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A STATISTICAL ANALYSIS OF PRODUCTION SCHEDULING SYSTEMS

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Abstract

This paper is concerned with the scheduling of a number of jobs to be processed on various machines. Due to its combinatorial nature, economical solutions to most scheduling problems have not yet been attained, even though computational abilities are increasing concomitantly with the development of more powerful and faster computers. A statistical analysis in which the original problem is decomposed into a series of smaller subproblems, has been carried out. Each of these problems is solved, and their solutions are combined together to form a decomposition solution of the original problem. Considerable experimentation has been conducted to study the statistical effects of decomposition solutions. The hypotheses which have been tested are: (1) the distribution of the schedule times shifts toward the optimal as the number of jobs in each subproblem, K, increases; (2) the decomposition solution is improved as K increases; and (3) the computation time required to obtain a decomposition solution decreases as K increases. These hypotheses are verified using a statistical test.

Introduction

The scheduling problem involves the selection of a sequence of jobs to be processed on various machines such that a certain criterion is optimized. This paper is concerned with the minimization of the schedule time as a criterion. The schedule time is the total time required to process the jobs on all machines. The scheduling problem is of special interest because of the large amount of computational effort invoved to obtain the optimal solution. One can appreciate the size and complexity of the problem by considering the fact that there are $(J!)^M$ sequences for a problem having J jobs to be performed on M machines. For example, a problem of six jobs and three machines has $(6!)^3$ or 373,248,000 sequences. A complete enumeration of these sequences would require a prohibitive amount of computer time. Many of these sequences; however, are not technologically feasible. Thus, the non-feasible sequences must be eliminated; and those which are feasible must be evaluated so that the sequence(s) with the minimum schedule time can be found.

Considerable research has been done in the study of the sequencing problem. Excellent expositions appear in [4,] [8], [9]. The purpose of most of the research is to develop efficient algorithms for arriving at optimal solution. No practical procedure has yet been developed that guarantees an optimal solution except for problems involving very few jobs and machines, even though computational abilities are increasing concomitantly with the development of more powerful and faster computers.

Progress has been made toward approximate computational procedures to solve this problem. Combinatorial analysis [2], [3], [10], mathematical programming [6], and heuristics [5], among other approaches, have been proposed. The efficiency of those approaches which represent quasi-enumeration techniques, depend on how effectively enumeration is curtailed.

Since most practical problems are too large to be solved economically by the existing procedures, a statistical analysis has been carried out.

In this analysis the original problem is decomposed into a series of subproblems. Each of these subproblems is then solved, and their solutions are combined together to form a decomposition solution of the original problem. To study the statistical effects of decomposition, considerable experimentation has been conducted.

Statistical Analysis

Consider a set of J jobs, $s=\{j|1\leq j\leq J$ and $J \neq$ a prime number$\}$, decompose the set s, in all possible ways, into G subsets J_g, $1\leq g\leq G$ such that the subset consists of K jobs with $K=J/G$ where K and G are positive integers. For each subset, the subsequence which minimizes the schedule time is obtained. The matched subsequences are next obtained in all possible ways to form different sequences over all J jobs. These sequences are then evaluated to obtain the corresponding schedule times. The minimum of these schedule times is the decomposition solution of the set s. Several decomposition solutions may be obtained as K changes such that $1<K<J$. Each of these solutions will be referred to as a decomposition solution with K equaling a certain number of jobs.

Obviously, the J jobs are scheduled simultaneously when the number of subsets equals the number of jobs or G equals J and, consequently, each subset consists of one job, or K equals one. The solution obtained in this case will be referred to as a complete or partial enumeration solution. It is necessary to distinguish between these two solutions. The complete enumeration solution, which is optimal, is obtained by enumerating and evaluating the complete set of sequences. However, the partial enumeration solution is obtained by generating and evaluating a subset of all sequences. In the experiments reported in this paper, complete enumeration is applied to flow-shop problems having up to six jobs, Partial enumeration is applied to flow-shop problems consisting of more than six jobs and job shop problems of any size.

In measuring the effectiveness of the different decomposition solutions, several aspects are considered. One aspect is to study the schedule time distributions obtained by complete or partial enumeration and decomposition. It has been shown in [7] that the schedule times obtained from scheduling the J jobs simultaneously is asymptotically normally distributed for a large number of jobs.

