

**A SERIALIZED LINEAR PROGRAMMING MODEL FOR  
OBTAINING SUCCESSIVE MINIMAX STRATEGIES**

**GRACE J. KELLEHER**

*Southern Methodist University*

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**Abstract**

Some problems need to be reduced to successively smaller dimensions to obtain a minimax (maximin) set solution. Such problems typically involve the redistribution of inventories, funds or populations to achieve minimax (maximin) densities over a total set while conforming to stated constraints. This paper presents an algorithm for obtaining such solutions through serial application of a linear program model to successively smaller subsets of the problem. The model is described and its use illustrated with results of a prototype study.

**Introduction**

In tackling what may appear to be a simple minimax (or miximin) problem, an analyst may find that in his initial solution in fact does

not accomplish the objective involved in the problem. Such was the case when the author attempted to optimize the redistribution of an urban population during emergencies in order to reduce their vulnerability to a randomly distributed threat. Such a threat might be posed, for example, by a hurricane that is moving toward a region by a rather erratic path or by an expected enemy attack in which population would be targeted, directly or indirectly.

### 1. The Problem

The algorithm to be presented was developed in the course of developing a minimax solution to the following problem. It is assumed that urban populations within a specific region of the United States will be evacuated to peripheral low-density areas in the event their lives are threatened if they remain in their home cities. The cities involved and their total populations are identified. Qualified reception areas and their permanent populations also are identified. The objective is to redistribute the urban populations among the reception areas in a manner that will minimax the resultant population density per square mile. The principal constraint is that no one will be required to move more than some number ( $M$ ) miles from their home city.

The prototype study involved a region encompassing eight urbanized areas of the United States. Fifteen (15) reception areas were identified, each located within  $M$  miles of one or more of the urbanized areas. The allocation problem, simply stated is to determine how many evacuees from each urbanized area should be assigned to which reception areas. As shown in the following table, as many as four urbanized areas qualified as contenders for a single reception area.

### 2. Notation

The following notation was applied in the model used to solve this problem :

$i=1, 2, \dots, M$ =index of cities to be evaluated.

$j=1, 2, \dots, N$ =index of identified reception areas.

$\bar{E}_i$ =number of residents to be evacuated from the  $i$ -th city.

$\bar{R}_j$ =pre-evacuation residents of the  $j$ -th reception area.

$\bar{A}_j$ =inhabitable area (in square miles) of the  $j$ -th reception area.

$\delta_{ij}$ =the movement distance criterion function where:

$\delta_{ij}=1$  if reception area  $j$  may receive evacuees from the  $i$ -th city (*i.e.*, the movement distance criterion is satisfied) and

$\delta_{ij}=0$  in all other cases.

$E_{ij}$ =evacuees from the  $i$ -th city who are assigned to the  $j$ -th reception area.

$V$ =the maximum post-evacuation population density of any reception area.

The first condition applied in the model is that the number of evacuees from a given city assigned to that city's reception areas sum to the total number of residents who must be evacuated from that city. In our notation:

$$\sum_{j=1}^N \delta_{ij} E_{ij} = \bar{E}_i \text{ for } i=1, 2, \dots, M.$$

Second, since  $V$  is the maximum post-evacuation population density of any reception area, it must, by definition, equal or exceed the actual density achieved in each reception area after evacuation including both evacuees and pre-evacuation residents. In symbols this condition appears as:

$$V - \sum_{i=1}^M \frac{\delta_{ij} E_{ij}}{\bar{A}_j} \geq \frac{\bar{R}_j}{\bar{A}_j} \text{ for } j=1, 2, \dots, N.$$

In addition to these restrictions we have the self-evident fact that neither the maximum density,  $V$ , nor the numbers of evacuees moving from cities to reception areas,  $E_{ij}$ , may be negative. That is:

**Table 1**  
Solutions  $(X_{i,j})$  Required for Regional Allocation Plan  
 $X$  = Number of Evacuees

The Reception Area $j=$	From Urbanized Area, $i=$							
	1	2	3	4	5	6	7	8
1	X							
2					X			
3						X		
4							X	
5								X
6	X	X						
7	X	X	X	X				
8	X		X					
9	X		X	X				
10			X	X				
11			X	X	X			
12				X	X			
13					X	X		
14						X	X	
15							X	X

$$V \geq 0 \text{ and } E_{ij} \geq 0 \text{ for } i=1, 2, \dots, M \text{ and } j=1, 2, \dots, N.$$

The objective suggested by the givens of the problem is to minimize  $V$ , the maximum population density of any reception area after evacuation. This, in fact, is the objective of this connected series of linear programming problems.

