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PHASE TYPE SERVICE QUEUES WITH TWO SERVERS IN BI-SERIES

PARKASH LAL MAGGU

Kurukshetra University, Kurukshetra, India

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Abstract

This paper studies a queueing system wherein units demanding one phase or two phase services arrive with a Poisson stream at two service facilities S_1 and S_2. The server S_1 attends to units which on being serviced may leave the system or enter the other server S_2 of the system. On the other hand, the server S_2 attends to units which on being serviced may leave the system or enter the other server S_1 of the system. The service discipline at each service channel is assumed to be 'first come, first served' and the service times have been assumed to be distributed exponentially with different service parameters. An explicit expression for the time-dependent probability generating function of the queue lengths of the system has been found out. Further the mean queue lengths in the steady-state of the system have been derived. Also, a few particular cases have been discussed at the end.

Introduction

In most of the studies concerning queues in series it has been assumed that each unit before being finally discharged from the system has to go through all the service channels beginning from the first c.f. Jackson [1], O'Brien [2]. However, in the system considered by us this condition has been relaxed. Such an arrangement of two channels with phase type service has been preferred by us to be designated as 'Channels' in Bi-series.

Thus for the present queueing model it has been assumed that units demanding one phase or two phase services arrive with Poisson mean rates λ and λ' and form two queues Q_1 and Q_2 infront of two service facilities S_1 and S_2, respectively. The units are served on a 'first come, first served' basis and the service times are assumed to be exponential with parameters μ and μ' at S_1 and S_2, respectively. Further units joining S_1 may leave the system or may enter the server S_2 of the system after being serviced by S_1. Similarly, the units joining S_2 may leave the system or may enter the server S_1 of the system after being serviced by S_2. Let p denote the probability that units serviced by S_1 leave the system and q the probability that units serviced by S_1 enter the server S_2 of the system, where $p, q>0$ or $p, q=0$ with $p+q=1$. On the other hand, let p' denote the probability that units on being discharged by S_2 leave the system and q' denote the probability that units discharged by S_2 enter the server S_1 of the system, where $p', q'\geq 0$ with $p'+q'=1$. Thus μp and μq are the respective service rates at which units serviced by S_1 leave the system and enter the server S_2 of the system. Similarly, $\mu' p'$ and $\mu' q'$ are the respective service rates at which units serviced by S_2 leave the system and enter the server S_1 of the system.

One of the physical situations in every day life which corresponds very closely to the above system is the case of a Barber shop. Units (customers) demanding hair-cut or shave services arrive at two service facilities S_1 and S_2. The server S_1 is attending to hair-cut and the server S_2 to that shaving. Thus S_1 attends to customers which on being serviced may leave the system or may enter the other server S_2 of the system. On the other hand, S_2 attends to customers which on being serviced may leave the system or enter the server S_1 of the system.

Formulation of Equations Describing the System and Their Solution:

Define,

$P(m, n, t)$ = The probability that at time t there are m units (which may leave the system or enter the server S_2 of the system after being serviced by S_1) waiting in Q_1 and n units (which may leave the system or enter the server S_1 of the system after being serviced by S_2) waiting in Q_2. $(m, n \geq 0)$

Connecting the state probabilities at time $t+\delta t$ with those at t, and then letting $\delta t \to 0$, the following set of difference-differential equations for the system can be seen easily to hold:

(1)
$$P'(m, n, t) = -(\lambda+\lambda'+\mu+\mu')P(m, n, t)+\lambda P(m-1, n, t) \\ +\lambda' P(m, n-1, t)+\mu p P(m+1, n, t) \\ +\mu q P(m+1, n-1, t)+\mu' p' P(m, n+1, t) \\ +\mu' q' P(m-1, n+1, t); \quad \text{for } m, n>0 .$$

(2)
$$P'(0, n, t) = -(\lambda+\lambda'+\mu')P(0, n, t)+\lambda' P(0, n-1, t) \\ +\mu p P(1, n, t)+\mu q P(1, n-1, t) \\ +\mu' p' P(0, n+1, t); \quad \text{for } m=0, n>0 .$$