As mentioned before, any schedule time over a set of J jobs obtained by decomposition is a result of combining the matched subsequences having the minimum schedule times. Consequently, the other subsequences are discarded. Thus, one would conjecture that the distribution of the combined schedule times shifts toward the minimum value. Furthermore, in fitting the schedule times of the matched subsequences within each arrangement of J jobs, there are idle times on the various machines between each two successive subsets. The amount of these idle times decreases and hence the combined schedule times become shorter as the number of subsets decreases and, consequently, as the number of jobs in each subset increases. An implication of the above is that the amount of distribution shift becomes larger as the number of jobs in each subset, K, increases where $1<K<J$. In the extreme case when all J jobs are in one subset or G equals one and, consequently, K equals J the schedule time distribution becomes a point distribution with all the weight at the minimum value. Figure 1 shows the con-

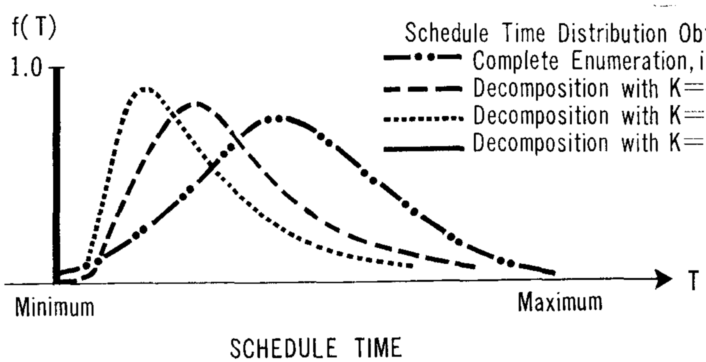

Fig. 1. Conjectured Schedule Time Distribution of Various Solutions to a Problem of Six Jobs.

jectured schedule time distribution obtained by complete enumeration (or, decomposition with $K=1$), decomposition with $K=2$, decomposition with $K=3$, and decomposition with $K=6$ for a problem of six jobs and an arbitrary number of machines.

According to the above reasoning one would also conjecture that the minimum schedule times obtained by decomposition with different K's, would be optimal or near-optimal. Moreover, the decomposition solutions would be improved as the number of jobs in each subset, K, increases. Therefore, the relative efficiencies of the various decomposition solutions are investigated. A relative efficiency is defined in this paper as the quotient of the complete or partial enumeration solution and the decomposition solution. This will clearly show the number of decomposition solutions that are strictly equal to the complete or partial enumeration solutions which may be obtained. One can also ascertain how close the decomposition solutions are to those obtained by complete or partial enumeration.

Furthermore, to obtain the decomposition solution with K equaling a certain number of jobs, $J!/(K!)^G$ arrangements in which a set of J jobs can be decomposed into G subsets, each having K jobs, are evaluated. The number of these arrangements decreases as the number of jobs in each subset increases. For example, in a flow show problem of six jobs, the number of all sequences $6!$ or 720; however, there are $6!/(2!)^3$ or 90 arrangements of six jobs involving three subsets, each having two jobs. The number of arrangements is reduced to $6!/(3!)^2$ or 20 when the number of jobs in each subset is three. As a result, one would conjecture that the computation time required to obtain a decomposition solution decreases as the number of jobs in each subset increases.

To summarize, the hypotheses to be tested may be stated as follows:
Hypothesis I: The mean of the schedule times obtained by complete or partial enumeration is greater than that obtained by decomposition. Moreover, the mean of the schedule times obtained by decomposition increases as the number of jobs in each subset decreases. In other

words, the mean of the schedule times shifts toward the minimum as K increases.

Hypothesis II: The minimum schedule times obtained by decomposition are equal or close to the optimal value. Furthermore, the decomposition solutions are improved as the number of jobs in each subset increases.

Hypothesis III: The computer time required to obtain a decomposition solution decreases as the number of jobs in each subset increases.

If the above hypotheses are verified, one would sample from one of the distributions obtained by decomposition. For a particular sample size, the shift of the distributions obtained by decomposition should increase the probability of determining an optimal solution. An obvious corrollary to the above would be that for a given probability of obtaining an optimal solution one should require a smaller sample size when sampling from the complete or partial enumeration distribution. Consequently, the computational effort required should be significantly reduced.