The solution to the first problem, however, is only a partially satisfactory assignment of evacuees to potential reception areas. Although the maximum population density of any reception area,  $V$ , has been minimized, the assignments are characteristically haphazard in one important respect. Among those reception areas which have not been filled to the density  $V$ , a great deal of reassigning may be possible which does

not alter the minimum value of  $V$ . The first problem, in short, does not usually possess a unique solution.

The second linear programming problem is designed to resolve this degeneracy in the assignment of evacuees to partially filled reception areas in a manner consistent with the original minimax objective. This is done by removing, in effect, potential reception areas which are filled and those evacuees who have been assigned to these areas. With this change in the problem the new objective becomes the minimization of the maximum post-evacuation density of any of the remaining reception areas.

Again, a degeneracy in the solution may appear. And this degeneracy is resolved with yet a third linear program. This process of solving linear programming problems and eliminating reception areas and evacuees continues until every evacuee and reception area is accounted for. The evacuation assignments which are ultimately obtained minimize an ordered (highest to lowest) series of maximum reception area densities.

**Table 2**  
Problem Dimensions

Problem No.	No. Reception Areas Involved	No. Urbanized Areas Involved
1	15	8
2	12	7
3	10	6
4 <sup>a</sup>	5	4

<sup>a</sup> This was the final solution required. It allocated the evacuees of 2 contending urbanized areas among 3 reception areas. Each of the 2 remaining urbanized areas qualified for only one of the 2 remaining reception areas and was assigned accordingly.

### **3. The Linear Programs**

The first linear programming problem is obtained directly from the initial set of restrictions. It is:

Minimize:  $V$

Subject to:  $\sum_{j=1}^N \delta_{ij} E_{ij} = \bar{E}_i \quad i=1, 2, \dots, M$

$$V - \sum_{i=1}^M \frac{\delta_{ij} E_{ij}}{\bar{A}_j} \geq \frac{\bar{R}_j}{\bar{A}_j} \quad j=1, 2, \dots, N$$

and:  $V \geq 0, E_{ij} \geq 0$  for  $i=1, 2, \dots, M$  and  
 $j=1, 2, \dots, N$ .

The second linear programming problem is derived from the first by using the solution to the first problem's dual program. If the dual price associated with a constraint is positive<sup>1</sup>, that constraint is binding. When a binding constraint is identified in this manner, the solution values of the variables  $E_{ij}$  which appear in it are saved but the constraint is omitted from the next problem. (This permits calculation of a lower value of  $V$  in solving the next smaller subset of the problem). Dual prices of zero indicate constraints which are not binding. These constraints remain and the solution values of the variables which appear in them are recomputed by solving the second problem. This second problem is identical in structure to the first, however, several indices and variables have slightly different definitions. These are:

$i=1, 2, \dots, M$  = index of cities whose evacuations were not determined in the solution to the first problem.

$j=1, 2, \dots, N$  = index of remaining potential reception areas.

$V$  = the maximum post-evacuation population density of any remaining reception area.

The solution to the dual of the second problem is employed in the construction of the third, and the third dual solution in the construction of the fourth, et cetera, until all constraints and variables are eliminated. The final assignment of evacuees to reception areas is constructed from

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<sup>1</sup> In the case of a reception area, this tells us that assignment of another evacuee to that area would cause us to exceed the minimax value established for  $V$ .

the solution values of the variables  $E_{ij}$  which were saved when the constraints were eliminated.

An evacuation assignment based on the solutions to the above series of linear programming problems is unique\*. Moreover, it has an attractive property shared by no other feasible assignment of evacuees to reception areas. It is impossible to reassign evacuees in such a manner that the post-evacuation population density of any given reception area can be decreased without increasing the density of another reception area which is already *as densely or more densely filled* than the reception area whose density is being decreased.

#### 4. Prototype Results

In the prototype study, the unique solution was achieved by solving a series of 4 problems, each a subset of the original problem. Thus  $V$  took on 4 successively lower values.

#### 5. Other Applications

The linear programming approach described in section 3 undoubtedly has many useful applications. An example is the redistribution of inventories from one set of existing supply points to another with the objective of minimaxing stockage at any one location. The constraints of the problem could be total cost, maximum transshipment distance, number of customers to be supported, etc.<sup>1</sup>

#### REFERENCES

- [1] Criffiths, D.O., "The Use of Regression Analysis in a Depot Location Exercise," *Applied Statistics*, 17, 1 (1968) 57-63.

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<sup>1</sup> An analog to this problem, the *initial* location of warehouses or depots to minimize total time required to satisfy projected customer demands, is solved via regression analysis in [1].

\* Generally this is not unique. (by the referee)