lambda t)$ $(t \geq 0)$.

In the above we note that the occurrence time of a need has the memoryless property because of a convenience of the analysis.

Initially at $t=0$, the system begins an operative period, given that a need does not occur. Our concern for the system is the time to 'disappointment' time starting from the initial state at $t=0$.

Derivation of the LS Transform

For the model just mentioned above we shall derive the LS transform of the time distribution to the 'disappointment' time. Focussing on the regeneration points of the failure and the repair time, we shall obtain the LS transform, applying a signal flow graph method [2].

In our model, we define the following three states, where a state is a regeneration point of the failure time or the repair time:

State S_0; the system begins an operative period, given that a need does not occur.

State S_1; the system fails and the repair of the system begins, given that a need does not occur.

State S_2; the 'disappointment' time, *i.e.*, the time to system failure during a usage period, or to occurrence of a need during a system inoperative period, whichever occurs first.

Since the signal flow graph method is based on a Markov renewal process, we should obtain the LS transform of each one-step time distribution from one state to the other, which corresponds to each branch gain of the signal flow graph.

The signal flow graph of the model is demonstrated in Fig. 1 using the above three states, where state $S_i (i=0, 1, 2)$ corresponds to node S_i, a new node S_α is added as a source, and a branch gain from state S_α to state S_0 is assumed to be 1. We shall obtain the remaining branch gains in the signal flow graph.

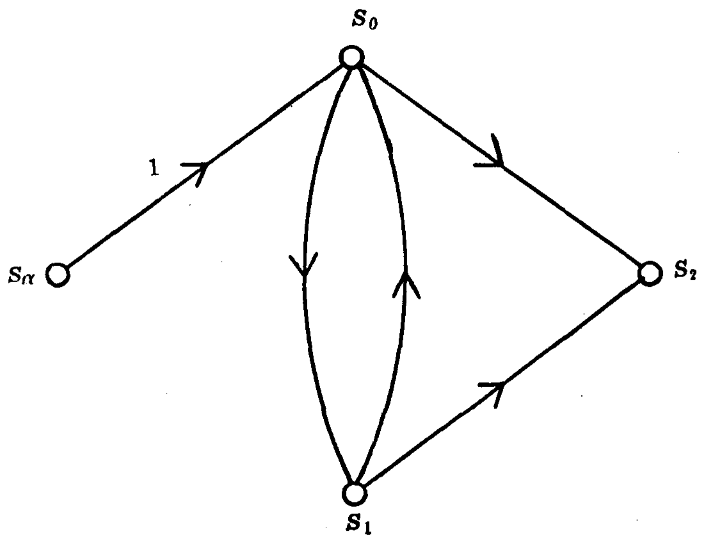

Fig. 1. Signal flow graph of the system.

We start the transitions from state S_0. In state S_0 two transitions can be considered; one is to state S_1, and the other is to state S_2.

First we consider the transition from state S_0 to state S_1. Consider the time interval $(0, t)$ of the failure time distribution $\alpha(t)$. The probability that the system fails in the infinitesimal interval $(t, t+dt)$ is $d\alpha(t)$. At time t, the probabilities that a need does not occur are $\bar{F}(t)$,

$F(t)*G(t)*\bar{F}(t)$, $F(t)*G(t)*F(t)*G(t)*\bar{F}(t)$, and so on, where $\bar{F}(t)=1-F(t)$, the survival probability of $F(t)$, and the asterisk * denotes the convolution operation. That is, $\bar{F}(t)$ means that a need never occurs in the interval $(0, t)$, $F(t)*G(t)*\bar{F}(t)$ means that a need does not occur at time t via occurrence of a need-termination of the holding time, and so on. These events are mutually exclusive. Thus, the one-step time distribution (which may be improper) from state S_0 to state S_1 is

$$(1) \qquad Q_{01}(t)=\int_0^t [\bar{F}(t)+F(t)*G(t)*\bar{F}(t) + F(t)*G(t)*F(t)*G(t)*\bar{F}(t)+\cdots]d\alpha(t) .$$