Experimental Investigation

For testing the hypotheses stated in the preceding section, considerable computational experiments were conducted. The number of problems solved was 128 selected with six to 40 jobs and three to ten machines. In Experiments I and II, the entries of the processing time matrices were generated uniformly between one and 30, inclusive. Different tests of randomness were applied to the entries of the processing time matrices of all problems in each of these two experiments. The uniform distributions of matrix entries were verified by applying the chi-square test. The runs test was applied to those entries of each experiment to test the randomness of their order. The serial correlation test was also employed to test against cyclical effects. These tests proved to be not significant.

In each experiment, the complete computation was applied to flowshop problems having up to six jobs. However, partial enumeration was applied to flow shop problems consisting of more than six jobs and job shop problems of any size, since experimenting with these problems requires an excessive amount of computer time.

The partial enumeration solution was obtained by sampling, with replacement, which is more convenient on the computer, and evaluating a certain number of sequences. The number of sequences, n, is selected so that there is at least a specified probability α, that at least one of the sequences will lie in the portion p of the schedule time distribution. Thus, the probability of obtaining the minimum schedule time is

$$1-(1-p)^n \geq \alpha.$$

With a selection of p as 0.001 and a probability of $(1-\alpha)$ or 0.95 of obtaining the minimum schedule time, the sample size, n, is approximately 3000 sequences. In reviewing the relative frequencies of the minimum schedule times in all problems solved by complete enumeration the range of p obtained was from 0.001 to 0.039. This supported the selection of the value of p. Since the partial enumeration does not guarantee obtaining the optimal solution, an estimate of the minimum schedule time was computed as in [5]. It should be pointed out that this estimate is less than or equal to the optimal solution. Next, the sampling experiments with their results are reported.

Sampling Experiments I

This experiment consists of 100 flow shop problems, each of which has six jobs and three machines. The solutions of these 100 problems were obtained by complete enumeration, and decomposition. By complete enumeration the 6! or 720 sequences for each of the 100 problems were constructed. Each sequence was evaluated to determine the corresponding schedule time. The 720 schedule times were then ordered and the minimum schedule time observed. Further, the mean, median, variance, moment coefficient of skewness and excess of kurtosis were computed.

Two decomposition solutions were obtained by using the decomposition algorithm developed for this investigation. The first is the decomposition solution with K equaling two which was obtained by partitioning the six jobs of the original problem into three subsets, each having two jobs. The second is the decomposition solution with K equaling three which was obtained by decomposing the six jobs to two subsets. Each subset has three jobs.

In the case of decomposition with K equaling two, 90 schedule times were obtained over all six jobs for each problem. However, with decomposition with K equaling three, 20 schedule times for each problem were obtained over all six jobs. In each of the above cases, the schedule times were ordered, the minimum schedule time observed, and the different statistics, as in complete enumeration, computed.

One of the objectives was to observe the statistical distributions of the minimum schedule times obtained from the 100 problems by different approaches. Thus, the frequencies of the optimal values obtained by complete enumeration, and those of the minimum schedule times produced by decomposition with K equaling two and K equaling three were tabulated. The observed relative cumulative frequencies of each solution were then plotted on the normal probability paper as shown in Figure 2. A comparison of the sample average and standard deviation obtained from the complete enumeration solutions with those obtained from the decomposition solutions appear in Table 1.

Upon examining the results in Table 1, and the plots in Figure 2, it seems evident that the decomposition solution with K equaling three yields a better approximation to the optimal value than the decomposition solution with K equaling two.

To verify Hypothesis II, mentioned in the preceding section, the efficiencies of the decomposition solutions with K equaling two and with K equaling three were computed. The results of those efficiencies are shown in Table 2, with the number of decomposition solutions with K equaling two and K equaling three. In this table, the results show that 92 decomposition solutions with K equaling three have efficiencies greater than or equal to 0.95; however, those with K equaling two have only 79 solutions.

