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SYSTEM RELIABILITY ANALYSIS BY MARKOV RENEWAL PROCESSES

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Abstract

It is an important problem to maintain a system with high reliability. Some policies to maintain a system can be considered: (i) Repair Maintenance. (ii) Redundancy Technique. (iii) Preventive Mantenance. This paper discussed some systems in which the above two or three policies are considered simultaneously. Our concern for the system is the first time to system down. That is, we shall consider the situations where the total system failure is a catastrophe. The recent large-scale and complicated systems enjoy such situations. We shall discuss the first time to system down throughout this paper.

Chapter 2 discusses the signal flow graph analysis for systems. The relationship between Markov renewal processes and signal flow

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graphs is investigated and some examples of the signal flow graph analysis for redundant systems are presented.

Chapter 3 discusses a two-unit standby redundant system with standby failure. By using the results obtained in Chapter 2, we shall derive the Laplace-Stieltjes transform of the first time distribution to system down and the mean time.

Chapter 4 discusses a two-unit standby redundant system with repair and preventive maintenance. Considering the repair and preventive maintenance policies for a two-unit standby redundant system, we shall obtain the Laplace-Stieltjes transform of the first time distribution to system down and the mean time. The analysis is made by using the signal flow graph method obtained in Chapter 2.

§1. Introduction

The remarkable progress of engineering techniques yields various kinds of systems. The systems are from a simple system of a machine tool to a large-scale system such as the Manned Spacecraft Center in Houston. As a simple example of systems, we consider a system of a machine tool. We shall below describe the problems of system reliability analysis by demonstrating the system of a machine tool.

For a system of a machine tool, the performance of the system is assumed to be defined as that the machine is operable. If the machine is down, we cannot perform its function. Then we should consider the maintenance problem. Before the discussion of the maintenance problem, we should know the failure law of the machine. That is, we should investigate the failure time distribution of the machine. The random variables occurring in such problems are all nonnegative. One of the simplest examples is a random failure law, *i.e.*, the exponential failure

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time distribution. Some of the failure time distributions can be further considered: The gamma, the Weibull, the extreme value, the truncated normal, the log normal, and the regular (constant time) distributions. In this paper, we shall discuss the failure time distributions as the exponential ones or the arbitrary ones. The analysis of systems with the exponential distributions is easy because of the "memoryless property [10, p. 411]."

To maintain the system, we can consider the following three policies:

(i) The machine is repairable.

(ii) The redundant machines are provided.

(iii) The inspection or the preventive repair is made before failure.

The first policy is that we have repair facilities of the machine. If the machine fails, then the repair of the failed machine is made. The repair time is also random. The repair time distributions can be similarly considered as described in the failure time distributions. After repair the machine recovers its function, *i.e.*, the machine can be operable.

The second policy is the redundancy technique. That is, if two or more machines are provided, we may use the provided machine instead of the failed machine. We can further consider the redundant repairable system, *i.e.*, the system in which the repair of the failed machine is made when any of the other machines are operable. This is a simple redundant repairable model. If the two machines are provided and they are used alternatively, this system is called a two-unit standby redundant model which will be discussed in this paper. We will further discuss some redundant models.

The third policy is the preventive maintenance one. If the failure time distribution of the machine has Increasing Failure Rate (IFR) [4, p. 12], *i.e.*, the probability of the failure increases as the elapsed time is longer, we should make the inspection or the preventive repair before failure since the inspection or the preventive repair is easier and shorter in time than that of the usual repair.

Systems considered in this paper are redundant systems of multiple

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units (or subsystems) each of which is repairable. The "unit" refers to a machine, a computer, a generator, and others. Systems with preventive maintenance are also considered.

In this paper, we shall consider the situations where the total system failure is a catastrophe. The recent large-scale and complicated systems enjoy such situations. Then our concern of the systems is the time to first system down starting in an initial state. We shall discuss the time to system down throughout this paper.

Chapter 2 discusses the signal flow graph analysis for systems. The relationship between Markov renewal processes and signal flow graphs is investigated and some examples of the signal flow graph analysis for redundant systems are presented.

Chapter 3 discusses a two-unit standby redundant system with standby failure. Taking account of the failure of a standby unit, we shall derive the Laplace-Stieltjes (LS) transform of the time distribution to first system down and its mean time. The analysis is made by using the signal flow graph method obtained in the preceding chapter.

Chapter 4 discusses a two-unit standby redundant system with repair and preventive maintenance. Considering the repair and preventive maintenance policies for a two-unit standby redundant system, we shall obtain the LS transform of the time distribution to first system down and its mean time. The analysis is also made by using the signal flow graph method obtained in chapter 2.

In the rest of the Introduction we review the literature on system reliability analysis. Many contributions to the reliability theory have been made and a large number of papers have be published in technical articles. Barlow and Proschan [4] summarized an excellent book emphasizing the mathematical theory in 1965. In 1965, the Russian mathematicians Gnedenko *et al.* summarized a book of the reliability theory and it was translated in English [17] in 1969.

The measures of reliability have been defined by many authors and summarized by Barlow and Proschan [4, pp. 5-8]. The measures of

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reliability are: Reliability, Pointwise Availability, Limiting Interval Availability, Interval Reliability, and so on. Hosford [20] has defined three measures of dependability of the system.

The reliability analysis of two-unit redundant systems has been discussed by Epstein and Hosford [9]. Gaver [14, 15] and Liebowitz [23] have discussed a two-unit paralleled (or standby) redundant system. Harris [19] has also discussed a two-unit paralleled redundant system in which the two units are correlated each other. Gnedenko *et al.* [17] and Srinivasan [49] have discussed a two-unit standby redundant system under the most generalized assumptions. Srinivasan [51] has discussed the same system with noninstantaneous switchover.

Multiple unit redundant systems have been discussed by Barlow [1], Halperin [18], Srinivasan [50], and Mine and Asakura [26]. Downton [8] has discussed m-out-of-n systems. The reliability analysis for the multiple unit redundant systems by the integral equatious of the renewal type has been discussed by Gnedenko [16].

The graphic representation of systems plays an important role in the system theory. In particular, signal flow graphs are applicable to the reliability analysis. The signal flow graphs have been first discussed by Mason [24, 25]. The applications of signal flow graphs are found in Huggins [21, 22]. The relationship between Markov processes and signal flow graphs in the reliability theory has been investigated by Dolazza [7] and Tin Htun [52].

The preventive maintenance theory has been discussed as the replacement problems by Barlow and Hunter [2], Barlow and Proschan [3, 4] and others. Flehinger [12, 13] has discussed some interesting maintenance policies as the marginal checking and marginal testing.

Finally we review mathematical tools used in this paper. Renewal processes are of importance throughout this paper. The theory of renewal processes are summarized in Smith [48], Cox [6], and Feller [11]. Markov renewal processes, which are extensions of renewal processes and Markov processes, play an important role throughout this paper. For Markov

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renewal processes, Smith [47] and Pyke [45, 46] have discussed in detail.

§2. Signal Flow Graph Analysis for Systems

2.1. Introduction

In this chapter we shall discuss the relationship between Markov renewal processes and signal flow graphs. The results obtained in this chapter will be used throughout this paper.

Markov chains are well-known as a mathematical tool for system analysis. Renewal processes are also used to analyze systems (in particular, maintainable systems). A Markov renewal process (or a semi-Markov process), which is a marriage of Markov chains and renewal processes, was first discussed by Lévy and Smith, independently, in 1954. A Markov renewal process is one of the most important mathematical tools for system reliability analysis. We shall discuss Markov renewal processes as a mathematical tool throughout this paper.

Graphical representations for systems are of great importance in system science. Especially, block diagrams, signal flow graphs, and wiring diagrams are generally used to represent systems graphically. Block diagrams are mainly used for control engineering, signal flow graphs are mainly used for electrical engineering (in particular, electrical circuit theory), and wiring diagrams are used for analogue computation (or simulation). The relationship among the above three graphs (or diagrams) are well-known (see, *e.g.*, Huggins [21, 22]). In this paper we shall discuss the relationship between Markov renewal processes and signal flow graphs, and show that deriving the Laplace-Stieltjes (LS) transform of the first passage time distribution from one state to the other in a Markov renewal process is obtaining the system gain by

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defining a starting state is a source and an ending state is a sink in the signal flow graph, where the signal flow graph is a corresponding state transition diagram of the Markov renewal process.

Using the relationship between Markov renewal processes and signal flow graphs, we shall finally obtain system reliability for some systems, *e.g.*, a two-unit standby redundant system, a two-unit standby redundant system with noninstantaneous switchover, and m-out-of-n systems. The use of the signal flow graphs for system analysis makes the system clear, and obtaining the system gain implies our desired result, which is an easy mechanical procedure since we can apply Mason's gain formula [24, 25].

2.2. Markov renewal processes

A Markov renewal process [45, 46], roughly speaking, is a stochastic process in which the state transitions obey the given transition probabilities and the sojourn time in a state is a random variable with any distribution depending on that state and the next visiting state, where the number of states may be denumerable. In this paper we restrict our attention to Markov renewal processes with finitely many states since our models discussed in this paper can be usually represented by Markov renewal processes with finitely many states. The detailed discussion of Markov renewal processes with finitely many states can be found in Pyke [46].

Here we shall describe the necessary definitions and properties of these processes. We denote the states of a Markov renewal process by the symbols $s_0, s_1, \cdots, s_N$. We define the transition probability p_{ij} from state s_i to state s_j for all $i, j=0, 1, 2, \cdots, N$. We also define the distribution $F_{ij}(t)$ $(t\geq 0)$ of the sojourn time in state s_i and the next visiting state s_j. We define

$$Q_{ij}(t)=p_{ij}F_{ij}(t), \qquad (i, j=0, 1, \cdots, N) \tag{2.1}$$

which satisfies the following two conditions:

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(2.2) $$Q_{ij}(0)=0 \qquad (i,\ j=0,\ 1,\ \cdots,\ N)$$

(2.3) $$\sum_{j=0}^{N} Q_{ij}(\infty)=\sum_{j=0}^{N} p_{ij}F_{ij}(\infty)=\sum_{j=0}^{N} p_{ij}=1. \qquad (i=0,\ 1,\ \cdots,\ N)$$

Defining the states of the process, we can find $Q_{ij}(t)$ for all i and j. Then we have all information on the process considered. In this paper we restrict our attention to the first passage time distribution from one state to the other. So, define an absorbing state s_N. Then the remaining states s_i, $i=0, 1, 2, \cdots, N-1$, are transient. We define the first passage time distribution $\Phi_i(t)$ $(i=0, 1, 2, \cdots, N-1)$ starting from state s_i at $t=0$ to the absorbing state s_N up to time t. We consider two (exclusive and exhaustive) cases for $\Phi_i(t)$: One is the immediate transition to the absorbing state s_N. The other is the transition to any transient state $s_j (j=0, 1, \cdots, N-1)$. These two events are mutually exclusive. In the latter case, the process after the transition to state s_j obeys $\Phi_j(t)$. Thus we have

(2.4) $$\Phi_i(t)=Q_{iN}(t)+\sum_{j=0}^{N-1} Q_{ij}(t)*\Phi_j(t), \qquad (i=0,\ 1,\ 2,\ \cdots,\ N-1)$$

where $*$ denotes the convolution operation. The small letters $\varphi_i(s)$ and $q_{ij}(s)$ denote the corresponding LS transforms of $\Phi_i(t)$ and $Q_{ij}(t)$, respectively. Defining the $N\times 1$ vector $\varphi(s)$ with component $\varphi_i(s)$ $(i=0, 1, 2, \cdots, N-1)$ and taking the LS transforms for (2.4), we have in matrix form

(2.5) $$\varphi(s)=q_N(s)+q(s)\varphi(s),$$

where $q_N(s)$ is the $N\times 1$ vector with component $q_{iN}(s)$ $(i=0, 1, \cdots, N-1)$ and $q(s)$ is the $N\times N$ matrix with element $q_{ij}(s)$ $(i, j=0, 1, \cdots, N-1)$. Solving (2.5) for $\varphi(s)$, we have

(2.6) $$\varphi(s)=[I-q(s)]^{-1}q_N(s),$$

where I is the $N\times N$ identity matrix. We note that the inverse matrix $[I-q(s)]^{-1}$ exists for $s>0$. Our concern is to find $\varphi(s)$ (in particular,

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$\varphi_0(s)$) for the models discussed below. The general first passage time distribution from one state to the other has been given by Pyke [46]. Noting that

$$(2.7) \qquad q_{Nj}(s)=0, \qquad (j=0,\ 1,\ \cdots,\ N-1)$$

and using the general result, we can also obtain (2.6). However, we derived (2.6) by using the intuitive method.

2.3. Signal flow graphs

In this section we shall consider an algorithm for deriving $\varphi_i(s)$ $(i=0,\ 1,\ 2,\ \cdots,\ N-1)$ by using signal flow graphs. The definitions and the notations of signal flow graphs obeys those of Chow and Cassignol [5].

Consider a system whose states are defined and their associated $q_{ij}(s)$'s are given. For the system, using the states and $q_{ij}(s)$, we can construct a state transition diagram which may be considered to be a signal flow graph of the system. In the graph each node corresponds to each state of the system and each branch gain to $q_{ij}(s)$. We shall consider an algorithm for deriving $\varphi_i(s)$ by using the signal flow graph. As is anticipated, $\varphi_i(s)$ given in (2.6) is derived by using Mason's gain formula [5, p. 63] in the signal flow graph, where we define that node s_i is a source and node s_N is a sink. We shall below verify the above fact by using the results of signal flow graphs [5].

