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GENERALLY APPLICABLE SOLUTIONS FOR TWO-PERSON MEDIAN GAME THEORY*

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Abstract

Two-person median game theory has application advantages over expected-value game theory. For example, median game theory is usable when the values in one or both payoff matrices do not satisfy the arithmetical operations (but can be ranked within each matrix). Also, the class of games where the players have optimum solutions is huge compared to (and includes) this class for expected-value game theory. Moreover, there is a much larger class where one player, but not necessarily the other, has an optimum solution (no expected-value analogue occurs). The overwhelmingly large class of median games, however, is that

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where at least one player does not have an optimum solution. That is, for one or both players, no strategy exists (pure or mixed) such that the player can simultaneously be as protective as possible for himself and as vindictive as possible toward the other player. To reasonably resolve such situations a "relative desirability" function, suitably chosen, is used to order pairs of payoffs, a payoff to each player, that occur for some of the combinations of pure strategies (according to increasing desirability to the player considered). This provides the basis for a compromise "optimum" solution and identification of a corresponding "optimum" strategy.

Introduction and Discussion

The case of two players with finite numbers of strategies is considered. Each player selects his strategy separately and independently of the strategy choice by the other player. Median game theory and its application advantages over expected-value game theory are discussed in [1] (also see [2]).

The class of median competitive games, and of one player median competitive (OPMC) games, are identified in [1]. These identifications are stated in terms of the pairs of payoffs that correspond to the strategy combinations for the players (called I and II). A game is OPMC for player I (II) if and only if he can assure, with probability at least 1/2, that a pair in set I (set II) occurs. Set I (set II consists of the pairs such that both the payoff to player I (II) is at least P_{I} (P_{II}) and the payoff to player II (I) is at most P'_{II} (P'_{I}). Here P_{I} (P_{II}) is determined as the largest value that player I (II), acting as protectively as possible, can assure himself with probability at least 1/2. Also, a smallest value P'_{I} (P'_{II}) occurs in the payoff matrix for player I (II) such that vindictive player II (I) can assure, with probability at least 1/2, that player I (II) receives at most this amount. A way of evaluating P_{I}, P_{II}, P'_{I}, P'_{II}, and of deciding whether a game is OPMC for a player, is given in [1].

A game is median competitive if and only if it is OPMC for both players.

When a game is OPMC for a player, the median optimum solution given in [1] would seem satisfactory for that player. However, the overwhelmingly large class of games is that where an OPMC situation does not occur for at least one player.

A generalization of the OPMC concept is needed to obtain "optimum" solutions for players in games that are not OPMC for them. A method is given for supplementing the pairs of set I (set II) for player I (II) until (the first time) some pair of this set can be assured with probability at least 1/2. Specifically, a suitable "relative desirability" function is chosen for selecting the pairs for the set (according to increasing desirability to the player considered).

This function is such that all pairs of set I (set II) have the maximum desirability for player I (II). Let (p_{I}, p_{II}) denote a general pair. For player I, relative desirability is a nonincreasing function of $P_{I}-p_{I}$ for fixed $p_{II}-P'_{II}$, and a nonincreasing function of $p_{II}-P'_{II}$ for fixed $P_{I}-p_{I}$. Likewise, for player II, relative desirability is a nonincreasing function of $P_{II}-p_{II}$ for fixed $p_{I}-P_{I}'$, and a nonincreasing function of $p_{I}-P_{I}'$ for fixed $P_{II}-p_{II}$.

Subject to these weak restrictions, which are intuitively justified, the relative desirability function for a player can be of any form. Of course, the relative desirability function for one player need not be even roughly the same as that for the other player. Also, neither player needs to know anything further about the relative desirability function for the other player. In fact a player does not need an explicit function as long as his ordering of pairs satisfies the restrictions.

A compromise median optimum solution is obtained through ordering of the pairs, separately for each player, according to increasing desirability. Also, a corresponding median optimum strategy is developed. These results and statements about their derivation are given in the next, and final, section.

Results

Since the game is not OPMC for the player considered, say player I, a pair of set I cannot be assured with probability at least 1/2. However, pairs not in set I can be ordered according to their desirability to player I.

Consider the payoff matrix for player I. The rows of this matrix correspond to the strategies for player I while the columns are the strategies for the other player. First mark the payoffs in this matrix that occur for the pairs of set I. Then, according to decreasing desirability, mark the payoffs for the other pairs, where all pairs of equal desirability are marked at the same time. Continue until the first time that marks in all columns can be obtained from two or fewer of the rows. Now, remove the marks for the least desirable pair(s) of those that received marks. Then, by the following method, determine whether some one of the remaining marked pairs can be assured with probability at least 1/2. The method is to replace every marked value in the matrix by unity and all others by zero. The resulting matrix of ones and zeroes is considered to be for a zero-sum game with an expected-value basis and is solved for the value of the game to player I. Some one of the remaining marked pairs can be obtained with probability at least 1/2 if and only if this game value is at least 1/2.

Suppose that player I cannot assure a remaining marked pair with probability at least 1/2. Then the maximum level of desirability that can be assured with probability at least 1/2 is the level corresponding to a pair that just had its markings removed. Otherwise, remove the marks for the least desirable pair(s) of those still having marks. Then, as just discussed, determine whether some one of the remaining marked pairs can be assured with probability at least 1/2. /f not, the maximum level of desirability that can be assured with probability at least 1/2 is the level corresponding to a pair that just had its markings removed (at this step). If a probability of at least 1/2 can be assured, continue in the same way until some one of the remaining marked pairs cannot be assured with probability at least 1/2. Then, the maximum desirability level that can be assured with probability at least 1/2 is the level for a pair that just had its markings removed (at this last step).

Use of "level of desirability," rather than statement of a lower bound for p_{I} and an upper bound for p_{II}, seems appropriate, even though these bounds have determined values. The reason is that knowledge of the lower value of p_{I} and the lower value of p_{II} for a set of pairs does not necessarily establish the minimum level of desirability for the pairs of this set.

Now consider determination of a median optimum strategy for player I. Use the matrix making that (ultimately) resulted in the smallest set of marked pairs such that a pair of this set can be assured with probability at least 1/2. Replace the marked values by unity and the others by zero, in the matrix for player I. Treat the resulting matrix as that for a zero-sum game with an expected-value basis. An optimum strategy for player I in this zero-sum game is a median optimum strategy for that player.

The verification for these results is of the same nature as for the results of [1]. Suitably interpreted, Theorems 1 and 2 of [1] provide this verification. For brevity, no specific justification is given.

REFERENCES

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