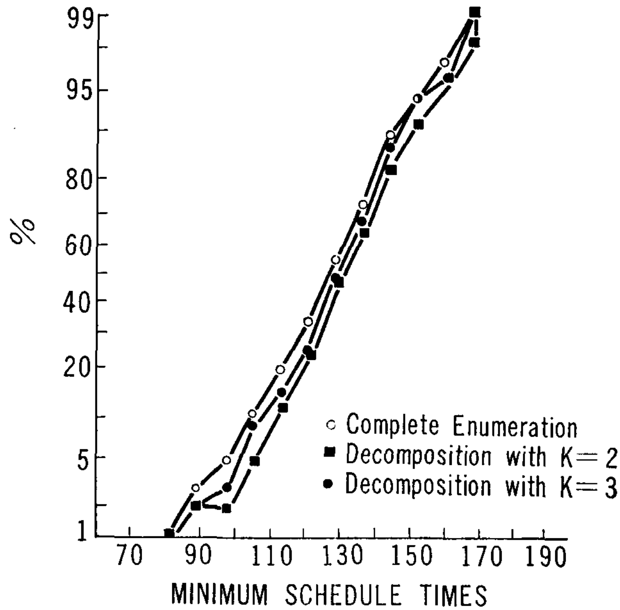

Fig. 2. Relative Cumulative Frequencies of Minimum Schedule Times Obtained by Complete Enumeration and Decomposition.

To observe the shift of the mean schedule times obtained by decomposition with K equaling three and K equaling two from that obtained by complete enumeration, the relative cumulative frequencies of the 100 schedule time means obtained by the different approaches were tabulated. They were then plotted on the normal probability graph paper as shown in Figure 3. The sample average of the mean schedule times and the standard deviation of complete enumeration, decomposition with K equaling two

Table 1: Comparison of Minimum Schedule Times Obtained by Complete Enumeration and Decomposition

Statistics	Complete Enumeration	Decomposition with K=2	Decomposition with K=3
Sample Average	127.14	131.37	129.54
Standard Deviation	17.02	16.51	16.76

Table 2: Efficiencies of Decomposition Solutions

Efficiency	Decomposition with K=2	Decomposition with K=3
1.00	21	43
0.99	12	19
0.98	16	9
0.97	10	10
0.96	8	6
0.95	12	5
0.94	6	2
0.93	6	1
0.92	5	1
0.91	1	1
0.90	2	1
0.89		
0.88		1
0.87		
0.86		1
0.85		
0.84		
0.83	1	
	100	100

and decomposition with K equaling three appear in Table 3. In terms of the sample mean distribution function, Figure 3 shows that the mean values of the decomposition with K equaling three are shifted toward the minimum. However, it is not the case with decomposition with K equaling two.

To test statistically Hypothesis I mentioned earlier, the Matched-Pairs t test was employed under the assumption that the random variable Difference 1, Difference 2 or Difference 3 is normally distributed. Difference 1 refers to the difference between the mean of the schedule times obtained from decomposition with K equaling two and that obtained by decomposition with K equaling three. Difference 3 refers to the difference between the mean of the schedule times obtained by complete

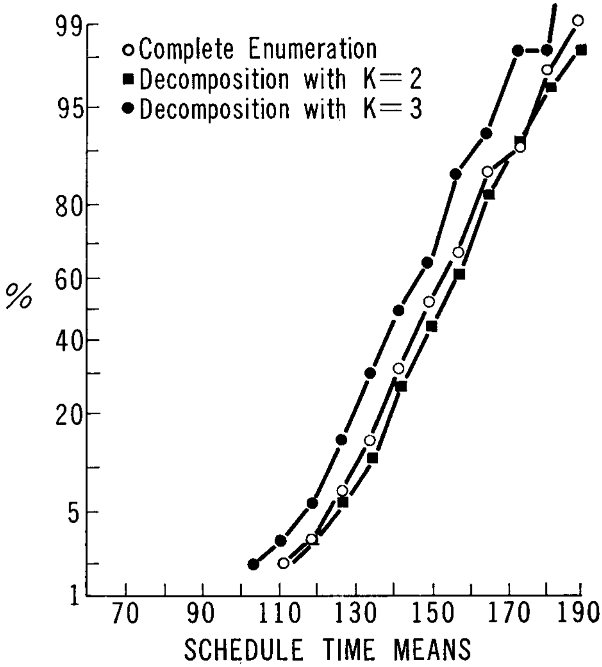

Fig. 3. Relative Cumulative Frequencies of Mean Schedule Times Obtained by Complete Enumeration and Decomposition.