Since node s_i in the graph has both incoming and outcoming branches, we define a new source s_α which has an outcoming branch to node s_i with its branch gain unity (see Fig. 2.1). Let's define the corresponding variables of nodes s_α, s_0, s_1, s_2, $\cdots$, s_N by x_α, x_0, x_1, x_2, $\cdots$, x_N. Each branch gain corresponds to each $q_{ij}(s)$ (in particular $q_{\alpha i}(s)=1$ and $q_{\alpha j}(s)=0$ for $j\neq i$). Using the rule of signal flow graphs, we have the following set of simultaneous linear equations:

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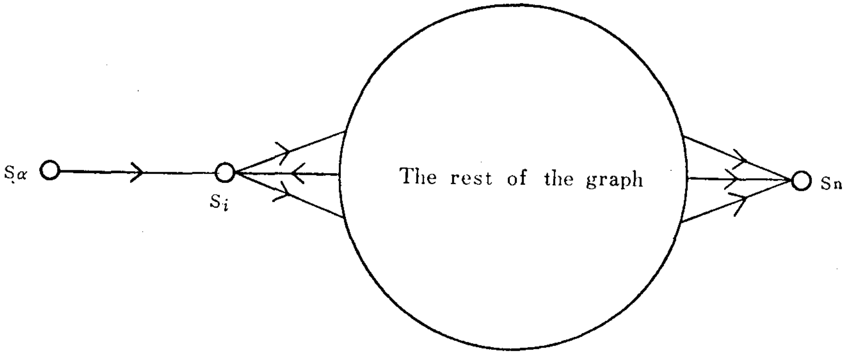

Fig. 2.1. Signal flow graph with the source connected only to one node of the system.

$$(2.7)\quad \begin{cases} x_0=q_{00}(s)x_0+q_{10}(s)x_1+\cdots+q_{N-1,\,0}(s)x_{N-1} \\ \cdots\cdots \\ x_i=q_{0i}(s)x_0+q_{1i}(s)x_1+\cdots+q_{N-1,\,i}(s)x_{N-1}+x_\alpha \\ \cdots\cdots \\ x_{N-1}=q_{0,\,N-1}(s)x_0+q_{1,\,N-1}(s)x_1+\cdots+q_{N-1,\,N-1}(s)x_{N-1}, \end{cases}$$

$$(2.8)\quad x_N=q_{0N}(s)x_0+q_{1N}(s)x_1+\cdots+q_{N-1,\,N}(s)x_{N-1}.$$

Define the $N\times 1$ vectors

$$(2.9)\quad x=\begin{pmatrix} x_0 \\ \vdots \\ x_{N-1} \end{pmatrix}, \qquad e_i=\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

where e_i is a vector with $(i+1)$th component unity and the other components zero. Using $q(s)$, $q_N(s)$, and (2.9), we have from (2.7) and (2.8) the following equations:

$$(2.10)\quad x=q(s)^{\mathrm{T}}x+e_ix_\alpha,$$

$$(2.11)\quad x_N=x^{\mathrm{T}}q_N(s),$$

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where the superscript $^{\mathsf{T}}$ denotes the transpose of the matrix. Noting that the inverse matrix $[I-q(\alpha)]^{-1}$ exists for $s>0$, we have the ratio x_N/x_α as follows:

$$x_N/x_\alpha = e_i^{\mathsf{T}}[I-q(s)]^{-1}q_N(s), \tag{2.12}$$

which is the system gain assuming that node s_α is a source and node s_N is a sink. That is, the system gain coincides with $\varphi_i(s)$ given in

$$\varphi(s)=[I-q(s)]^{-1}q_N(s). \tag{2.13}$$

As described above, deriving the LS transform of the first passage time distribution from state s_i to state s_N in a Markov renewal process is obtaining the system gain assuming that state s_i is a source and state s_N is a sink, where the signal flow graph is the state transition diagram in the Markov renewal process and each branch gain corresponds to each $q_{ij}(s)$. To obtain the system gain in the graph, we can apply Mason's gain formula [5, p. 63], which is an easy mechanical procedure. In particular it is more efficient to obtain the system gain for the complicated systems.

We shall derive the system reliability by using the signal flow graph method. In the above analysis we make an additional state s_x which is a source. We should always consider state s_x. However, we omit state s_α and we regard state s_i as a source in the analysis below.

2.4. Mean time and the higher moments

As described in the preceding section, the LS transform $\varphi_i(s)$ can be obtained by using Mason's gain formula. In this section we shall consider an algorithm for deriving the mean time or the higher moments of $\Phi_i(t)$ by using signal flow graphs.

To obtain the mean time, we rewrite (2.5) as follows:

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$$
(2.14)\quad \begin{cases} \varphi_0(s)=q_{0N}(s)+q_{00}(s)\varphi_0(s)+\cdots+q_{0,\,N-1}(s)\varphi_{N-1}(s) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \varphi_i(s)=q_{iN}(s)+q_{i0}(s)\varphi_0(s)+\cdots+q_{i,\,N-1}(s)\varphi_{N-1}(s) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \varphi_{N-1}(s)=q_{N-1,\,N}(s)+q_{N-1,\,0}(s)\varphi_0(s)+\cdots+q_{N-1,\,N-1}(s)\varphi_{N-1}(s). \end{cases}
$$

Denote the mean time from state s_i to state s_N by

$$
(2.15)\quad \hat{T}_i=-\left.\frac{d\varphi_i(s)}{ds}\right|_{s=0}. \qquad (i=0,\ 1,\ \cdots,\ N-1)
$$

Differentiating (2.14) with respect to s, inverting the sign, and setting $s=0$, we have

$$
(2.16)\quad \begin{cases} \hat{T}_0=\xi_0+q_{00}(0)\hat{T}_0+\cdots+q_{0,\,N-1}(0)\hat{T}_{N-1} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \hat{T}_i=\xi_i+q_{i0}(0)\hat{T}_0+\cdots+q_{i,\,N-1}(0)\hat{T}_{N-1} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \hat{T}_{N-1}=\xi_{N-1}+q_{N-1,\,0}(0)\hat{T}_0+\cdots+q_{N-1,\,N-1}(0)\hat{T}_{N-1}, \end{cases}
$$

where

$$
(2.17)\quad \xi_i=-\sum_{j=0}^{N}\left.\frac{dq_{ij}(s)}{ds}\right|_{s=0} \qquad (i=0,\ 1,\ 2,\ \cdots,\ N-1)
$$

is the *unconditional* mean in state s_i. That is,

$$
(2.18)\quad H_i(t)=\sum_{j=0}^{N}Q_{ij}(t) \qquad (i=0,\ 1,\ 2,\ \cdots,\ N-1)
$$

is the distribution regardless the next visiting state. So, it is called the *unconditional* distribution. Then (2.17) is the mean time of the unconditional distribution in state s_i.

Comparing with (2.16), we can obtain an algorithm for deriving the mean time $\hat{T}_i$ by using signal flow graphs. That is, to derive the mean time $\hat{T}_i$ ($i=0,\ 1,\ 2,\ \cdots,\ N-1$), we can obtain the system gain by using

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Mason's gain formula, where node s_i is a source, node s_N is a sink, each branch gain corresponds to $q_{ij}(0)$ $(i, j=0, 1, 2, \cdots, N-1)$ except that each branch gain from state s_i to state s_N corresponds to ξ_i given in (2.17). In other words, we can obtain the system gain in the signal flow graph discussed in the preceding section by setting $q_{ij}(0)$ $(i, j=0, 1, 2, \cdots, N-1)$ instead of $q_{ij}(s)$ and ξ_i $(i=0, 1, 2, \cdots, N-1)$ instead of $q_{iN}(s)$.

Next we shall consider the higher moments of $\Phi_i(t)$ $(i=0, 1, 2, \cdots, N-1)$. Define

$$\hat{T}_i^{(n)}=\int_0^\infty t^n d\Phi_i(t)=(-1)^n \left.\frac{d^n\varphi_i(s)}{ds^n}\right|_{s=0} \qquad (n\geq 2;\ i=0, 1, \cdots, N-1) \tag{2.19}$$

For example, the variance in state s_i is given by

$$\sigma_i^2=\hat{T}_i^{(2)}-(\hat{T}_i)^2. \qquad (i=0, 1, \cdots, N-1) \tag{2.20}$$

Differentiating (2.14) two times with respect to s and setting $s=0$, we have

$$\hat{T}_i^{(2)}=\xi_i^{(2)}+2\sum_{j=0}^{N-1} p_{ij}b_{ij}\hat{T}_j+\sum_{j=0}^{N-1} q_{ij}(0)\hat{T}_j^{(2)}, \qquad (i=0, 1, 2, \cdots, N-1) \tag{2.21}$$

where

$$\xi_i^{(2)}=\int_0^\infty t^2 dH_i(t)=\sum_{j=0}^{N} \left.\frac{d^2q_{ij}(s)}{ds^2}\right|_{s=0} \qquad (i=0, 1, \cdots, N) \tag{2.22}$$

is the second moment of $H_i(t)$, and

$$p_{ij}\,b_{ij}=\int_0^\infty t dQ_{ij}(t)=-\left.\frac{dq_{ij}(s)}{ds}\right|_{s=0} \qquad (i, j=0, 1, \cdots, N-1) \tag{2.23}$$

is the mean time of $Q_{ij}(t)$. Thus we can obtain an algorithm for deriving $\hat{T}_i^{(2)}(i=0, 1, \cdots, N-1)$. That is, we apply the same algorithm for deriving the mean time $\hat{T}_i(i=0, 1, 2, \cdots, N-1)$, where each branch

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gain corresponds to each $q_{ij}(0)$ (i, $j=0, 1, \cdots, N-1$) and each branch gain from node s_i ($i=0, 1, \cdots, N-1$) to node s_N corresponds to $\xi_i^{(2)}+2\sum_{j=0}^{N-1} p_{ij} b_{ij} \hat{T}_j$.

We can further obtain the higher moments $\hat{T}_i^{(n)}$($n\geq 3$; $i=0, 1, \cdots, N-1$) of $\Phi_i(t)$ by using signal flow graphs, where $\hat{T}_i^{(n)}$ ($i=0, 1, 2, \cdots, N-1$) can be represented by $\hat{T}_i^{(n-1)}$, $\hat{T}_i^{(n-2)}$, $\cdots$, $\hat{T}_i$ ($i=0, 1, 2, \cdots, N-1$) and

$$p_{ij}\, b_{ij}^{(n-1)}=\int_0^{\infty} t^{n-1}\, dQ_{ij}(t). \qquad (n\geq 2;\ i, j=0, 1, 2, \cdots, N-1) \tag{2.24}$$

2.5. System reliability

In the preceding two sections we discussed the relationship between Markov renewal processes and signal flow graphs. Markov renewal processes are of great use for system analysis. In particular, the processes are used in the reliability theory. We encouter a problem that the total failure of a system yields a catastrophe. Our concern in the problem is the first passage time distribution to system down. For the problem we can apply the signal flow graph method to obtain the LS transform of the first passage time distribution. We shall show some examples of systems. The systems have been investigated by Gaver [14, 15], Srinivasan [49, 50, 51], Mine and Osaki [34], Downton [8], and others. However, the signal flow graph method discussed in this paper is simple and elegant.

A two-unit standby redundant system

A two-unit standby redundant system with instantaneous switchover has been investigated by Gnedenko *et al.* [17], and Srinivasan [49] under the most generalized assumptions that both the failure and repair time distributions are arbitrary. Appropriately labeling the number of the two units, we may call them units 1 and 2. The failure time of unit

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i ($i=1, 2$) is a random variable with an arbitrary distribution $F_i(t)$ and the repair time of unit i is also a random variable with an arbitrary distribution $G_i(t)$. These random variables are nonnegative and mutually independent.

Initially, at $t=0$, unit 1 begins to be operative and unit 2 is in standby (state s_0). As soon as unit 1 fails, unit 2 begins to be operative and unit 1 undergoes repair (state s_1). When the repair of unit 1 is completed before unit 2 fails, unit 1 is in standby and then as soon as unit 2 fails, unit 1 in standby begins to be operative and unit 2 undergoes repair (state s_2). While in state s_1 we consider the other case that unit 2 fails before the repair completion of unit 1, which implies the system down (state s_3). In state s_2 we can consider the following two cases: One is the repair completion of unit 2 before unit 1 fails, which goes to state s_1. The other is the failure of unit 1 before the repair completion of unit 2, which goes to state s_3. The system behaves from the operating unit 2 to the operating unit 1 and so on until the occurrence of the system down. In the system considered, we assume that each switchover time is instantaneous.

Fig. 2. 2 shows the signal flow graph of the system. Then each branch gain is easily obtained from the discussion just mentioned above as follows:

(2. 25) $$q_{01}(s)=\int_0^\infty e^{-st}dF_1(t)\equiv f_1(s),$$

(2. 26) $$q_{12}(s)=\int_0^\infty e^{-st}G_1(t)dF_2(t),$$

(2. 27) $$q_{13}(s)=\int_0^\infty e^{-st}\bar{G}_1(t)dF_2(t),$$

(2. 28) $$q_{21}(s)=\int_0^\infty e^{-st}G_2(t)dF_1(t),$$

(2. 29) $$q_{23}(s)=\int_0^\infty e^{-st}\bar{G}_2(t)dF_1(t),$$

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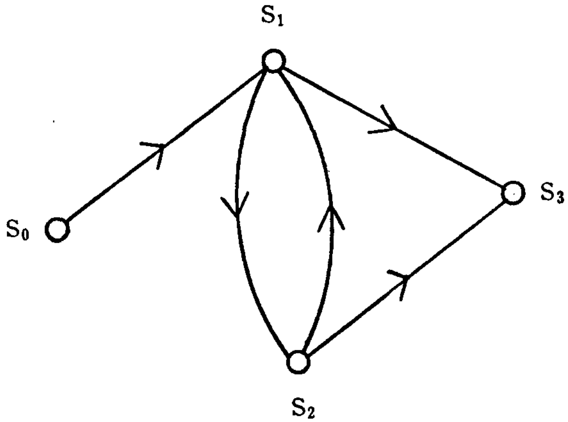

Fig. 2.2. Signal flow graph of a two-unit standby redundant system.

where $\bar{G}_i(t)=1-G_i(t)$ is the survival probability function. In the graph in Fig. 2. 2, node s_0 is a source and node s_3 is a sink. Thus, we have from Mason's gain formula [5, p. 63] the following system gain

$$(2.30) \qquad \varphi_0(s)=\frac{q_{01}(s)q_{13}(s)+q_{01}(s)q_{12}(s)q_{23}(s)}{1-q_{12}(s)q_{21}(s)},$$

which is the LS transform of the first passage time distribution from state s_0 to state s_3. The mean time is given by using the signal flow graph as follows:

$$(2.31) \qquad \hat{T}_0=\frac{1}{\lambda_1}+\frac{1/\lambda_2+q_{12}(0)/\lambda_1}{1-q_{12}(0)q_{21}(0)},$$

where

$$(2.32) \qquad 1/\lambda_i=\int_0^\infty t dF_i(t) \qquad (i=1,\ 2)$$

is the mean failure time of unit i.