We introduce the following notation:

$$(2) \qquad (1-A(t))^{(-1)}=\sum_{n=0}^{\infty}[A(t)]^{n*} ,$$

where

$$(3) \qquad [A(t)]^{n*}=\begin{cases}1 & (n=0;\ \text{the Heaviside step function})\\ \overbrace{A(t)*A(t)*\cdots*A(t)}^{n} & (n>0) .\end{cases}$$

Using the above notation (2), we can rewrite

$$(4) \qquad Q_{01}(t)=\int_0^t [\bar{F}(t)*(1-F(t)*G(t))^{(-1)}]d\alpha(t) .$$

Taking the LS transforms, we have

$$(5) \qquad q_{01}(s)=\int_0^{\infty} e^{-st}[\bar{F}(t)*(1-F(t)*G(t))^{(-1)}]d\alpha(t) .$$

Second we consider the transition from state S_0 to state S_2. In a similar way of deriving $q_{01}(S)$, we have

$$(6) \qquad q_{02}(s)=\int_0^{\infty} e^{-st}[F(t)*\bar{G}(t)*(1-F(t)*G(t))^{(-1)}]d\alpha(t) .$$

We consider the transitions from state S_1. In state S_1 two transitions can be considered; one is to state S_0 and the other is to state S_2.

Applying the well-known techniques, we have the following two branch gains:

$$q_{10}(s)=\int_0^\infty e^{-st}\bar{F}(t)d\beta(t)=\beta(s+\lambda)\ , \tag{7}$$

$$q_{12}(s)=\int_0^\infty e^{-st}\beta(t)dF(t)=\frac{\lambda}{s+\lambda}[1-\beta(s+\lambda)]\ , \tag{8}$$

where $\beta(s)$ is the LS transform of $\beta(t)$ and $F(t)=1-\exp(-\lambda t)$.

We define $\varphi_0(s)$, the LS transform of the time distribution to the 'disappointment' time starting from state S_0 at $t=0$. Using the signal flow graph in Fig. 1 and each branch gain prepared above, we shall obtain the system gain, corresponding to $\varphi_0(s)$, by Mason's gain formula [2], where we assume that a source is node S_1 and a sink is node S_2. Then we have

$$\varphi_0(s)=\frac{q_{02}(s)+q_{01}(s)q_{12}(s)}{1-q_{01}(s)q_{10}(s)}\ . \tag{9}$$

We can easily verify that $\varphi_0(s)\rightarrow 1$ as $s\rightarrow 0$. The mean time is also easily given by

$$\hat{T}=-\frac{d\varphi_0(s)}{ds}\bigg|_{s=0}=\frac{E(X_1)+q_{01}(0)[1-\beta(\lambda)]/\lambda}{1-q_{01}(0)q_{10}(0)}\ , \tag{10}$$

where

$$E(X_1)=\int_0^\infty t d\alpha(t)\ . \tag{11}$$

Special Case

In the above analysis we obtained the LS transform of the time distribution to the 'disappointment' time under the assumption that the type of the holding time distribution of a need is arbitrary. As a special case, we consider the model with the exponential holding time of a need, that is,

$$G(t)=1-\exp(-\mu t)\ . \tag{12}$$

Then we have in (5)

$$F(t)*(1-F(t)*G(t))^{(-1)}=\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}, \quad (13)$$

and in (6)

$$F(t)*\bar{G}(t)*(1-F(t)*G(t))^{(-1)}=\frac{\lambda}{\lambda+\mu}[1-e^{-(\lambda+\mu)t}]. \quad (14)$$

Thus, the results (9) and (10) coincide with (17) and (18) in Gaver [1], respectively.

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REFERENCES

[1] Gaver, D. P., "A Probability Problem Arising in Reliability and Traffic Studies," *Operations Research*, 12 (1964), 534-542.

[2] Mason, S.J., "Feedback Theory: Some Properties of Signal-Flow Graphs," *Proc. IRE*, 41 (1953), 1144-1156.