Table 3: Comparison of Mean Schedule Times Obtained by Complete Enumeration and Decomposition

Statistics	Complete Enumeration	Decomposition with K=2	Decomposition with K=3
Sample Average of the Means	151.22	153.65	143.98
Standard Deviation	16.97	17.08	16.91

enumeration and that by decomposition with K equaling three.

The results of the above tests are summarized in Table 4. Test 3 proved to be statistically significant. The null hypothesis that the expected value of Difference 3 is zero is rejected at 0.01 level of significance in favor of the alternative hypothesis that the expected value is

Table 4: Test of Significance for Experiment I

Test No.	Hypothesis	t	$t_{99}(.01)$	Decision
1	$H_0: \mu_1=\mu_2$ $H_1: \mu_1>\mu_2$	−3.354	2.326	reject H_1
2	$H_0: \mu_2=\mu_3$ $H_1: \mu_2>\mu_3$	31.844	2.326	accept H_1
3	$H_0: \mu_1=\mu_3$ $H_1: \mu_1>\mu_3$	11.147	2.326	accept H_1

positive. A similar result is obtained in Test 2. Therefore, one concludes that at this level of significance the mean of the schedule times obtained from decomposition with K equaling three shifts toward the minimum.

However, Test 1 proved to be not significant. In this case, the null hypothesis that the expected value of Difference 1 is zero is accepted at 0.01 level of significance against the alternative hypothesis that the expected value is positive. The above statistical test indicates no significant difference in location between the distribution of the schedule times obtained from the decomposition with K equaling two and that from complete enumeration.

In addition to the above, two important features of the decomposition solutions were observed. First, the maximum schedule time decreased as the number of jobs in each subset increased. Second, the variance of the schedule times produced to find the minimum value decreased as the number of jobs in each subset increased.

Finally, Hypothesis III mentioned in the preceding section was also verified, since the computer time spent to obtain the solutions by complete enumeration, decomposition with K equaling two, and decomposition with K equaling three were approximately, 19.2, 7.1 and 3.7 seconds, respectively for each of these 100 problems.

Summarizing the results of Eexperiment I, one concludes that the distribution of the schedule times obtained by decomposition with K equaling three shifts toward the minimum. Furthermore, the decomposition solution with K equaling three is superior to that with K equaling two in the minimum schedule times obtained and the computer time required.

Sampling Experiment II

To obtain more conclusive results on the decomposition solutions, another experiment was conducted. This experiment included ten problems. Each problem had eight jobs and three machines. The solutions of these ten problems were obtained by partial enumeration and decomposition. The schedule times obtained by partial enumeration for each problem were ordered and the schedule time observed. Then, as in Experiment I, various statistics were computed.

The decomposition solution with K equaling four was obtained for each problem by decomposing the original problem of the eight jobs to two subgroups of four jobs each. In this case, 70 schedule times were ordered, the minimum schedule time observed, and different statistics computed.

It should be pointed out that the decomposition solution with K equaling two was not investigated. This is because it appeared from Experiment I that the mean of the schedule time distribution obtained by decomposition with K equaling two did not shift toward the minimum.

Summarizing the results obtained, Table 5 shows the minimum schedule times obtained by partial enumeration and decomposition with K equaling four for each problem. Also, the relative efficiencies, as defined earlier, and the estimates of the optimal values are shown in the same table.

In reviewing the results in Table 5, the solutions obtained by either enumeration decomposition with K equaling four, or Prolems 1, 2, 6, 7 and 10 are optimal. This is because these solutions are equal to their corresponding minimum estimates. It is also interesting that the decomposition solutions obtained for Problems 8 and 9 are less than those

Table 5: Different Solutions of Problems in Experiment II

Problem No.	Partial Enumeration	Estimate of Optimal Value	Decomposition with K=4	Efficiency
1	157	157	157	1.00
2	130	130	130	1.00
3	195	187	195	1.00
4	145	144	148	.98
5	157	151	159	.99
6	160	160	160	1.00
7	173	173	173	1.00
8	196	177	191	1.03
9	126	118	122	1.03
10	172	172	172	1.00

Table 6: Test of Significance of Experiment II

Test No.	Hypothesis	t	$t_9(.01)$	Decision
1	$H_0: \mu_1=2$ $H_1: \mu_1>2$	5.501	2.821	accept H_1

obtained by partial enumeration. Specifically, the efficiency of each of these two problems is 1.03. Only two problems in this experiment have solutions that yield less than 100 percent efficiency. In particular, the solutions of Problems 4 and 5 yield 0.98 and 0.99 efficiency, respectively.