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A two-unit standby redundant system with noninstantaneous switchover

Here we shall discuss a two-unit standby redundant system with noninstantaneous switchover, which has been discussed by Srinivasan [51]. He discussed a simple case that the two units are identical, but we shall discuss the more generalized system that the two units are dissimilar. In a similar way of the preceding system, the two units are denoted by $i=1, 2$. The failure time of unit i is a random variable with distribution $F_i(t)=1-\exp(-\lambda_i t)$ and the repair of unit i is a random variable with an arbitrary distribution $G_i(t)$. Here we assume the memory-less property of the failure time distribution. Whenever unit i ($i=1, 2$) is active and the other j ($j\neq i$) in standby, action is initiated on the latter of T_j unit of time in order to bring it to the operating standby state. The switchover time from the action to the operating standby state of unit j is a random variable with an arbitrary distribution $\Gamma_j(t)$. These random variables are nonnegative and mutually independent. The five states of each unit are active, repair, standby, switchover, and operating standby, and are denoted by the symbols 0, 1, 2, 3, and 4, respectively. The state of the system will be specified by the states of unit 1 and unit 2 together. The possible states of the system are enumerated in Table 2. 1, where states s_8, s_9, s_{10}, s_{11}, and s_{12} denote the system down and these states are combined in an absorbing state s_a.

Table 2.1. Possible states of the system.

State of system	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}
State of unit 1	0	0	0	1	2	3	4	0	1	1	1	3	2
State of unit 2	2	3	4	0	0	0	0	1	1	2	3	1	1

Fig. 2. 3 shows the signal flow graph of the system using s_0-s_7 and s_a. Each branch gain of the system is given by

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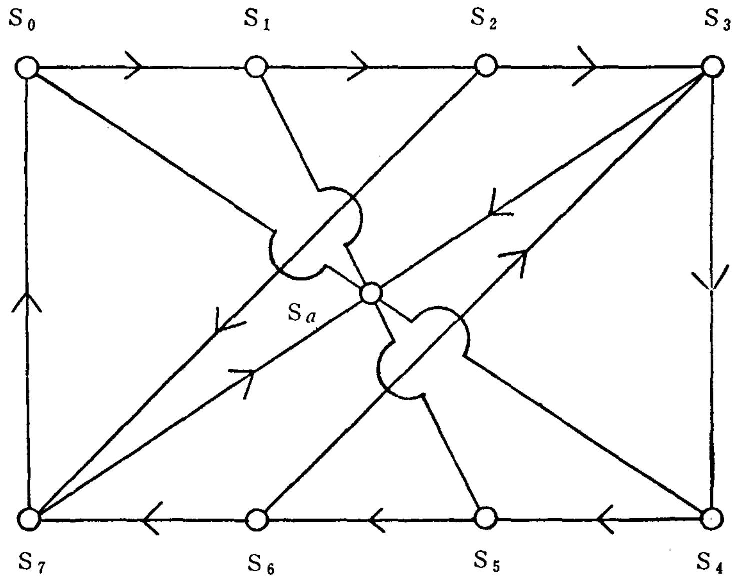

Fig. 2.3. Signal flow graph of a two-unit standby redundant system with noninstantaneous switchover.

(2.33) $q_{01}(s)=e^{-(s+\lambda_1)T_2}$, $\quad q_{45}(s)=e^{-(s+\lambda_2)T_1}$

(2.34) $q_{0a}(s)=\dfrac{\lambda_1}{s+\lambda_1}[1-e^{-(s+\lambda_1)T_2}]$, $\quad q_{4a}(s)=\dfrac{\lambda_2}{s+\lambda_2}[1-e^{-(s+\lambda_2)T_1}]$

(2.35) $q_{12}(s)=\gamma_2(s+\lambda_1)$, $\quad q_{56}(s)=\gamma_1(s+\lambda_2)$

(2.36) $q_{1a}(s)=\dfrac{\lambda_1}{s+\lambda_1}[1-\gamma_2(s+\lambda_1)]$, $\quad q_{5a}(s)=\dfrac{\lambda_2}{s+\lambda_2}[1-\gamma_1(s+\lambda_2)]$

(2.37) $q_{23}(s)=\dfrac{\lambda_1}{s+\lambda_1+\lambda_2}$, $\quad q_{67}(s)=\dfrac{\lambda_2}{s+\lambda_1+\lambda_2}$

(2.38) $q_{27}(s)=q_{67}(s)$, $\quad q_{63}(s)=q_{23}(s)$

(2.39) $q_{34}(s)=g_1(s+\lambda_2)$, $\quad q_{70}(s)=g_2(s+\lambda_1)$

(2.40) $q_{3a}(s)=\dfrac{\lambda_2}{s+\lambda_2}[1-g_1(s+\lambda_2)]$, $\quad q_{7a}(s)=\dfrac{\lambda_1}{s+\lambda_1}[1-g_2(s+\lambda_1)]$,

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where $g_i(s)$ and $\gamma_i(s)$ ($i=1$, 2) are the LS transforms of $G_i(t)$ and $\Gamma_i(t)$, respectively. We define that node s_0 is a source and node s_a is a sink in the graph of Fig. 2. 3. From Mason's gain formula the system gain is given by

$$\varphi_0(s)=N/(1-q_{01}q_{12}q_{27}q_{70}-q_{45}q_{56}q_{63}q_{34}), \tag{2.41}$$

where

$$\begin{aligned} N=&q_{0a}(1-q_{45}q_{56}q_{63}q_{34})+q_{01}q_{1a}(1-q_{45}q_{56}q_{63}q_{34}) \\ &+q_{01}q_{12}q_{23}q_{3a}+q_{01}q_{12}q_{27}q_{7a} \\ &+q_{01}q_{12}q_{23}q_{34}q_{4a}+q_{01}q_{12}q_{23}q_{34}q_{45}q_{5a}. \end{aligned} \tag{2.42}$$

Here we use the abbreviated notation q_{ij} instead of $q_{ij}(s)$. We further note that the result (2.41) is simplified by using the relations (2.33)-(2.40). We have obtained $\varphi_i(s)$, which is the LS transform of the first passage time distribution to system down from state s_0. We can obtain $\varphi_i(s)$ ($i=1, \cdots, 7$) by assuming a source s_i in a similar fashion.

For the mean time we have from Mason's gain formula

$$\hat{T}_0=M/(1-q_{01}q_{12}q_{27}q_{70}-q_{45}q_{56}q_{63}q_{34}), \tag{2.43}$$

where

$$\begin{aligned} M=&(1-q_{45}q_{56}q_{63}q_{34})[1-e^{-\lambda_1T_2}]/\lambda_1+q_{01}q_{12}q_{23}q_{34}[1-e^{-\lambda_2T_1}]/\lambda_2 \\ &+q_{01}(1-q_{45}q_{56}q_{63}q_{34})[1-\gamma_2(\lambda_1)]/\lambda_1+q_{01}q_{12}q_{23}q_{34}q_{45}[1-\gamma_2(\lambda_1)]/\lambda_2 \\ &+q_{01}q_{12}q_{23}[1-g_1(\lambda_2)]/\lambda_2+q_{01}q_{12}q_{27}[1-g_2(\lambda_1)]/\lambda_1 \\ &+q_{01}q_{12}/(\lambda_1+\lambda_2)+q_{01}q_{12}q_{23}q_{34}q_{45}q_{56}/(\lambda_1+\lambda_2). \end{aligned} \tag{2.44}$$

Here we use the abbreviated notation q_{ij} instead of $q_{ij}(0)$.

Srinivasan [51] considered a special case of the above system. That is, he considered a simple case that the two units are identical. In this case we assume that the failure time distribution is $F(t)=1-\exp(-\lambda t)$, the repair time distribution $G(t)$, the switchover time distribution $\Gamma(t)$, and the required time to bring a standby unit to the operating standby state T. Noting that states s_0 and s_4, s_1 and s_5, s_2 and s_6, and s_3 and s_7

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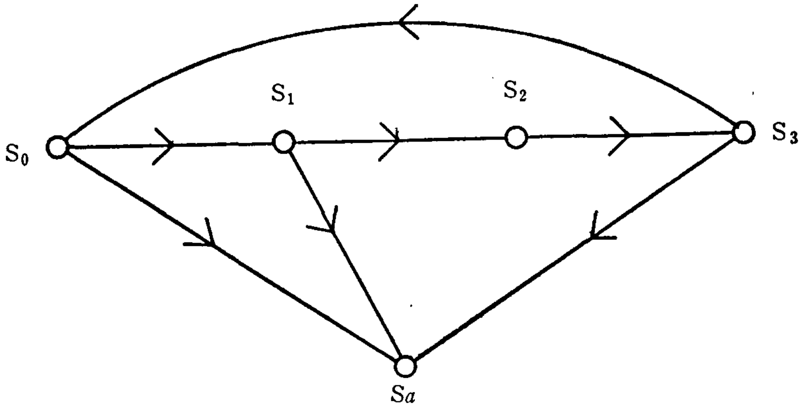

Fig. 2.4. Signal flow graph of the system of identical units.

are identical, we have the reduced signal flow graph in Fig. 2. 4. Then each branch gain is given by

(2.45) $$q_{01}(s)=e^{-(s+\lambda)T},$$

(2.46) $$q_{0a}(s)=\frac{\lambda}{s+\lambda}[1-e^{-(s+\lambda)T}],$$

(2.47) $$q_{12}(s)=\gamma(s+\lambda),$$

(2.48) $$q_{1a}(s)=\frac{\lambda}{s+\lambda}[1-\gamma(s+\lambda)],$$

(2.49) $$q_{23}(s)=2\lambda/(s+2\lambda),$$

(2.50) $$q_{30}(s)=g(s+\lambda),$$

(2.51) $$q_{3a}(s)=\frac{\lambda}{s+\lambda}[1-g(s+\lambda)].$$

Assuming that node s_0 is a source and node s_a is a sink, we have the system gain from Mason's gain formula as follows:

(2.52) $$\varphi_0(s)=\frac{q_{0a}(s)+q_{01}(s)q_{1a}(s)+q_{01}(s)q_{12}(s)q_{23}(s)q_{3a}(s)}{1-q_{01}(s)q_{12}(s)q_{23}(s)q_{30}(s)}.$$

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The mean time is given by

$$(2.53) \qquad \hat{T}_0 = \frac{1}{\lambda}\left\{1 + \frac{1}{2[1-\gamma(\lambda)g(\lambda)e^{-\lambda T}]}\gamma(\lambda)e^{-\lambda T}\right\}.$$

m-out-of-n systems

An m-out-of-n system is a redundant system composed of n parallel units ($n \geq m$). When m units are simultaneously under failure or repair, the system down occurs [8]. We assume that the system considered has one repair facility and the failed unit may be waiting for the repair if the repair facility is busy. We also assume that each switchover time is instantaneous.

First, we shall consider a 2-out-of-n system of dissimilar units. Appropriately labeling the number of units, we may call them units 1, 2, $\cdots$, $n-1$, and n. The failure time of units i (i=1, 2, $\cdots$, n) is a random variable with exponential distribution $F_i(t)=1-\exp(-\lambda_i t)$ and the repair time of the failed unit i is a random variable with an arbitrary distribution $G_i(t)$, where we assume the memoryless property of the failure time distribution for the convonience of analysis. These random variables are nonnegative and mutually independent. In the system considered state s_0 denotes one that all n units are operative, state i (i=1, 2, $\cdots$, n) denotes one that unit i is under repair and the remaining units are operative, and state s_{n+1} denotes one that at least two units are under repair or failure simultaneously (*i.e.*, state s_{n+1} denotes the system down). Fig. 2. 5 shows the signal flow graph of the system. For the system we obtain easily each branch gain as follows:

$$(2.54) \qquad q_{0i}(s)=\lambda_i/(s+\Lambda), \qquad (i=1,\ 2,\ \cdots,\ n)$$

$$(2.55) \qquad q_{i0}(s)=\int_0^\infty e^{-st}e^{-\lambda_i^* t}dG_i(t)$$

$$=g_i(s+\lambda_i^*), \qquad (i=1,\ 2,\ \cdots,\ n)$$

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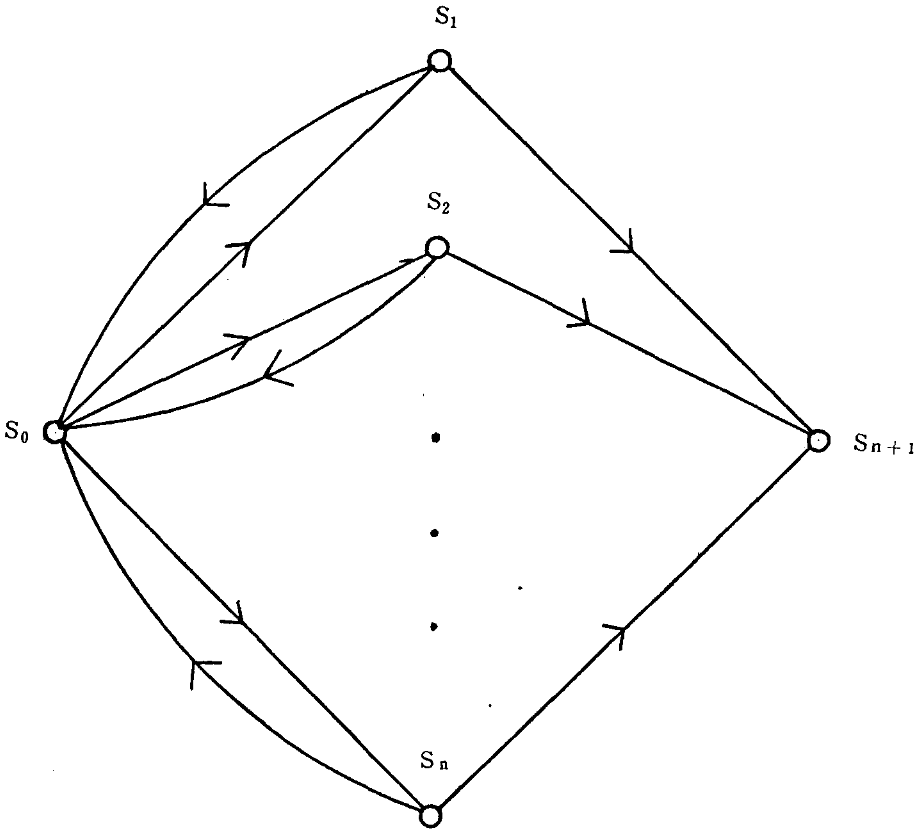

Fig. 2.5. Signal flow graph of a 2-out-of-n system of dissimilar units.

$$(2.56)\qquad q_{i,\,n+1}(s)=\int_0^\infty e^{-st}\bar{G}_i(t)\lambda_i{}^*e^{-\lambda_i{}^*t}dt$$

$$=\frac{\lambda_i{}^*}{s+\lambda_i{}^*}[1-g_i(s+\lambda_i{}^*)],\qquad (i=1,\ 2,\ \cdots,\ n)$$

where

$$(2.57)\qquad \Lambda=\sum_{i=1}^n \lambda_i\,,$$

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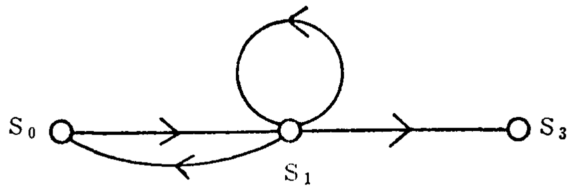

Fig. 2.6. Signal flow graph of a 3-out-of-n system of identical units.