(3)
$$R'(m, 0, t) = -(\lambda+\lambda'+\mu)P(m, 0, t)+\lambda P(m-1, 0, t) \\ +\mu p P(m+1, 0, t)+\mu' p' P(m, 1, t) \\ \mu' q' P(m-1, 1, t); \quad \text{for } m>0, \quad n=0 .$$

(4)
$$P'(0, 0, t) = -(\lambda+\lambda')P(0, 0, t)+\mu p P(1, 0, t) \\ +\mu' p' P(0, 1, t); \quad m, n=0 .$$

Assume that

(5)
$$P(m, n, 0) = 1, \quad m=n=0 \\ = 0, \quad \text{otherwise} .$$

Let the Laplace transform of $P(m, n, t)$ be $P^*(m, n, s)$, where

$$P^*(m, n, s) = \int_0^\infty P(m, n, t) \exp(-st) dt \quad (Re(s) \geq 0) .$$

Taking the Laplace Transform of (1) through (4) and employing the initial condition (5), we have

(6)
$$(s+\lambda+\lambda'+\mu+\mu')P^*(m, n, s)=\lambda P^*(m-1, n, s)$$
$$+\lambda' P^*(m, n-1, s)+\mu p P^*(m+1, n, s)$$
$$+\mu q P^*(m+1, n-1, s)+\mu' p' P^*(m, n+1, s)$$
$$+\mu' q' P^*(m-1, n+1, s) . \quad (m, n>0)$$

(7)
$$(s+\lambda+\lambda'+\mu')P^*(0, n, s)=\lambda' P^*(0, n-1, s)+\mu p P^*(1, n, s)$$
$$+\mu q P^*(1, n-1, s)+\mu' p' P^*(0, n+1, s) .$$
$$(m=0 , \quad n>0)$$

(8)
$$(s+\lambda+\lambda'+\mu)P^*(m, 0, s)=\lambda P^*(m-1, 0, s)$$
$$+\mu p P^*(m+1, 0, s)+\mu' p' P^*(m, 1, s)$$
$$+\mu' q' P^*(m-1, 1, s) . \quad (m>0 , \quad n=0)$$

(9)
$$(s+\lambda+\lambda')P^*(0, 0, s)=1+\mu p P^*(1, 0, s)+\mu' p' P^*(0, 1, s) .$$
$$(m, n=0)$$

Introducing the generating function:

(10)
$$F(x, y, s)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P^*(m, n, s)x^m y^n$$
$$(|x|<1 , \quad |y|<1) .$$

The following Partial generating functions are also used:

(11)
$$F_n(x, s)=\sum_{m=0}^{\infty} P^*(m, n, s)x^m \qquad (|x|<1) .$$

(12)
$$G_m(y, s)=\sum_{n=0}^{\infty} P^*(m, n, s)y^n \qquad (|y|<1) .$$

Now multiplying (6) by x^m, summing over m from 0 to ∞ and using (7) and (11), we obtain:

$$(13)\qquad (s+\lambda+\lambda'+\mu+\mu')F_n(x,s)-\mu P^*(0,n,s)=\lambda xF_n(x,s)$$
$$+\lambda' F_{n-1}(x,s)+\mu p/xF_n(x,s)-\mu p/xP^*(0,n,s)$$
$$+\mu q/xF_{n-1}(x,s)-\mu q/xP^*(0,n-1,s)$$
$$+\mu' p' F_{n+1}(x,s)+\mu' q' xF_{n+1}(x,s)\,.$$

Multiplying (8) by x^m, summing over m from 0 to ∞ and using (9) and (11), we have:

$$(14)\qquad (s+\lambda+\lambda'+\mu)F_0(x,s)-\mu P^*(0,0,s)=1+\lambda xF_0(x,s)$$
$$+\mu p/xF_0(x,s)-\mu p/xP^*(0,0,s)+\mu' p' F_1(x,s)$$
$$+\mu' q' xF_1(x,s)\,.$$

Multiplying (13) by y^n, summing over n from 0 to ∞, using (10), (11), (12) and (14) and on simplification, we have:

(15)
$$F(x,y,s)=\frac{1+(\mu'-\mu' p'/y-\mu' q' x/y)F_0(x,s)+(\mu-\mu p/x-\mu q y/x)G_0(y,s)}{s+\lambda+\lambda'+\mu+\mu'-\lambda x-\lambda' y-\mu p/x-\mu q y/x-\mu' p'/y-\mu' q' x/y}\,.$$

Applying Rouche's theorem to the denominator of right hand of (15), it is easily seen that a single zero lies inside the unit circle $|x|=1$. Let this zero be denoted by $x=\alpha(y,s)$, say, where α is the zero lying inside $|x|=1$ and given by the following equation:

$$(16)\qquad (\lambda+\mu' q'/y)x^2-(s+\lambda+\lambda'+\mu+\mu'-\lambda' y-\mu' p'/y)x$$
$$+(\mu p+\mu q y)=0\,.$$

Now, since $F(x,y,s)$ is regular inside or on $|x|=1$, therefore, numerator of (15) vanishes for $x=\alpha$, giving us:

$$(17)\qquad G_0(y,s)=\frac{1+(\mu'-\mu' p'/y-\mu' q'/y)F_0(\alpha,s)}{\mu p/\alpha+\mu q y/\alpha-\mu}\,.$$

Again using Rouche's theorem to (17), it is clear that the denominator has a single zero within the unit circle $|y|=1$ which we denote by $y=\beta(s)$, say, at which the numerator vanishes. This gives us:

(18) $$F_0(\alpha_1, s) = \frac{\beta}{\mu' p' + \mu' q' \alpha_1 - \mu' \beta},$$

where $\alpha_1 \equiv \alpha(\beta, s)$.

Applying Rouche's theorem to (15) it is easily seen that the right hand denominator has a single zero within the unit circle $|y|=1$ which we denote by $y=\gamma(x, s)$, say, at which the numerator vanishes, since the *g.f.* $F(x, y, s)$ is regular inside or on $|y|=1$. This yields

(19) $$F_0(x, s) = \frac{1+\mu(1-p/x-\mu q\gamma/\mu x)G_0(\gamma, s)}{\mu'(p'/\gamma + q'x/\gamma - 1)}$$

where γ in (19) is the zero within or on $|y|=1$ and given by the equation

(20) $$(\lambda' + \mu q/x)y^2 - (s+\lambda+\lambda'+\mu+\mu'-\lambda x-\mu p/x)y + (\mu' p' + \mu' q' x) = 0.$$

Since γ is a function of x in particular, therefore changing x to α_1 will change γ to γ_1 (say) with $\gamma_1 = \gamma(\alpha_1, s)$. Hence in (19) letting $x=\alpha_1$ and correspondingly replacing γ by γ_1, and using (18), we have

(21) $$G_0(\gamma_1, s) = \frac{[\beta(p'\gamma_1 + q'\alpha_1/\gamma_1 - 1)/(p' - q'\alpha_1 - \beta)] - 1}{\mu(1-p/\alpha_1 - q\gamma_1/\alpha_1)}.$$

Also, since x and y are arbitrary non-negative variables therefore they can be replaced by α_1 and γ_1, respectively. Hence replacing x by α_1 and y by γ_1 in (15) and using (18) and (21), it can be easily seen that the probability generating function (*p.g.f.*) of the queue lengths is completely determined in terms of Laplace transforms.

Steady-State Solutions

The time independent steady-state of the queueing system considered here exists for $\rho<1$ and $\rho'<1$, where ρ and ρ' are given by,

(22) $$\rho = \frac{\lambda + \lambda' q'}{\mu(1-qq')}, \qquad \rho' = \frac{\lambda' + \lambda q}{\mu'(1-qq')}.$$

To find out the steady-state solution of the system, we equate the time-derivatives to zero in equations (1) through (4) and then solve the resulting steady-state equations, we get for $P(m, n)$, the steady-state probabilities of the system, as follows:

$$P(m, n)=\rho^m\rho'^n P(0, 0)\,, \tag{23}$$

where $P(0, 0)$ is obtained from the following condition:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P(m, n)=1 \tag{24}$$

Now taking the sum over m and n on both sides of (23) and using (24), we obtain:

$$P(0, 0)=(1-\rho)(1-\rho')\,. \tag{25}$$

Thus by virtue of (25), (23) gives us:

$$P(m, n)=\rho^m\rho'^n(1-\rho)(1-\rho') \tag{26}$$

where ρ, ρ' are given from (22).