Further, the hypothesis that the mean of the schedule times obtained by decomposition with K equaling four shifts toward the minimum was tested. In this experiment, difference equals the mean of the schedule times obtained from partial enumeration minus the mean obtained from decomposition with K equaling four. Table 6 shows the result of this test. The null hypothesis that the expected value of Difference is zero is rejected at the 0.01 level of significance in favor of the alternative hypothesis that the expected value is positive. Therefore, one concludes, at this level of statistical significance, that the distribution of the schedule times obtained by decomposition with K equaling four shifts toward the minimum.

Finally, the solutions of each problem carried out in this experiment required, on the average, 147.2 and 81.7 seconds for partial enumeration and decomposition with K equaling four, respectively.

Further Experiments

Various size flow shop and job shop problems were solved by complete or partial enumeration and decomposition to gain more computational experience. Since the computation time involved increases rapidly as the number of jobs increases, it was [necessary to sample a subset from the set of all sequences required to obtain the decomposition solution. This procedure was applied to Problems having more than eight jobs. In Experiment I and II it was observed that the relative frequency of the shortest schedule time increased as the number of jobs in each subset increased. Therefore, probabilities ranging between 0.003 to 0.06 of observing an optimal schedule time were assumed to obtain the different decomposition solutions for each problem.

For comparison, the different solutions with their efficiencies, the schedule time means, and the computer time spent to obtain these solutions are shown for Problems 1-18 in Tables 7, 8, and 9, respectively.

Upon examining the results of Problems 1-6 in Tables 7-9, it appears, as in Experiment I, that decomposition solution is optimal or very close to the optimal value with less computational effort involved. Furthermore, the distribution of the schedule times obtained by decomposition with K equaling three shifts toward the minimum; however, the shift of that obtained from decomposition with K equaling two was not significant.

The results of Problems 7-18 support our conjectures. However, in Problem 15, the computer spent more time in obtaining each of the decomposition solutions that it did to obtain the partial enumeration solution. This was for two reasons: First, in decomposition the subsets were solved by complete enumeration; however, the partial enumeration solution was obtained by generating only 3000 sequences. Second, large

Table
Schedule Time Solutions with Their Efficiencies

Problem No.	Size	Complete Enumeration	Partial Enumeration	Estimate of Optimal Value*	Decomposition with K=2#	Decomposition with K=3
1	6×3	57		53	57 (1.00)	57 (1.00)
2	6×3	68		63	70 (0.97)	69 (0.99)
3	6×3	63		63	63 (1.00)	63 (1.00)
4	6×3	64		64	67 (0.96)	65 (0.98)
5	6×3	69		69	77 (0.90)	72 (0.96)
6	6×3	76		67	78 (0.97)	77 (0.99)
7	6×3	28		28	32 (0.87)	29 (0.96)
8	6×4	151		133	152 (0.99)	151 (1.00)
9	6×5	97		75	101 (0.96)	99 (0.98)
10	6×10	74		70	79 (0.94)	79 (0.94)
11+	6×3		98		118 (0.83)	105 (0.93)
12+	8×3		115			
13	8×3		117	117		
14	8×4		174	168		
15	12×4		254	239		266 (0.95)
16	20×3		93	89		
17	20×5		106	91		
18	40×3		181	179		

* The estimate of the optimal solution is computable only for flow shop problems
\# The efficiencies appear between brackets.
† The processing time matrices of Problem 1 - 6 appear in [4] and that of
\+ Problems 11 and 12 are the only problems of the job shop type.

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of Several Sequencing Problems

Decompo-sition with K=4	Decompo-sition with K=5	Decompo-sition with K=6	Decompo-sition with K=8	Decompo-sition with K=10	Decompo-sition with K=20	Processing Time Range†
						0–9
						1–30
						0–9
						0–30
115 (1.00)						0–30
117 (1.00)						0–30
176 (0.99)						1–30
258 (0.98)		247 (1.03)				1–30
	97 (0.96)			90 (1.03)		0–9
	110 (0.96)			106 (1.00)		0–9
			189 (0.96)	185 (0.98)	180 (1.01)	0–9

[5].

Problem 9 appears in [3].