(2.58) $\lambda_i^* = \Lambda - \lambda_i. \quad (i=1, 2, \cdots, n)$

Assuming that node s_0 is a source and node s_{n+1} is a sink, we have from Mason's gain formula

(2.59) $$\varphi_0(s) = \sum_{i=1}^{n} q_{0i}(s) q_{i,n+1}(s) / [1 - \sum_{i=1}^{n} q_{0i}(s) q_{i0}(s)].$$

The mean time is immediately given by

(2.59) $$\hat{T}_0 = \frac{1 + \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_i^*} [1 - g_i(\lambda_i^*)]}{\Lambda - \sum_{i=1}^{n} \lambda_i g_i(\lambda_i^*)} .$$

The results obtained above contains some interesting results as special cases. For example, we can consider a case $n=2$. Then the system is a two-unit paralleled redundant system which has been discussed by Gaver [14, 15]. We can further show that the results given by Gnedenko [16] are derived from the above results. The detailed discussion can be found in Mine and Osaki [34].

Second, we shall consider a 3-out-of-n system. In the system considered, we shall only consider a simple case that all units are identical. The failure time distribution of each unit is $F(t) = 1 - \exp(-\lambda t)$ and the repair time distribution of each unit is $G(t)$. The state s_i ($i=0, 1, 3$) of the system denotes the corresponding number of the failed units. In the system we need not to consider a state s_2 since we take notice of the regeneration point of the repair time distribution. Fig. 2. 6 shows

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the signal flow graph of the system. Each branch gain is given by

$$q_{01}(s)=n\lambda/(s+n\lambda), \tag{2.60}$$

$$q_{10}(s)=\int_0^\infty e^{-st}e^{-(n-1)\lambda t}dG(t)=g(s+[n-1]\lambda), \tag{2.61}$$

$$\begin{aligned} q_{11}(s)&=\binom{n-1}{1}\int_0^\infty e^{-st}(1-e^{-\lambda t})e^{-(n-2)\lambda t}dG(t) \\ &=(n-1)\{g(s+[n-2]\lambda)-g(s+[n-1]\lambda)\}, \end{aligned} \tag{2.62}$$

$$\begin{aligned} q_{13}(s)&=\binom{n-1}{2}\int_0^\infty e^{-st}\bar{G}(t)e^{-(n-3)\lambda t}d[(1-e^{-\lambda t})^2] \\ &=\frac{\lambda(n-1)(n-2)}{s+(n-2)\lambda}[1-g(s+[n-2]\lambda)] \\ &\quad-\frac{\lambda(n-1)(n-2)}{s+(n-1)\lambda}[1-g(s+[n-1]\lambda)]. \end{aligned} \tag{2.63}$$

Assuming that node s_0 is a source and node s_3 is a sink, we have from Mason's gain formula

$$\varphi_0(s)=\frac{q_{01}(s)q_{13}(s)}{1-q_{01}(s)q_{10}(s)-q_{11}(s)}. \tag{2.64}$$

The mean time is immediately given by

$$\hat{T}_0=\frac{[1-q_{11}(0)]/n\lambda+\xi_1}{1-q_{01}(0)q_{10}(0)-q_{11}(0)}, \tag{2.65}$$

where

$$\xi_1=\frac{n-1}{(n-2)\lambda}[1-g([n-2]\lambda)]-\frac{n-2}{(n-1)\lambda}[1-g([n-1]\lambda)]. \tag{2.66}$$

As the third model we shall consider a 4-out-of-n system. In the system we shall also consider a simple case that all units are identical. The state s_i (i=0, 1, 2, 4) of the system denotes the corresponding number of the failed units. In the same reason of the preceding model we need not to consider a state s_3. Fig. 2. 7 shows the signal flow

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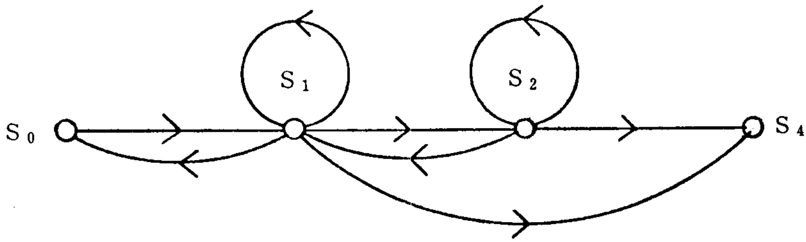

Fig. 2.7. Signal flow graph of a 4-out-of-n system of identical units.

graph of the system. We shall derive each branch gain. For $q_{01}(s)$, $q_{10}(s)$, and $q_{11}(s)$, we have obtained in (2.60), (2.61), and (2.62), respectively, of a 3-out-of-n system. For the other $q_{ij}(s)$, we have by focussing on the regeneration point of the repair time distribution

(2.67) $$q_{12}(s)=\frac{(n-1)(n-2)}{2}\times[g(s+[n-3]\lambda)-2g(s+[n-2]\lambda)+g(s+[n-1]\lambda)],$$

(2.68) $$q_{14}(s)=\frac{\lambda(n-1)(n-2)}{s+(n-2)\lambda}[1-g(s+[n-2]\lambda)]-\frac{\lambda(n-1)(n-2)}{s+(n-1)\lambda}[1-g(s+[n-1]\lambda)],$$

(2.69) $$q_{21}(s)=g(s+[n-2]\lambda),$$

(2.70) $$q_{22}(s)=(n-2)[g(s+[n-3]\lambda)-g(s+[n-2]\lambda)],$$

(2.71) $$q_{24}(s)=\frac{\lambda(n-2)(n-3)}{s+(n-3)\lambda}[1-g(s+[n-3]\lambda)]-\frac{\lambda(n-2)(n-3)}{s+(n-2)\lambda}[1-g(s+[n-2]\lambda)].$$

Assuming that node s_0 is a source and node s_4 is a sink, we have from Mason's gain formula

(2.72) $$\varphi_0(s)=\frac{q_{01}q_{12}q_{24}+q_{01}q_{14}(1-q_{22})}{1-q_{01}q_{10}-q_{11}-q_{12}q_{21}-q_{22}+q_{01}q_{10}q_{22}+q_{11}q_{22}},$$

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where we also use the abbreviated notation q_{ij} instead of $q_{ij}(s)$. The mean time is given by

$$(2.73)\qquad \hat{T}_0=\frac{(1-q_{11}-q_{12}q_{21}-q_{22}+q_{11}q_{22})/n\lambda+q_{01}(1-q_{22})\xi_1+q_{01}q_{12}\xi_2}{1-q_{01}q_{10}-q_{11}-q_{12}q_{21}-q_{22}+q_{01}q_{10}q_{22}+q_{11}q_{22}},$$

where we use the abbreviated notation q_{ij} instead of $q_{ij}(0)$. Here ξ_1 and ξ_2 are given by

$$(2.74)\qquad \xi_1=\frac{n-1}{(n-2)\lambda}[1-g([n-2]\lambda)]-\frac{n-2}{(n-1)\lambda}[1-g([n-1]\lambda)],$$

$$(2.75)\qquad \xi_2=\frac{n-2}{(n-3)\lambda}[1-g([n-3]\lambda)]-\frac{n-3}{(n-2)\lambda}[1-g([n-2]\lambda)].$$

Multiple-unit standby redundant systems

As a final example of this section, we consider a multiple-unit standby redundant system. We have considered a two-unit standby redundant system in the first example of this section. As an extension of a two-unit standby redundant system, we consider a multiple-unit standby redundant system of $(r+1)$ repairable units. We assume that all $(r+1)$ $(r\geq 1)$ units are identical. The failure time of each unit obeys an arbitrary distribution $F(t)$ and the repair time of each failed unit obeys the exponential distribution $1-\exp(-\mu t)$. At $t=0$ all $(r+1)$ units are operable, where one unit begins to be operative and the other units are in standby. When the operative unit fails, one unit in standby begins to be operative, the remaining operable units are in standby and the repair of the failed unit begins. We assume that it is possible to make the repairs simultaneously for all the failed units. We also assume that after repair unit recovers its function perfectly. Our concern for the system is the first emptiness, *i.e.*, the time instant that all $(r+1)$ units are under repair simultaneously.

To apply the signal flow graph, we define that state s_i $(i=0, 1, 2, \cdots)$ denotes the corresponding number of the failed units. That is, state s_i denotes that i units are under repair simultaneously and the remaining

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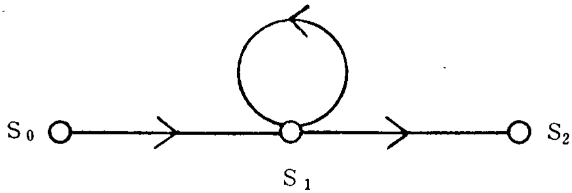

Fig. 2.8. Signal flow graph of a multiple-unit standby redundant system ($r=1$).

units are operative or in standby.

As a simple case, we consider $r=1$. Then the signal flow graph is shown in Fig. 2. 8. Each branch gain is given by

(2. 76) $$q_{01}(s)=\int_0^\infty e^{-st}dF(t)\equiv f(s),$$

(2. 77) $$q_{11}(s)=\int_0^\infty e^{-st}(1-e^{-\mu t})dF(t)=f(s)-f(s+\mu),$$

(2. 78) $$q_{12}(s)=\int_0^\infty e^{-st}e^{-\mu t}dF(t)=f(s+\mu),$$

where $f(s)$ is the LS transform of $F(t)$. Thus we have

(2. 79) $$\varphi_0(s)=q_{01}(s)q_{12}(s)/[1-q_{11}(s)],$$

and

(2. 80) $$\hat{T}_0=\frac{1}{\lambda}+\frac{1}{\lambda[1-q_{11}(0)]},$$

where

(2. 81) $$1/\lambda=\int_0^\infty tdF(t).$$

Next we consider $r=2$. The signal flow graph is shown in Fig. 2. 9. Each branch gain is given by

(2. 82) $$q_{01}(s)=\int_0^\infty e^{-st}dF(t)\equiv f(s),$$

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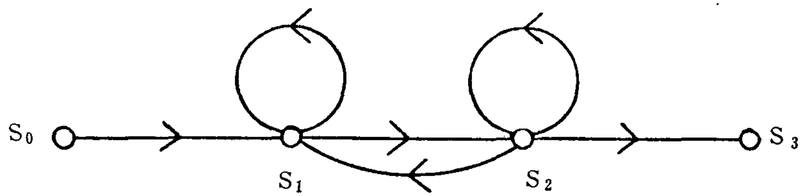

Fig. 2.9. Signal flow graph of a multiple-unit standby redundant system ($r=2$).

$$q_{11}(s)=\int_0^\infty e^{-st}(1-e^{-\mu t})dF(t)=f(s)-f(s+\mu), \tag{2.83}$$

$$q_{12}(s)=\int_0^\infty e^{-st}e^{-\mu t}dF(t)=f(s+\mu), \tag{2.84}$$

$$q_{21}(s)=\int_0^\infty e^{-st}(1-e^{-\mu t})^2dF(t)=f(s)-2f(s+\mu)+f(s+2\mu), \tag{2.85}$$

$$q_{22}(s)=\int_0^\infty e^{-st}\binom{2}{1}e^{-\mu t}(1-e^{-\mu t})dF(t)=2f(s+\mu)-2f(s+2\mu), \tag{2.86}$$

$$q_{23}(s)=\int_0^\infty e^{-st}e^{-2\mu t}dF(t)=f(s+2\mu). \tag{2.87}$$

From Mason's gain formula we have

$$\varphi_0(s)=\frac{q_{01}(s)q_{12}(s)q_{23}(s)}{1-q_{11}(s)-q_{22}(s)-q_{12}(s)q_{21}(s)+q_{11}(s)q_{22}(s)}, \tag{2.88}$$

and

$$\hat{T}_0=\frac{1}{\lambda}+\frac{1-q_{22}(0)+q_{12}(0)}{\lambda[1-q_{11}(0)-q_{22}(0)-q_{12}(0)q_{21}(0)+q_{11}(0)q_{22}(0)]}. \tag{2.89}$$

We can further obtain the LS transform and the mean time by using Mason's gain formula for the general multiple-unit standby redundant system of $(r+1)$ units, but we omit the results. Srinivasan [50] discussed multiple-unit standby redundand systems by using the supplementary variable techniques. He obtained the LS transform and the mean time by using the binomial moments. The detailed discussion can be found in Srinivasan [50].

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A Two-Unit Standby Redundant System with Standby Failure

3.1. Introduction

A two-unit standby redundant system and a two-unit paralleled redundant system are well-known and have been investigated by many authors. Gnedenko *et al.* [17] and Srinivasan [49] analysed the most generalized model for a two-unit standby redundant system, and Gaver [14, 15] analysed the model for a two-unit paralleled redundant system.