Now computing for ρ and ρ' from (22) in (26), we have:

$$P(m, n)=[(\lambda+\lambda'q')/\mu(1-qq')]^m[(\lambda'+\lambda q)/\mu'(1-qq')]^n \times\left[1-\frac{\lambda+\lambda'q'}{\mu(1-qq')}\right]\left[1-\frac{\lambda'+\lambda q}{\mu'(1-qq')}\right] \quad (m, n\geq 0)\,. \tag{27}$$

The probability p_m (say) that there are m units of both types (*i.e.* units which may leave the system or may enter the server S_2 of the system after being serviced by S_1)in Q_1 before the server S_1 is obtained by summing (27) over n from 0 to ∞. This gives us:

$$P_m=\left[\frac{\lambda+\lambda'q'}{\mu(1-qq')}\right]^m\left[1-\frac{\lambda+\lambda'q'}{\mu(1-qq')}\right] \tag{28}$$

Similarly, the probability q_n (say) of there being n units of both types (which may leave the system or may enter the server S_1 of the system after being serviced by S_2) in Q_2 before the server S_2 is:

(29) $$q_n=\left[\frac{\lambda'+\lambda q}{\mu'(1-qq')}\right]^n\left[1-\frac{\lambda'+\lambda q}{\mu'(1-qq')}\right]$$

Now the mean queue length L_1 (say) before the server S_1 is obtained by setting the value of p_m from (30) in the formula:

(30) $$L_1=\sum_{m=0}^{\infty} mp_m=\sum_{0}^{\infty} m\left[\frac{\lambda+\lambda'q'}{\mu(1-qq')}\right]^m\left[1-\frac{\lambda+\lambda'q'}{\mu(1-qq')}\right]$$
$$=\frac{\lambda+\lambda'q'}{\mu(1-qq')-(\lambda+\lambda'q')}\,.$$

Similarly, the mean queue length L_2 (say) before the server S_2 is:

(31) $$L_2=\frac{\lambda'+\lambda q}{\mu'(1-qq')-(\lambda'+\lambda q)}\,.$$

Thus, the total mean number of units in the system is,

$$L_1+L_2=L \text{ (say)}\,.$$

Whence by virtue of (30) and (31), we have

(32) $$L=\frac{\lambda+\lambda'q'}{\mu(1-qq')-(\lambda+\lambda'q')}+\frac{\lambda'+\lambda q}{\mu'(1-qq')-(\lambda'+\lambda q)}\,.$$

Particular Cases

(I).

If one lets

$$\lambda'=0\,,\quad q=1\,,\quad q'=0\,;\quad \text{with}\quad \lambda<\mu\,,\quad \lambda<\mu'\,,\quad \text{then}$$

the expression (32) for the steady-state mean length of the system becomes:

(33) $$L=\frac{\lambda}{\mu-\lambda}+\frac{\lambda}{\mu'-\lambda}\,.$$

Now mean length (33) is that given by R.R.P. Jackson [1] and developed in the case of queues in tandem.

(II).

If one lets

$\lambda'=0$, $q=1$ in (29), with assuming,

$\lambda<\mu(1-qq')$; $\lambda<\mu'(1-qq')$, (where $q=1$)

then we have:

(34)
$$P(m, n)=\left(\frac{\lambda}{\mu-\mu q'}\right)^m\left(\frac{\lambda}{\mu'-\mu' q'}\right)^n \times\left(1-\frac{\lambda}{\mu-\mu q'}\right)\left(1-\frac{\lambda}{\mu'-\mu' q'}\right).$$

Thus (34) gives the solution of cyclic queues with terminal feedback and represents the particular case for $j=1, 2$. (except for the natations used) of Finch, P.D. [3].