Table

Schedule Time Means of

Problem No.	Size	Complete Enumeration	Partial Enumeration	Decomposition with K=2	Decomposition with K=3
1	6×3	65.84		65.73	62.15
2	6×3	74.91		75.49	72.90
3	6×3	72.42		73.95	69.65
4	6×3	87.05		91.91	83.65
5	6×3	84.52		88.80	80.75
6	6×3	92.90		97.00	91.00
7	6×3	38.39		40.00	36.20
8	6×4	173.66		174.02	164.95
9	6×5	124.67		131.00	119.40
10	6×10	88.01		91.29	86.95
11*	6×3		132.88	142.54	114.95
12*	8×3		150.76		
13	8×3		142.41		
14	8×4		215.61		
15	12×4		298.94		307.22
16	20×3		112.67		
17	20×5		130.75		
18	40×3		207.28		

* Problems 11 and 12 are the only problems of the job shop type.

number of generated sequences for decomposition were chosen; however, small number of distinct schedule times were obtained, see Table 11.

Summary and Conclusions

The purpose of this paper has been to study the statistical effectiveness of the various decomposition solutions to the sequencing problem. In particular, the following has been investigated: (1) the frequency distributions of the schedule times obtained when the original problem is partitioned into G subproblems, each having K jobs for $1<K<J$; (2) the relative efficiencies of the various solutions, and the correlation among

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Several Sequencing Problems

Decomposition with K=4	Decomposition with K=5	Decomposition with K=6	Decomposition with K=8	Decomposition with K=10	Decomposition with K=20
12[illegible].20					
123.53					
204.41					
291.92		270.98			
	109.96			103.90	
	132.02			119.76	
			207.71	203.31	193.25

those solutions and the corresponding number of jobs in each subproblem, K; and (3) the computation time required to obtain each solution.

In reviewing the results of the various experiments conducted, the most significant conclusions that may be drawn are as follows: (1) The number of distinct schedule times is much less than the number of sequences evaluated. Furthermore, these distinct schedule times have lower frequencies of occurrence as the number of jobs in each subproblem, K, increases. Thus, as K increases the redundancies of the sequences are reduced; (2) The maximum schedule time decreases as K increases; (3) The variance of the schedule times decreases as K in-

Table

Computer Time Spent to Obtain Solutions

Problem No.	Size	Complete Enumeration	Partial Enumeration	Decomposition with K=2	Decomposition with K=3
1	6×3	0.33		0.11	0.06
2	6×3	0.33		0.11	0.06
3	6×3	0.33		0.11	0.06
4	6×3	0.33		0.11	0.06
5	6×3	0.33		0.11	0.06
6	6×3	0.33		0.11	0.06
7	6×3	0.30		0.10	0.05
8	6×4	0.35		0.13	0.06
9	6×5	0.47		0.21	0.09
10	6×10	0.84		0.36	0.14
11†	6×3		3.15	0.47	0.21
12†	8×3		4.70		
13	8×3		2.45		
14	8×4		3.16		
15	12×4		4.81		12.82
16	20×3		6.00		
17	20×5		10.33		
18	40×3		10.91		

* The computer time spent, in minutes, to obtain the different solutions for Program (MSCP), developed by the author, in FORTRAN Language for the

† Problems 11 and 12 are the only problems of the job shop type.

creases; (4) The schedule time distribution shifts toward the optimal value as K increases, except in the cases where the values of K are very close to one; (5) The minimum schedule time is improved as K increases; and (6) The computer time required to obtain the schedule time solution, generally, decreases as K increases.

Indeed, the result of this investigation is that the decomposition of the original problem into two subproblems, each having $J/2$ jobs is more effective than other decomposition procedures. It is possible to evaluate alternative types of processing times. The objective of our further re-

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for Several Sequencing Problems*

Decompo-sition with K=4	Decompo-sition with K=5	Decompo-sition with K=6	Decompo-sition with K=8	Decompo-sition with K=10	Decompo-sition with K=20
2.45					
1.36					
1.60					
11.40		8.04			
	9.78			4.83	
	21.67			9.97	
			32.89	18.30	6.89

each problem is based on the general Purpose Machine Scheduling Computer IBM 7044 computer.

search is to investigate the statistical effectiveness of decomposition when the processing times are generated from an exponential or normal distribution.

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