In this chapter we shall discuss a two-unit standby redundant model. In the model of Gnedenko *et al.* [17] and Srinivasan [49], it is assumed that a standby unit never fails in the standby interval. However, in this chapter, we assume that a standby unit may fail in the standby interval. Our model considered here has the following three advantages:

(i) In the actual situations we should consider failure of the unit in standby.

(ii) Our results obtained here include those of a two-unit standby redundant system given by Gnedenko *et al.* [17] and Srinivasan [49] and of a two-unit paralleled redundant system given by Gaver [14, 15] as special cases.

(iii) A two-unit paralleled redundant system was only analysed under the assumptions that the failure time distribution of the unit is exponential and the repair time distribution is arbitrary. Our model is an approximate model of a two-unit paralleled redundant system with arbitrary failure and repair time distributions.

The above advantages and their related discussions will be presented in Section 3.5.

3.2. Model

Consider a system of two units (or subsystems), and we may call

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them units 1 and 2. The failure time of the operative unit i ($i=1, 2$) obeys an arbitrary distribution $F_i(t)$, the repair time of the failed unit i obeys an arbitrary distribution $G_i(t)$, and the failure time of the standby unit i obeys an arbitrary distribution $H_i(t)$. That is, if a unit i is operative, the failure law of the unit obeys $F_i(t)$, and if a unit i is in standby, the failure law of the unit obeys $H_i(t)$. The assumption, we note, is that the life of the operative unit enjoys $F_i(t)$, regardless of how long it had been operating in standby. The question concerning the assumption will be answered below. The repair time law always obeys $G_i(t)$ whether a unit i fails in the operative interval or in the standby interval. It is assumed that after repair the failed unit recovers its function perfectly. We assume that these random variables are mutually independent and are nonnegative. The switchover times from the operative state to the repair, from the repair completion to the standby state, and from the standby state to the operative state are assumed to be instantaneous.

We assume that at $t=0$ unit 1 begins to be operative and unit 2 begins to be in standby. The behavior of the model is: Upon failure of unit 1, unit 2 begins to be operative and unit 1 undergoes repair. Upon failure of unit 2, unit 1 (if it is in standby at that time) begins to be operative and unit 2 undergoes repair. (If unit 1 is under repair at that time, the system down occurs.) The model behaves in a similar fashion. Our concern for the model is the first time to system down (*i.e.*, the time instant that two units are under repair or failure simultaneously).

3.3. Derivation of the LS transform

In this section we assume for simplicity of analysis that the two units are identical. That is, we assume that $F(t)\equiv F_i(t)$, $G(t)\equiv G_i(t)$, and $H(t)\equiv H_i(t)$ ($i=1, 2$). We denote $\bar{G}(t)=1-G(t)$ and $\bar{H}(t)=1-H(t)$ as the survival functions of $G(t)$ and $H(t)$, respectively. To analyse the model, we define the following three states of the model (where state s_0 is a starting point, state s_2 is an ending point, and state s_1 is a regeneration

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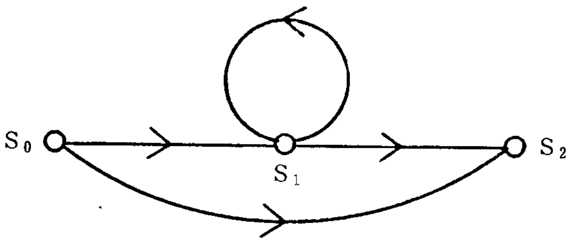

Fig. 3.1. Signal flow graph of the first model of identical units.

point):

State s_0, one unit begins to be operative and the other unit begins to be in standby.

State s_1; one unit begins to be operative and the other unit begins to be repaired.

State s_2; two units are under repair or failure simultaneously. This state denotes the system down.

We note that these states denote the time instants (or epochs) of the model. The state transition diagram (which becomes a signal flow graph) of the model is given in Fig. 3. 1.

We shall derive each branch gain of the system. In state s_0 two transitions can be considered; one is to state s_1, and the other is to state s_2.

First we consider the transition from state s_0 to state s_1 in the time interval $(0, t)$. The probability that the operative unit fails first in the time interval $(t, t+dt)$ is $dF(t)$. In the time interval $(0, t)$, the probabilities that the other unit is in standby up to time t are $\bar{H}(t)$, $H(t)*G(t)*\bar{H}(t)$, $H(t)*G(t)*H(t)*G(t)*\bar{H}(t)$, and so on, where $*$ denotes the convolution operation. We note that $\bar{H}(t)$ means that one unit never fails in $(0, t)$, $H(t)*G(t)*\bar{H}(t)$ means that a unit is in standby up to time t via its failure and repair, $H(t)*G(t)*H(t)*G(t)*\bar{H}(t)$ means that a unit is in standby up to time t via two failures and two repairs, and so on. These events are mutualy exclusive. Thus, the one step distribution (which

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may be improper [10, p. 129]) from state s_0 to state s_1 is

$$Q_{01}(t)=\int_0^t [\bar{H}(t)+H(t)*G(t)*\bar{H}(t) + H(t)*G(t)*H(t)*G(t)*\bar{H}(t)+\cdots]dF(t). \tag{3.1}$$

Introduce the notation

$$(1-M(t))^{(-1)}=\sum_{n=0}^{\infty}[M(t)]^{n*}, \tag{3.2}$$

where

$$[M(t)]^{n*}=\begin{cases}\overbrace{M(t)*M(t)*\cdots*M(t)}^{n} & (n\geq 1)\\ 1. & (n=0;\text{ a Heaviside step function})\end{cases} \tag{3.3}$$

Using the notation (3.2), (3.1) can be rewritten

$$Q_{01}(t)=\int_0^t [\bar{H}(t)*(1-H(t)*G(t))^{(-1)}]dF(t). \tag{3.4}$$

The LS transform of $Q_{01}(t)$ (which is the branch gain) is given by

$$q_{01}(s)=\int_0^\infty e^{-st}[\bar{H}(t)*(1-H(t)*G(t))^{(-1)}]dF(t). \tag{3.5}$$

Now we consider the state transition from state s_0 to state s_2 in the time interval $(0, t)$. The probability that the operative unit fails first in the time interval $(t, t+dt)$ is $dF(t)$. In the time interval $(0, t)$ the probabilities that the other unit is under repair up to time t are $H(t)*\bar{G}(t)$, $H(t)*G(t)*H(t)*\bar{G}(t)$, $H(t)*G(t)*H(t)*G(t)*H(t)*\bar{G}(t)$, and so on. Thus the branch gain, *i.e.*, the LS transform of the one step distribution from state s_0 to state s_2, is given by

$$q_{02}(s)=\int_0^\infty e^{-st}[H(t)*\bar{G}(t)*(1-H(t)*G(t))^{(-1)}]dF(t). \tag{3.6}$$

Finally, we consider the state transitions from state s_1. In state s_1 two transitions can be considered; one is back to state s_1, and the other

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is to state s_2.

First we consider the state transition from state s_1 to state s_1. In this case we can consider that the operative unit fails in the time interval $(t, t+dt)$ and the other unit is in standby up to time t. The probabilities that the other unit is in standby up to time t are $G(t)*\bar{H}(t)$, $G(t)*H(t)*G(t)*\bar{H}(t)$, and so on. Thus, we have

$$q_{11}(s)=\int_0^\infty e^{-st}[G(t)*\bar{H}(t)*(1-H(t)*G(t))^{(-1)}]dF(t). \tag{3.7}$$

Next we consider the state transition from state s_1 to state s_2. In a similar way we consider that the operative unit fails in the time interval $(t, t+dt)$ and the other unit is under repair up to time t. Thus we have

$$q_{12}(s)=\int_0^\infty e^{-st}[\bar{G}(t)*(1-H(t)*G(t))^{(-1)}]dF(t). \tag{3.8}$$

We define $\varphi_i(s)$ $(i=0, 1)$, the LS transform of the first time distribution to system down starting from state s_i at $t=0$. Assuming that node s_0 is a source and node s_2 is a sink, we have from Mason's gain formula

$$\varphi_0(s)=q_{02}(s)+q_{01}(s)q_{12}(s)/[1-q_{11}(s)]. \tag{3.9}$$

The mean time is also derived from Mason's gain formula

$$\hat{T}_0=1/\lambda+q_{01}(0)/\lambda[1-q_{11}(0)], \tag{3.10}$$

where

$$1/\lambda=\int_0^\infty tdF(t). \tag{3.11}$$

3.4. A case with the exponential failure time distribution

In this section we shall consider a special case that the failure time distribution of the operative unit is exponential. That is, we assume that the failure time of the operative unit obeys $F(t)=1-\exp(-\lambda t)$ if a unit is operative and the other unit is *in standby*. Further we assume

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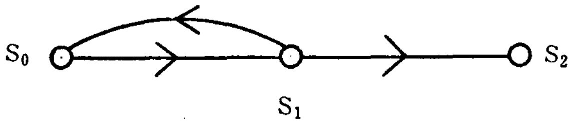

Fig. 3.2. Signal flow graph of the second model of identical units.

that the failure time of the operative unit obeys $A(t)$ if a unit is operative and the other unit is *under repair*. The same assumptions are imposed for the model except the failure time of the operative unit. We define the same three states (where states s_0 and s_1 are regeneration points) in the preceding model. We note that state s_0 is a regeneration point for the model since we assume the memoryless property of the failure time when a unit is operative and the other unit is in standby.

The signal flow graph is shown in Fig. 3. 2. In a similar way of deriving $q_{ij}(s)$ in the preceding section, we have

$$(3.12) \qquad q_{01}(s)=\int_0^\infty e^{-st}\bar{H}(t)\,dF(t)+\int_0^\infty e^{-st}\bar{F}(t)dH(t)$$

$$=\frac{\lambda}{s+\lambda}[1-h(s+\lambda)]+h(s+\lambda),$$

$$(3.13) \qquad q_{10}(s)=\int_0^\infty e^{-st}\bar{A}(t)dG(t),$$

$$(3.14) \qquad q_{12}(s)=\int_0^\infty e^{-st}\bar{G}(t)dA(t),$$

where $h(s)$ is the LS transform of $H(t)$.

We define $\varphi_i(s)$ $(i=0, 1)$, the LS transform of the distribution of the first time to system down starting from state s_i at $t=0$. From Mason's gain formula we have

$$(3.15) \qquad \varphi_0(s)=q_{01}(s)q_{12}(s)/[1-q_{01}(s)q_{10}(s)].$$

The mean time is given by

$$(3.16) \qquad \hat{T}_0=\left[\xi_1+\frac{1-h(\lambda)}{\lambda}\right]/[1-q_{10}(0)],$$

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where

$$\xi_1 = -\left.\frac{dq_{10}(s)}{ds}\right|_{s=0} - \left.\frac{dq_{12}(s)}{ds}\right|_{s=0}. \tag{3.17}$$

In particular we consider a case that $A(t)=1-\exp(-\lambda t)$, *i.e.*, the failure time of the operative unit always obeys $F(t)=1-\exp(-\lambda t)$ either the other unit is under repair or in standby. In the model of the preceding section we assume that $F(t)=1-\exp(-\lambda t)$. Then we have from (3.12), (3.13), and (3.14)

$$q_{01}(s) = \frac{\lambda}{s+\lambda}[1-h(s+\lambda)]+h(s+\lambda). \tag{3.18}$$

$$q_{10}(s) = q(s+\lambda), \tag{3.19}$$

$$q_{12}(s) = \frac{\lambda}{s+\lambda}[1-g(s+\lambda)], \tag{3.20}$$

where $g(s)$ is the LS transform of $G(t)$. The LS transform of the distribution of the first time to system down starting from state s_0 at $t=0$ is given by

$$\varphi_0(s) = q_{01}(s)q_{12}(s)/[1-q_{01}(s)q_{10}(s)], \tag{3.21}$$

and the mean time is given by

$$\hat{T}_0 = \frac{1}{\lambda} + \frac{1-h(\lambda)}{\lambda[1-g(\lambda)]}. \tag{3.22}$$

3.5. Special cases and discussions

We have already described the three advantages of our model in the first part of this chapter. In this section we shall study the three advantages and derive special cases. The three advantages are the following:

(i) In the actual situations we should consider failure of the unit in standby. Our model enjoys this situation. However, we note that the life of the new operative unit enjoys $F(t)$, regardless of how long it

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had been operating in standby since the moment of last repair completion. This assumption is not suitable. Thus we assume that

$$H(t)=1-\exp(-\lambda' t). \tag{3.23}$$

i.e., the failure time of the standby unit has the memoryless property. This model with the exponential failure time in standby is an interesting one and we can obtain the distribution of the first time to system down and the mean time $\hat{T}_0$ from (3.9) and (3.10). This model is a generalization of Gnedenko *et al.* [17] and Srinivasan [49].

Further we assume that the failure time of the operative unit obeys $F(t)=1-\exp(-\lambda t)$ if the other unit is in standby, and the failure time of the operative unit obeys an arbitrary distribution $A(t)$ if the other unit is under repair. Then we have the desired LS transform from (3.12), (3.13), (3.14), and (3.15). The mean time is given by

$$\hat{T}_0=\left[\xi_1+\frac{1}{\lambda+\lambda'}\right]/[1-q_{10}(0)]. \tag{3.24}$$

Finally, we consider a case that $A(t)=1-\exp(-\lambda t)$. Then we have from (3.21) and (3.22)

$$\varphi_0(s)=\frac{\dfrac{\lambda(\lambda+\lambda')}{(s+\lambda)(s+\lambda+\lambda')}g(s+\lambda)[1-g(s+\lambda)]}{1-\dfrac{\lambda+\lambda'}{s+\lambda+\lambda'}g(s+\lambda)}, \tag{3.25}$$

$$\hat{T}_0=\frac{1}{\lambda}+\frac{1}{(\lambda+\lambda')[1-g(\lambda)]}. \tag{3.26}$$

(ii) We shall give two special cases of the results (3.9) and (3.10).