Steady-State Solution with Limited Space

If there are a finite number of input sources N or, equivalently, if there are N identical input sources and a single unit has a probability $\lambda\ \delta t$ of joining Q_1 infront of server S_1 and a probability $\lambda'\ \delta t$ of joining Q_2 infront of the server S_2 during a small interval time δt and that of more than a single arrival during δt is negligible, and if everything else remains the same, the transient differential-difference equations for the system with limited waiting space N are as follows:

(35) The same as (1) $(m, n>0;\ m+n<N)$.

(36)
$$P'(m, n, t)=-(\mu+\mu')P(m, n, t)+\lambda P(m-1, n, t) +\lambda' P(m, n-1, t)+\mu q P(m+1, n-1, t) +\mu' q' P(m-1, n-1, t).$$
$(m, n>0;\ m+n=N)$

(37) The same as (2) $(m=0, n>0;\ n<N)$.

(38)
$$P'(0, N, t)=-\mu' P(0, N, t)+\lambda' P(0, N-1, t) +\mu q P(1, N-1, t).\quad (m=0,\ n=N>0).$$

(39) The same as (3) $(m>0\,,\quad n=0\,;\quad m<N)$.

(40)
$$P'(N,0,t)=-\mu P(N,0,t)+\lambda P(N-1,0,t) \\ +\mu' q' P(N-1,1,t)\quad (n=0\,;\quad m=N>0)\,.$$

(41) The same as (4) $(m=0\,,\quad n=0)$.

Thus, one may obtain the steady-state equations by equating the time-derivatives to zero in equations from (35) through (41). Solving the resulting equations recursively for $P(m,n)$, the steady-state probabilities corresponding to $P(m,n,t)$ as defined earlier, we have:

(42)
$$P(m,n)=\left[\frac{\lambda+\lambda' q'}{\mu(1-qq')}\right]^{m}\left[\frac{\lambda'+\lambda q}{\mu'(1-qq')}\right]^{n} P(0,0)$$

Now summing (42) over m and n from 0 to N $(m+n\leq N)$ and equating to unity, we have:

(43)
$$P(0,0)=\frac{(\rho-\rho')(1-\rho)(1-\rho')}{\rho-\rho'-\rho^{N+2}+\rho'^{N+2}+\rho\rho'(\rho^{N+1}-\rho'^{N+1})}\,,$$

where ρ,ρ' in (43) are given from (22).

The steady-state mean-length L (say) with $m+n\leq N$ is computed by multiplying both sides of (42) by $m+n$ and summing over m and then over n. This gives us:

(44)
$$L=\frac{P(0.0)[\rho^2(1-\rho')^2(1-\overline{N+1}\rho^N+N\rho^{N+1})-\rho'^2(1-o)^2(1-\overline{N+1}\rho'^N+N\rho'^{N+1})]}{(\rho-\rho')(1-\rho)^2(1-\rho')^2}\,,$$

where ρ,ρ' and $P(0,0)$ in (44) are given from (22) and (43), respectively, If we now make $N\to\infty$, then (44) reduces to (32) as it should, with the provision $\rho^N,\rho'^N\to 0$ as $N\to\infty$.

Particular Case

(III).

If one lets $\lambda'=0$, $q=1$ with assuming $\lambda<\mu(1-q')$; $\lambda<\mu'(1-q')$; then the steady-state solution (44) becomes

$$P(m,n)=\left[\frac{\lambda}{\mu(1-q')}\right]^m\left[\frac{\lambda}{\mu'(1-q')}\right]^n P(0,0) \tag{45}$$

where $P(0,0)$ in (47) is given by (45) now where ρ and ρ' are:

$$\rho=\frac{\lambda}{\mu(1-q')}; \qquad \rho'=\frac{\lambda}{\mu'(1-q')}.$$

Thus the result (45) is the particular case for $j=1,2$ (except for the notations used here) of the result that given by P.D. Finch [3] and developed in the case of cyclic queues with terminal feedback with limited waiting space $(m+n\leq N)$ undertaken for the system. Now, finally, if one makes in (45) $N\to\infty$ with the provision $\rho^N, \rho'^N\to 0$ in the limit as $N\to\infty$, then (45) corresponds to (34) as it should.

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