One special case we shall consider is a two-unit standby redundant system with no standby failure. In this case, we set $H(t)\equiv 0$ ($\bar{H}(t)\equiv 1$) in (3.5)-(3.8). Then we have

$$q_{01}(s)=\int_0^\infty e^{-st}dF(t), \tag{3.27}$$

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$$q_{11}(s)=\int_0^\infty e^{-st}G(t)dF(t), \tag{3.28}$$

$$q_{12}(s)=\int_0^\infty e^{-st}\bar{G}(t)dF(t). \tag{3.29}$$

The LS transform of the distribution of the first time to system down starting from state s_0 at $t=0$ is given by

$$\varphi_0(s)=q_{01}(s)q_{12}(s)/[1-q_{11}(s)]. \tag{3.30}$$

The mean time is given by

$$\hat{T}_0 = \frac{1}{\lambda}+\frac{1}{\lambda[1-q_{11}(0)]}, \tag{3.31}$$

where

$$1/\lambda=\int_0^\infty tdF(t). \tag{3.32}$$

The results (3.35) and (3. 36) have already been obtained by Gnedenko *et al.* [17].

The other special case is a two-unit paralleled redundant system. If the failure time of the standby unit also obeys $F(t)$, *i.e.*, $F(t)\equiv H(t)$ in the first model, the model becomes a two-unit paralleled redundant system. However, we note that the failure time distribution $F(t)$ of the operative unit is independent of the time duration in the standby state. To make use of this [10, p. 411], we assume that $F(t)\equiv H(t)\equiv 1-\exp(-\lambda t)$. Then we have from (3.9) and (3.10)

$$\varphi_0(s) = \frac{2\lambda^2[1-g(s+\lambda)]}{(s+\lambda)[s+2\lambda-2\lambda g(s+\lambda)]}, \tag{3.33}$$

$$\hat{T}_0 = \frac{1}{\lambda}+\frac{1}{2\lambda[1-g(\lambda)]}, \tag{3.34}$$

where $g(s)$ is the LS transform of $G(t)$. The results can be easily derived from (3.21) and (3.22) (also from (3.15) and (3.16)). The results (3.33) and (3.34) agree with those obtained by Gaver [14].

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(iii) A two-unit paralleled redundant system was only analysed under the assumptions that the failure time distribution of the unit is exponential and the repair time distribution is arbitrary [14, 15]. We shall show that our model is an approximate model of a two-unit paralleled redundant system with arbitrary failure and repair time distributions. We note that the first model is analysed under the assumptions that $F(t)$, $G(t)$, and $H(t)$ are all arbitrary.

As an application of the first model, we consider a special case that the failure time of the operative unit obeys $F(t)=1-\exp(-\lambda t)$ if the other unit is *in standby*, the failure time of the standby unit obeys $H(t)=1-\exp(-\lambda t)$, and the failure time of the operative unit is $A(t)$ if the other unit is *under repair*. That is, we consider the second model by setting $F(t)\equiv H(t)\equiv 1-\exp(-\lambda t)$. Then we can obtain each $q_{ij}(s)$ and the LS transform from (3.12), (3.13), (3.14), and (3.15). The mean time is given by

$$\hat{T}_0=\left[\xi_1+\frac{1}{2\lambda}\right]/[1-q_{10}(0)], \tag{3.35}$$

where ξ_1 is given in (3.17).

Consider a two-unit paralleled redundant system. In the earlier analysis [14], though one unit is under repair, the other unit is assumed to obey the same distribution $F(t)=1-\exp(-\lambda t)$. In practical situations, if one unit is under repair, the operative unit may fail shorter than that two units are operative. If we apply the results $\varphi_0(s)$ and $\hat{T}_0$, we can obtain the results considering the situations. As an example, we assume that $A(t)$ is the modified extreme value distribution [4, p. 13] with the density

$$a(t)=\lambda\exp[-\lambda(e^t-1)+t], \tag{3.36}$$

where $1/\lambda$ is the mean failure time of a unit when two units are operative. That is, when two units are operative, each unit obeys $F(t)=1-\exp(-\lambda t)$, and when a unit is under repair, the remaining

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operative unit obeys $A(t)$. This example is an approximate model of a two-unit paralleled redundant system. We can consider the other model by assuming the suitable distributions $F(t)$, $G(t)$, and $H(t)$. This fact is of use to analyse or approximate the two-unit paralleled redundant model.

3.6. Dissimilar unit case

We have derived the LS transform for the simple model with two identical units. In this section we shall extend the model to one of dissimilar units. We shall consider the first model with arbitrary distributions $F_i(t)$, $G_i(t)$, and $H_i(t)$. Then we define the following four states of the model (where state s_0 is a starting point, state s_3 is an ending point, and states s_1 and s_2 are regeneration points):

State s_0; unit 1 begins to be operative and unit 2 begins to be in standby.

State s_1; unit 1 begins to be repaired and unit 2 begins to be operative.

State s_2; unit 1 begins to be operative and unit 2 begins to be repaired.

State s_3; two units are under repair or failure simultaneously. This state denotes the system down.

The signal flow graph of the system is shown in Fig. 3. 3. Each branch gain is given by

$$q_{01}(s)=\int_0^\infty e^{-st}[\bar{H}_2(t)*(1-H_2(t)*G_2(t))^{(-1)}]dF_1(t), \tag{3.37}$$

$$q_{03}(s)=\int_0^\infty e^{-st}[H_2(t)*\bar{G}_2(t)*(1-H_2(t)*G_2(t))^{(-1)}]dF_1(t), \tag{3.38}$$

$$q_{12}(s)=\int_0^\infty e^{-st}[G_1(t)*\bar{H}_1(t)*(1-H_1(t)*G_1(t))^{(-1)}]dF_2(t), \tag{3.39}$$

$$q_{13}(s)=\int_0^\infty e^{-st}[\bar{G}_1(t)*(1-H_1(t)*G_1(t))^{(-1)}]dF_2(t), \tag{3.40}$$

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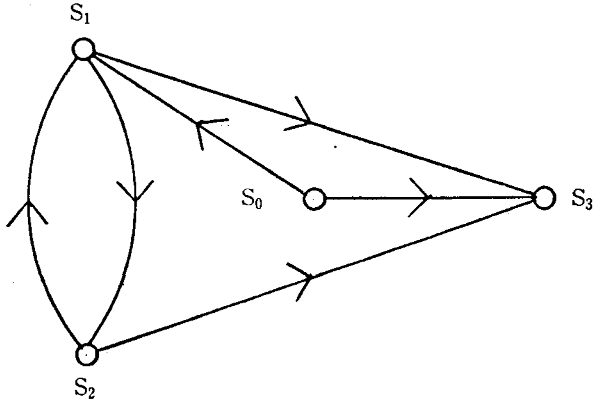

Fig. 3.3. Signal flow graph of the first model of dissimilar units.

(3.41) $$q_{21}(s)=\int_0^\infty e^{-st}[G_2(t)*\bar{H}_2(t)*(1-H_2(t)*G_2(t))^{(-1)}]dF_1(t),$$

(3.42) $$q_{23}(s)=\int_0^\infty [\bar{G}_2(t)*(1-H_2(t)*G_2(t))^{(-1)}]dF_1(t).$$

We also define $\varphi_i(s)$ $(i=0, 1, 2)$, the LS transform of the distribution of the first time to system down starting from state s_i at $t=0$.

Assuming that node s_0 is a source and node s_3 is a sink, we have from Mason's gain formula

(3.43) $$\varphi_0(s)=q_{03}(s)+\frac{q_{01}(s)q_{13}(s)+q_{01}(s)q_{12}(s)q_{23}(s)}{1-q_{12}(s)q_{21}(s)},$$

and the mean time is given by

(3.44) $$\hat{T}_0=\frac{1}{\lambda_1}+\frac{q_{01}(0)/\lambda_2+q_{01}(0)q_{12}(0)/\lambda_1}{1-q_{12}(0)q_{21}(0)},$$

where

(3.45) $$1/\lambda_i=\int_0^\infty t dF_i(t). \qquad (i=1,\ 2)$$

Second we shall consider a special case with the exponential failure

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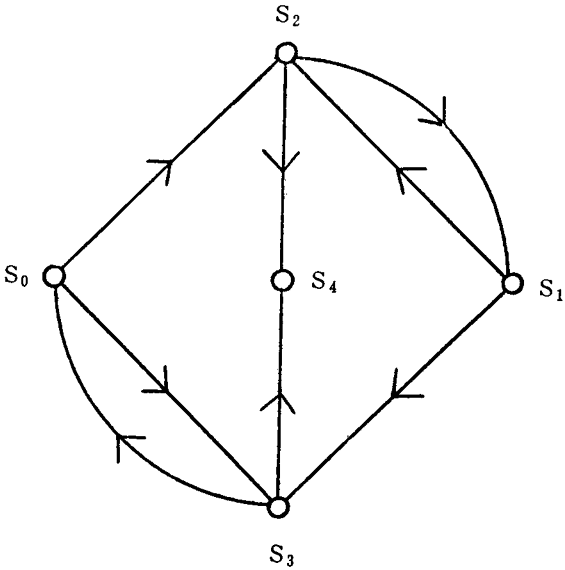

Fig. 3.4. Signal flow graph of the second model of dissimilar units.

time distributions of dissimilar units. That is, we assume that the failure time of the operative unit i (i=1, 2) obeys $F_i(t)=1-\exp(-\lambda_i t)$ if the other unit is *in standby*, the failure time of the operative unit i obeys an arbitrary distribution $A_i(t)$ if the other unit is *under repair*, and the remaining distributions $G_i(t)$ and $H_i(t)$ (i=1, 2) are arbitrary. Then we have the following five states of the model (where state s_4 is an ending point and the remaining states are regeneration points):

State s_0; unit 1 begins to be operative and unit 2 begins to be in standby.

State s_1; unit 1 begins to be in standby and unit 2 begins to be operative.

State s_2; unit 1 begins to be repaired and unit 2 begins to be operative.

State s_3; unit 1 begins to be operative and unit 2 begins to be repaired.

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State s_4; two units are under repair or failure simultaneously. This state denotes the system down.

The signal flow graph of the system is shown in Fig. 3. 4. Each branch gain is given by

$$q_{02}(s)=\int_0^\infty e^{-st}\bar{H}_2(t)dF_1(t)=\frac{\lambda_1}{s+\lambda_1}[1-h_2(s+\lambda_1)]. \tag{3.46}$$

$$q_{03}(s)=\int_0^\infty e^{-st}\bar{F}_1(t)dH_2(t)=h_2(s+\lambda_1), \tag{3.47}$$

$$q_{13}(s)=\int_0^\infty e^{-st}\bar{H}_1(t)dF_2(t)=\frac{\lambda_2}{s+\lambda_2}[1-h_1(s+\lambda_2)], \tag{3.48}$$

$$q_{12}(s)=\int_0^\infty e^{-st}\bar{F}_2(t)dH_1(t)=h_1(s+\lambda_2), \tag{3.49}$$

$$q_{24}(s)=\int_0^\infty e^{-st}\bar{G}_1(t)dA_2(t), \tag{3.50}$$

$$q_{21}(s)=\int_0^\infty e^{-st}\bar{A}_2(t)dG_1(t), \tag{3.51}$$

$$q_{34}(s)=\int_0^\infty e^{-st}\bar{G}_2(t)dA_1(t), \tag{3.52}$$

$$q_{30}(s)=\int_0^\infty e^{-st}\bar{A}_1(t)dG_2(t). \tag{3.53}$$

We also define $\varphi_i(s)$ $(i=0, 1, 2, 3)$, the LS transform of the distribution of the first time to system down starting from state s_i at $t=0$. In a similar way of deriving $\varphi_0(s)$ in (3.43), we have

(3. 54)

$$\varphi_0(s)=\frac{q_{02}(s)q_{24}(s)+q_{02}(s)q_{21}(s)q_{13}(s)q_{34}(s)+q_{03}(s)q_{34}(s)(1-q_{21}(s)q_{12}(s))}{1-q_{21}(s)q_{12}(s)-q_{03}(s)q_{30}(s)-q_{02}(s)q_{21}(s)q_{13}(s)q_{30}(s)+q_{21}(s)q_{12}(s)q_{03}(s)q_{30}(s)},$$

and the mean time can be also given by the signal flow graph method.

We can also consider a special case that $A_i(t)=1-\exp(-\lambda_i t)$ $(i=1, 2)$. That is, we assume that the failure time of the operative unit i $(i=1, 2)$ always obeys $F_i(t)=1-\exp(-\lambda_i t)$ either the other unit is under repair or in standby. The LS transform $\varphi_0(s)$ and the mean time $\hat{T}_0$ can be obtained from (3.46)-(3.53).

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Two special cases are easily derived. The first case is a two-unit standby redundant system with no standby failure. Setting $H_i(t)\equiv 0$ ($\bar{H}_i(t)\equiv 1$), we obtain the results given by Srinivasan [49]. The second case is a two-unit paralleled redundant system. Setting $F_i(t)\equiv H_i(t)\equiv 1-\exp(-\lambda_i t)$, the results given by Gaver [15] are obtained.

§4. A Two-Unit Standby Redundant System with Repair and Preventive Maintenance

4.1. Introduction

It is an important problem to operate a system in a specified long time without failure. We have known some policies to maintain a system. In particular, the following two policies are well-known:

(i) We make the system *redundant*.

(ii) We make the system *preventively maintainable*.

For the models using (i), a two-unit standby (or paralleled) redundant system is found in many fields and is well-known. The detailed discussion of such a system has been described in the preceding chapters. For the models using (ii), Barlow and Proschan [3, 4] have discussed as replacement problems. They have studied in detail a random replacement, an age replacement, a block replacement, and other replacement models.

In this chapter we shall consider a system which combines the above two policies. As a redundant model, we shall consider a two-unit standby redundant system with repair maintenance. As a preventive maintenance policy, we shall adopt a random one for an operative unit of the system. That is, an operative unit stops its operation after a time duration for the preventive maintenance. Combining the two policies just mentioned above, we call the system a *two-unit standby redundant*

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system with repair and preventive maintenance. Our concern for the system is the first time to system down.

First, we shall consider a system of two identical units. That is, we consider a system of two units in which the two units have the same statistical properties. Defining the states of the system, and focussing on the regeneration point of the failure or inspection time, we shall derive the Laplace-Stieltjes (LS) transform of the time distribution to first system down. The mean time will be also derived from it. We shall further show that the mean time derived here is greater than that of a two-unit standby redundant system with only repair maintenance under the suitable conditions.

Second, we shall consider a system of two dissimilar units. That is, we consider a system in which the statistical properties of the two units are different. In the same way we shall analyse the system by using the signal flow graph method.

4.2. Model

Consider a system of two identical unit (or subsystems). The failure time distribution of each unit is an arbitrary $F(t)$ and the repair time distribution is also an arbitrary $G(t)$. We assume that after the repair completion a unit recovers its function perfectly. We also assume that the switchover times from the failure to the repair, from the repair completion to the standby state, and from the standby state to the operative state of each unit are all instantaneous. The behavior of the system obeys the usual two-unit standby redundant system (see Gnedenko *et al.* [17, p. 329] and Srinivasan [49].)

Next we shall consider the preventive maintenance policy. When an operative unit goes to a specified time t and it is free from failure in that interval, the unit undergoes inspection as the preventive maintenance policy. We assume that the time distribution to the inspection is an arbitrary $A(t)$. The time distribution from the inspection to the inspection completion (or the preventive repair completion) is assumed

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to be an arbitrary $B(t)$. We assume that after inspection completion a unit recovers its function perfectly. We also assume that $G(t) \leq B(t)$ for all t so as to make the preventive maintenance policy effective. We shall further consider a special situation: When an operative unit goes to the inspection time before the repair completion of the other failed unit (or the inspection completion of the other unit under inspection), we make no inspection for an operative unit since the inspection of the operative unit yields the system down. That is, the inspection of an operative unit is only made if the other unit is in standby. We assume that the switchover times occurring in the inspection are all instantaneous. We also assume that all random variables are mutually independent and nonnegative. We should naturally assume that the failure time distribution of an operative unit has IFR (see Barlow and Proschan [5, p. 12]) so as to make the preventive maintenance policy effective.

Our concern for the system is the LS transform of the first time distribution to system down. We shall derive the LS transform.

4.3. Analysis

Consider the time instants of the failure or inspection of the units for the analysis of the system. We shall consider the following four states (which are the time instants of the system):

State s_0; one unit begins to be operative and the other is in standby.

State s_1; one unit begins to be operative instead of the other failed unit and the failed unit begins to be repaired.

State s_2; one unit begins to be operative instead of the inspection of the other unit and the inspection of the other unit begins.

State s_3; the two unit are under failure, inspection, or repair simultaneously, which state denotes the system down.

We shall consider the time distribution to first system down (*i.e.*, state s_3) starting from state s_0 at $t=0$. Then we shall consider each transition time distribution from one state to another.

In state s_0, we can consider the following two (exclusive and exhaus-

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tive) cases:

(i) An operative unit fails before the inspection time comes.

(ii) The inspection time of an operative unit comes before an operative unit fails.

In case (i) the system goes to state s_1. Its distribution becomes

$$Q_{01}(t)=\int_0^t \bar{A}(t)dF(t), \tag{4.1}$$

where $\bar{A}(t)=1-A(t)$ denotes the survival probability function. In general, the upper bar of the distribution denotes the survival probability function throughout this chapter. Applying the LS transforms for (4.1), we have

$$q_{01}(s)=\int_0^\infty e^{-st}\bar{A}(t)dF(t). \tag{4.2}$$

In case (ii) the system goes to state s_2. The LS transform of the time distribution from state s_0 to state s_2 becomes

$$q_{02}(s)=\int_0^\infty e^{-st}\bar{F}(t)dA(t). \tag{4.3}$$

In state s_1, we consider the following three (exclusive and exhaustive) cases:

(i) After the repair completion of a failed unit, an operative unit fails.

(ii) After the repair completion of a failed unit, the inspection time comes.

(iii) An operative unit fails before the repair completion of a failed unit.

In case (i) we can further consider the following two (exclusive and exhaustive) cases; (A) after the repair completion of a failed unit, an operative unit fails and that the inspection time does not come in that interval. Then its distribution becomes $\int_0^t \bar{A}(t)G(t)dF(t)$. (B) The inspection time comes before the repair completion of a failed unit. In this

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case the inspection is not made as we have described in Section 4. 2. Then the probability that the repair of a failed unit is completed up to time x after the inspection time comes is $\int_0^x A(y)dG(y)$. The time distribution that an operative unit fails after the repair completion (and that the inspection is not made) becomes $\int_0^t \left[\int_0^x A(y)dG(y)\right] dF(x)$. Thus we have the LS transform of the time distribution from state s_1 to state s_1 as follows:

$$(4.4) \qquad q_{11}(s)=\int_0^\infty e^{-st}\bar{A}(t)G(t)dF(t)+\int^\infty e^{-st}\left[\int_0^t \bar{A}(y)dG(y)\right]dF(t).$$

In case (ii), after the repair completion of a failed unit, the inspection time comes and it is free from failure of an operative unit in that interval. Then the system goes to state s_2. Its LS transform becomes

$$(4.5) \qquad q_{12}(s)=\int_0^\infty e^{-st}\bar{F}(t)G(t)dA(t).$$

In case (iii) the system goes to state s_3. Its LS transform becomes

$$(4.6) \qquad q_{13}(s)=\int_0^\infty e^{-st}\bar{G}(t)dF(t).$$

In state s_2, we can consider the following three (exclusive and exhaustive) cases:

(i) After the inspection is completed, the inspection time of an operative unit comes.

(ii) After the inspection is completed, an operative unit fails.

(iii) An operative unit fails before the inspection is completed.

In case (i), after the inspection is completed, the inspection time of an operative unit comes and it is free from failure of an operative unit in that interval. Then the system goes to state s_2. Its LS transform becomes

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(4.7) $$q_{22}(s)=\int_0^\infty e^{-st}\bar{F}(t)B(t)dA(t).$$

In case (ii) we can further consider the following two (exclusive and exhaustive) cases: (A) An operative unit fails after the inspection completion and that the inspection time does not come in that interval. (B) The inspection time of an operative unit comes before the inspection completion. In this case the inspection is not made so as to avoid the system down. Then the inspection is completed before an operative unit fails. In both cases (A) and (B), the system goes to state s_1. In a similar way of deriving (4.4), we have

(4.8) $$q_{21}(s)=\int_0^\infty e^{-st}\bar{A}(t)B(t)dF(t)+\int_0^\infty e^{-st}\left[\int_0^t A(y)dB(y)\right]dF(t).$$

In case (iii), the system goes to state s_3 (*i.e.*, the system down). Its LS transform becomes

(4.9) $$q_{23}(s)=\int_0^\infty e^{-st}\bar{B}(t)dF(t).$$

Thus we have all branch gains of the signal flow graph of the system in Fig. 4. 1, where each branch gain is given by (4.2)-(4.9). Assuming that state s_0 is a source and state s_3 is a sink in the graph, and applying Mason's gain formula, we have

(4.10)
$$\varphi_0(s)=\frac{q_{01}(s)q_{13}(s)[1-q_{22}(s)]+q_{01}(s)q_{12}(s)q_{23}(s)+q_{02}(s)q_{23}(s)[1-q_{11}(s)]+q_{02}(s)q_{21}(s)q_{13}(s)}{1-q_{11}(s)-q_{22}(s)+q_{11}(s)q_{22}(s)-q_{12}(s)q_{21}(s)},$$

which is the LS transform of the first time distribution to system down starting from state s_0 at $t=0$.

To prove that the above distribution is a proper one [11, p. 129], we should prove $\varphi_0(0)=1$. Then we should verify

(4.11) $$q_{01}(0)+q_{02}(0)=1,$$

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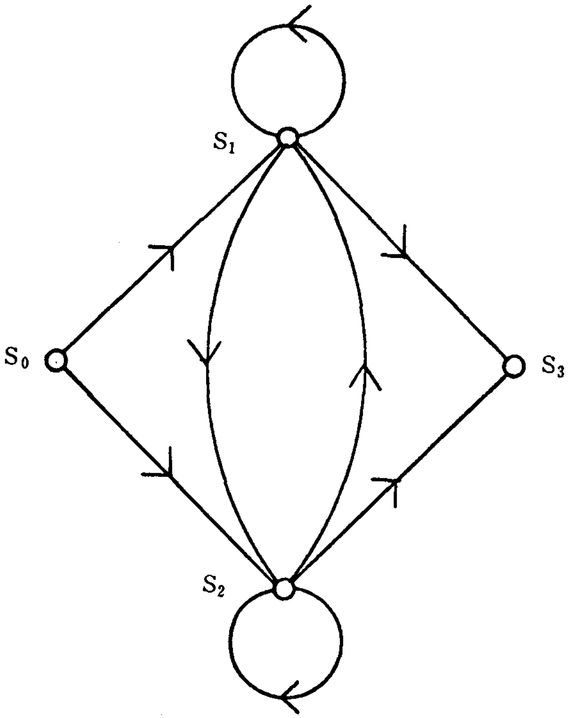

Fig. 4.1. Signal flow graph of the system of identical units.

(4. 12) $$q_{11}(0)+q_{12}(0)+q_{13}(0)=1,$$

(4. 13) $$q_{21}(0)+q_{22}(0)+q_{23}(0)=1.$$

As we have described in deriving $q_{ij}(s)$, we have considered all the possibilities in state s_i ($i=0$, 1, 2). Thus we have verified that (4.11), (4.12), and (4.13) hold. We can also verify analytically that (4.11), (4.12), and (4.13) hold but we omit the proof.

4.4. Mean time and discussions

In this section we shall derive the mean time to first system down.

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We shall further discuss some properties concerning the mean time. To simplify the notation, we introduce the notations

$$q_{ij}\equiv q_{ij}(0), \qquad (i,\ j=0,\ 1,\ 2,\ 3) \tag{4.14}$$

$$\xi_0=-\left.\frac{dq_{01}(s)}{ds}\right|_{s=0}-\left.\frac{dq_{02}(s)}{ds}\right|_{s=0}, \tag{4.15}$$

$$\xi_i=-\sum_{j=1}^{3}\left.\frac{dq_{ij}(s)}{ds}\right|_{s=0}. \qquad (i=1,\ 2) \tag{4.16}$$

Using the above notations (4.14), (4.15), and (4.16), we have the mean time to first system down by using Mason's gain formula (see Section 2. 4)

$$\hat{T}=\xi_0+\frac{[q_{01}(1-q_{22})+q_{02}q_{21}]\xi_1+[q_{02}(1-q_{11})+q_{01}q_{12}]\xi_2}{1-q_{11}-q_{22}-q_{12}q_{21}+q_{11}q_{22}}. \tag{4.17}$$

We have discussed a random preventive maintenance policy. We further consider an age preventive maintenance policy. In practical situations we should adopt an age preventive maintenance policy since it is suitable for the actual policy. The we assume that

$$A(t)=\begin{cases}0 & \text{for } t<t_0\\ 1 & \text{for } t\geq t_0.\end{cases} \tag{4.18}$$

In this case we introduce the following notations

$$\theta_1\equiv\int_0^\infty \bar{G}(t)dF(t), \qquad \theta_2\equiv\int_0^\infty \bar{B}(t)dF(t), \tag{4.19}$$

$$\beta_1\equiv F(t_0), \qquad \beta_2\equiv\bar{F}(t_0), \tag{4.20}$$

$$\gamma\equiv\int_0^{t_0}\bar{F}(t)dt, \tag{4.21}$$

$$1/\lambda\equiv\int_0^\infty\bar{F}(t)dt=\int_0^\infty tdF(t), \tag{4.22}$$

$$G\equiv G(t_0), \qquad B\equiv B(t_0). \tag{4.23}$$

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Using the above notations (4.19)-(4.23), we have for an age preventive maintenance policy

(4.24) $$\hat{T}_0 = \frac{1+\theta_1}{\lambda\theta_1} + \frac{(\theta_1+G)\{\theta_1\gamma-(\beta_1\theta_1+\beta_2\theta_2)/\lambda\}}{\theta_1\{\theta_1+\beta_2(G\theta_2-B\theta_1)\}},$$

where the first term of the above equation denotes the mean time without the preventive maintenance and the second term denotes the effect of the preventive maintenance.

As a special case of the system discussed in this chapter, we shall consider a two-unit standby redundant system with only repair maintenance. In this case we may only consider the states s_0, s_1, and s_3. We need not to consider a state s_2 since the inspection is not made. Each LS transform of the transition time distribution from one state to the other is given by

(4.25) $$q_{01}(s) = \int_0^\infty e^{-st} dF(t),$$

(4.26) $$q_{11}(s) = \int_0^\infty e^{-st} G(t) dF(t),$$

(4.27) $$q_{13}(s) = \int_0^\infty e^{-st} \bar{G}(t) dF(t).$$

These results can be also obtained by setting $A(t)\equiv 0$ ($\bar{A}(t)\equiv 1$) for all t in (4.2)-(4.9). We further note that these results is a special case of a two-unit standby redundant system of dissimilar unit, which has been given in Section 2. 5. The LS transform of the time distribution to first system down is given by

(4.28) $$\varphi_0(s) = q_{01}(s) q_{13}(s)/[1-q_{11}(s)],$$

where each $q_{ij}(s)$ is defined in (4.25)-(4.27). The mean time to first system down is given by

(4.29) $$\hat{T}_0 = \frac{1+\theta_1}{\lambda\theta_1}.$$

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The result (4.29) is equal to the first term of the right-hand side of equation (4.24) as is shown above. Thus the second term of equation (4.24) denotes the effect of the preventive maintenance policy.

We shall finally discuss the following theorem that the preventive maintenance policy is effective in the sense of the mean time.

Theorem 4. 1. The mean time (4.24) for the system with repair and preventive maintenance is greater than that of (4.29) for the system with only repair maintenance on the assumptions that the failure rate $r(t)$ of the failure time distribution is strictly incresing and there exists a $t_0{}^*$ such that

$$r(t_0{}^*)=\lambda\theta_1/(\theta_1-\theta_2), \tag{4.30}$$

and that we adopt a suitable inspection interval t_0.

Proof. To prove the theorem, we should verify that the second term of the right-hand side of equation (4.24) is positive on the above assumptions. We should only consider the second term of the right-hand side of equation (4.24). We shall only show that the denominator and the numerator of the second term are both positive on the above assumptions. It is evident from (4.19) and (4.23) that θ_1 and θ_1+G are both positive. The brackets of the denominator become

$$\begin{aligned}&\theta_1+\beta_2(G\theta_2-B\theta_1)\\&=(1-\bar{F}(t_0)B(t_0))\theta_1+\bar{F}(t_0)G(t_0)\theta_1>0,\end{aligned} \tag{4.31}$$

from $\bar{F}(t_0)<1$ and $B(t_0)<1$. Define $p(t_0)$ by the brackets of the numerator, which is a function of t_0. Then we have

$$\begin{aligned}p(t_0)&=\theta_1\gamma-(\beta_1\theta_1+\beta_2\theta_2)/\lambda\\&=\theta_1\int_0^{t_0}\bar{F}(t)dt-\{(1-\bar{F}(t_0))\theta_1+\bar{F}(t_0)\theta_2\}/\lambda\\&=(\theta_1-\theta_2)\bar{F}(t_0)/\lambda+\theta_1\int_{t_0}^{\infty}\bar{F}(t)dt.\end{aligned} \tag{4.32}$$

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For $p(t_0)$, we have $p(0)=-\theta_2/\lambda<0$, $p(\infty)=0$. Differentiating $p(t_0)$ with respect to t_0, we have

$$\text{(4.33)} \qquad \frac{dp(t_0)}{dt}=-(\theta_1-\theta_2)f(t_0)/\lambda+\theta_1\bar{F}(t_0)$$
$$=\bar{F}(t_0)\{\theta_1-(\theta_1-\theta_2)r(t_0)/\lambda\},$$

where $f(t_0)=dF(t_0)/dt_0$ and $r(t_0)=f(t_0)/\bar{F}(t_0)$. By using the assumptions that $r(t_0)$ is an increasing function of t_0 (*i.e.*, $r(t_0)\uparrow$ as $t_0\uparrow$), we can show that there exists a $t_0{}^*$ such that $dp(t_0)/dt_0=0$, that is,

$$\text{(4.34)} \qquad r(t_0{}^*)=\frac{\lambda\theta_1}{\theta_1-\theta_2},$$

where $\theta_1-\theta_2>0$ if $B(t)\geq G(t)$ and $B(t)\not\equiv G(t)$ for all t. Using also $p(0)<0$, $p(\infty)=0$, and $p(t_0)$ is a unimodal function of $t_0>0$, there exists a $\hat{t}_0$ such that $p(\hat{t}_0)=0$. Thus, if we choose a $t_0>\hat{t}_0$, we have $p(t_0)>0$. That is, the second term of the right-hand side of equation (4.24) is positive if we choose a $t_0(>\hat{t}_0)$, which proves the theorem.

4.5. Dissimilar unit case

In this section we shall further consider a two-unit standby redundant system with repair and preventive maintenance in which the two units are different in their statistical properties. We shall simply describe the necessary definitions of the system.

The two units can be labeled by the integers $i=1, 2$. The failure time distribution of unit i ($i=1, 2$) is an arbitrary $F_i(t)$ and the repair time distribution of unit i is also an arbitrary $G_i(t)$. The time distribution from the beginning of the operation to the inspection of the operative unit i is also an arbitrary $A_i(t)$. The time distribution from the inspection to the inspection completion (or the preventive repair completion) of unit i under inspection is an arbitrary $B_i(t)$. We also assume that all random variables are mutually independent and nonnegative. The same assumptions of the system are imposed as is described in Section 4. 2.

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For example, we should assume that $F_i(t)$ $(i=1, 2)$ has IFR, $B_i(t)\geq G_i(t)$ $(i=1, 2)$, and so on.

Consider the time instants of the failure or inspection of the units for the analysis of the system. We shall consider the following six states (which are the time instants of the system):

State s_0; unit 1 begins to be operative and unit 2 begins to be in standby.

State s_1; unit 2 begins to be operative instead of the failed unit 1 and the failed unit 1 begins to be repaired.

State s_2; unit 1 begins to be operative instead of the failed unit 2 and the failed unit 2 begins to be repaired.

State s_3; unit 2 begins to be operative instead of the other unit 1 and the inspection of unit 1 begins.

State s_4; unit 1 begins to be operative instead of the other unit and the inspection of unit 2 begins.

State s_5; the two units are under failure, inspection, or repair simultaneously, which state denotes the system down.

We shall consider each LS transform of the transition time distribution from one state to the other. Each LS transform is derived in a similar way of the system of identical units. Then we have

(4.35) $$q_{01}(s)=\int_0^\infty e^{-st}\bar{A}_1(t)dF_1(t),$$

(4.36) $$q_{03}(s)=\int_0^\infty e^{-st}\bar{F}_1(t)dA_1(t),$$

(4.37) $$q_{12}(s)=\int_0^\infty e^{-st}\bar{A}_2(t)G_1(t)dF_2(t)+\int_0^\infty e^{-st}\left[\int_0^t A_2(y)dG_1(y)\right]dF_2(t),$$

(4.38) $$q_{14}(s)=\int_0^\infty e^{-st}\bar{F}_2(t)G_1(t)dA_2(t),$$

(4.39) $$q_{15}(s)=\int_0^\infty e^{-st}\bar{G}_1(t)dF_2(t),$$

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(4.40) $$q_{21}(s)=\int_0^\infty e^{-st}\bar{A}_1(t)G_2(t)dF_1(t)+\int_0^\infty e^{-st}\left[\int_0^t A_1(y)dG_2(y)\right]dF_1(t),$$

(4.41) $$q_{23}(s)=\int_0^\infty e^{-st}\bar{F}_1(t)G_1(t)dA_2(t),$$

(4.42) $$q_{25}(s)=\int_0^\infty e^{-st}\bar{G}_2(t)dF_1(t),$$

(4.43) $$q_{34}(s)=\int_0^\infty e^{-st}\bar{F}_2(t)B_1(t)dA_2(t),$$

(4.44) $$q_{32}(s)=\int_0^\infty e^{-st}\bar{A}_2(t)B_1(t)dF_2(t)+\int_0^\infty e^{-st}\left[\int_0^t A_2(y)dB_1(y)\right]dF_2(t),$$

(4.45) $$q_{35}(s)=\int_0^\infty e^{-st}\bar{B}_2(t)dF_1(t),$$

(4.46) $$q_{43}(s)=\int_0^\infty e^{-st}\bar{F}_1(t)B_2(t)dA_1(t),$$

(4.47) $$q_{41}(s)=\int_0^\infty e^{-st}\bar{A}_1(t)B_2(t)dF_1(t)+\int_0^\infty e^{-st}\left[\int_0^t A_1(y)dB_2(y)\right]dF_1(y),$$

(4.48) $$q_{45}(s)=\int_0^\infty e^{-st}\bar{B}_1(t)dF_2(t).$$

The signal flow graph of the system is demonstrated in Fig. 4. 2, where each branch gain is given in (4.35)-(4.48).

Thus, assuming that state s_0 is a source and state s_5 is a sink, and applying Mason's gain formula, we have immediately

(4.49) $$\varphi_0(s)=N/D,$$

where

(4.50) $$\begin{aligned}D=1&-q_{12}(s)q_{21}(s)-q_{14}(s)q_{41}(s)-q_{34}(s)q_{43}(s)-q_{23}(s)q_{32}(s)\\&+q_{12}(s)q_{21}(s)q_{34}(s)q_{43}(s)+q_{14}(s)q_{41}(s)q_{23}(s)q_{32}(s)\\&-q_{12}(s)q_{23}(s)q_{34}(s)q_{41}(s)--q_{21}(s)q_{14}(s)q_{43}(s)q_{32}(s),\end{aligned}$$

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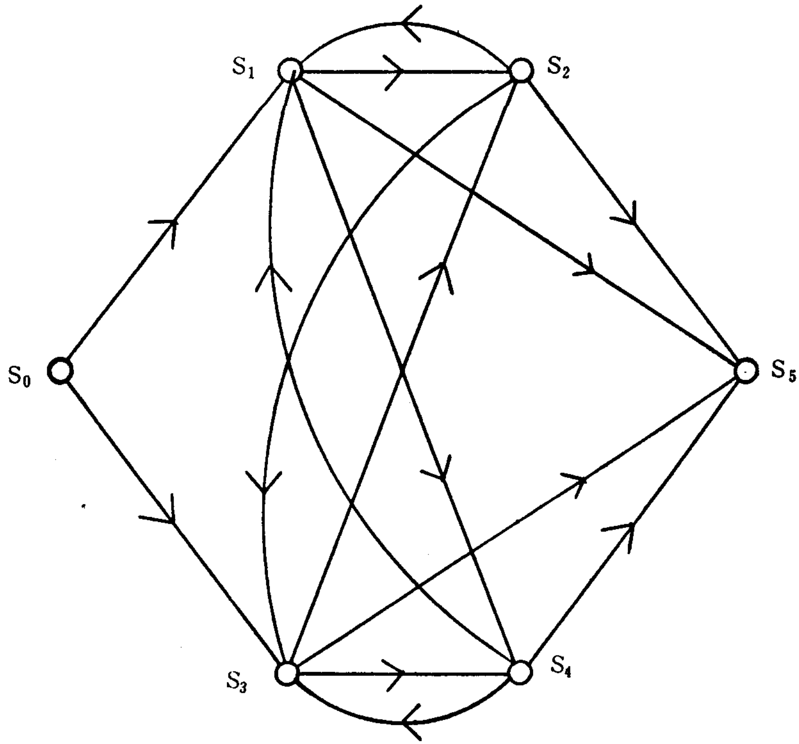

Fig. 4.2. Signal flow graph of the system of dissimilar units.

$$N=q_{01}(s)q_{12}(s)q_{25}(s)[1-q_{34}(s)q_{43}(s)]+q_{01}(s)q_{12}(s)q_{23}(s)q_{34}(s)q_{45}(s)$$
$$+q_{01}(s)q_{15}(s)[1-q_{34}(s)q_{43}(s)-q_{23}(s)q_{32}(s)]+q_{01}(s)q_{12}(s)q_{23}(s)q_{35}(s)$$
$$+q_{01}(s)q_{14}(s)q_{45}(s)[1-q_{23}(s)q_{32}(s)]$$
$$+q_{03}(s)q_{34}(s)q_{45}(s)[1-q_{12}(s)q_{21}(s)]+q_{03}(s)q_{34}(s)q_{41}(s)q_{12}(s)q_{25}(s)$$
$$+q_{03}(s)q_{35}(s)[1-q_{12}(s)q_{21}(s)-q_{14}(s)q_{41}(s)]+q_{03}(s)q_{34}(s)q_{41}(s)q_{15}(s)$$
$$+q_{03}(s)q_{32}(s)q_{25}(s)[1-q_{14}(s)q_{41}(s)]. \tag{4.51}$$

We have obtained $\varphi_0(s)$, the LS transform of the first time distribution to system down. The mean time $\hat{T}_0$ can be easily obtained by using Mason's gain formula (see Section 2. 4). The similar result of Theorem

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4. 1 will hold for the system under the suitable assumptions, but we omit the form of the theorem here.

§5. Conclusion

In this paper we have described some systems arising in reliability theory and obtained the LS transform of the first time distribution to system down and the mean time by using Markov renewal processes. A Markov renewal process, which is a marriage of renewal processes and Markov chains, is one of the most powerful mathematical tools for analyzing systems. The signal flow graph method used throughout this paper is of great interest for system designers and engineers. We cannot understand at a glance a large-scale and complicated system. The signal flow graph method makes us suggestive and helpful, and the required quantities can be obtained by using Mason's gain formula which is an easy mechanical procedure.

In Chapter 2 we have discussed the relationship between Markov renewal processes and signal flow graphs. The relationship between continuous time Markov processes and signal flow graphs have been discussed by Tin Htun [52] and Dolazza [7]. So far as we know, we have found no paper describing the relationship between Markov renewal processes and signal flow graphs. Markov renewal processes are of great use for the analysis in system science since the processes are generalizations of Markov processes and renewal processes, and have the fruitful results [45, 46]. We believe that the results obtained in this paper are of great use and may be applicable to many other fields.

The signal flow graph approach is intuitive and obtaining the required quantities is an easy mechanical procedure from Mason's gain formula. Thus, the LS transform of the first passage time distribution and the

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mean time can be automatically obtained if we can give the signal flow graph and its associated branch gains.

In Chapter 3 we have discussed a two-unit standby redundant system with standby failure. In the earlier analysis for the system it is assumed that a standby unit never fails in the standby interval. Our model in Chapter 3, however, is assumed that a standby unit may fail in the standby interval. Our model considered in Chapter 3 has three advantages (see Section 3. 5). For the system we have obtained the LS transform of the first time distribution to system down under the most generalized assumptions that all distributions are arbitrary. Thus our results may be applicable in the actual fields.

In Chapter 4 we have considered a two-unit standby redundant system with repair and preventive maintenance. For the system we have obtained the LS transform of the time distribution to first system down and its mean time. We have further shown that the preventive maintenance policy is effective in the sense of the mean time under the suitable assumptions. For the failure, repair, and inspection time distributions, we have assumed arbitrary distributions. Thus, our results obtained in Chapter 4 are available by assuming suitable distributions.

In a recent paper, Mine and Asakura [26] have discussed a multiple-unit standby redundant system with repair and preventive maintenance. They have derived the LS transform of the time distribution to the first emptiness and the mean time under the assumptions that the repair and the inspection time distributions are exponential. In Chapter 4, we have derived the LS transform under the assumptions that all distributions are arbitrary, where we have considered a two-unit standby redundant system.

In many fields we may use a two-unit standby redundant system. In this situations, if the failure time distribution has IFR, we should adopt the preventive maintenance policy. Then Theorem 4. 1 states that the preventive maintenance policy is effective under the suitable assumptions. In the actual situations, these assumptions may be satisfied.

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In Theorem 4. 1, we have adopted an age maintenance policy. However, we believe that a random preventive maintenance policy (which includies an age one) is effective under the suitable assumption of the random inspection distribution.

In the rest of the Conclusion, we shall simply describe further problems of system reliability analysis. Reliability analysis of redundant repairable systems has many fruitful studies. This paper discussed only simple models and we restricted our attention to the first passage times. In the actual situations we should consider more complicated models and further discuss the mixed configurations of the models. Our concerns are also extended to not only the first passage times but also the transition probabilities, the limiting probabilities, *etc.*

We did not consider the factors of costs, weight, capacities, etc., which were associated with the model, for the analysis of redundant systems. In the actual situations such factors will be imposed. We should consider the optimization problems of attaining the maximal reliability subject to the suitable constraints on such factors.

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