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# A NOTE ON A TWO-UNIT STANDBY REDUNDANT SYSTEM

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**Abstract**

This note dicusses a two-unit standby redundant system, where the standby unit is subject to failure in the standby interval. The Laplace-Stieltjes transform of the first time distribution to system down is derived by using the signal flow graph method. The mean time is also derived from it. The special cases of the results obtained in this note coincide with the earlier results given by some authors.

## 1. Introduction

One of the most important redundant systems is a two-unit paralleled (or standby) redundant system. Gaver [4, 5] gave the Laplace-Stieltjes transform (LS) of the first time distribution to system down for a two-unit paralleled redundant system. Srinivasan [7] gave the LS transform of the same one for a two-unit standby redundant system.

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Srinivasan [7], however, has not considered the failure of the standby unit, *i.e.*, he has assumed that the standby unit is not subject to failure in the interval of the standby state.

In this note we shall consider a two-unit standby redundant system in cosideration of the failure of the standby unit and give the LS transform of the first time distribution to system down by using the signal flow graph method [6]. We shall further show that the special cases of the results obtained in this note coincide with the earlier ones given by Gaver [4, 5] and Srinivasan [7].

## 2. Model

We consider a two-unit standby redundant system of two dissimilar units, which is considered to be the most generalized model. Appropriately labelling the number of two units, we may call unit $i(i=1, 2)$. The failure time of the operative unit $i$ is subject to the exponential distribution $F_i(t)=1-\exp(-\lambda_i t)(t\geqq 0)$, the repair time of unit $i$ is subject to an arbitrary distribution $G_i(t)(t\geqq 0)$, and the failure time of the standby unit $i$ in the standby interval is subject to an arbitrary distribution $H_i(t)(t\geqq 0)$. Here we assume that the failure time of an operative unit has the "memoryless property [2]," *i.e.*, the failure time distribution is exponential. We need the memoryless property for the analysis as it is shown below. We define the survival distribution $\bar{H}_i(t)\equiv 1-H_i(t)$, which denotes the probability that the standby unit $i$ does not fail up to time $t$ in the standby interval. We also define the similar survival distributions

$$\bar{F}_i(t)\equiv 1-F_i(t) \text{ and } \bar{G}_i(t)\equiv 1-G_i(t).$$

We assume that the switchover times from the operative state to the repair, from the repair completion to the standby state, and from the standby state to the operative state are instantaneous. We further assume that the repair completion of a unit recovers its functioning perfectly. We note that, when the standby unit $i$ is put into operation, the failure time of the operative unit $i$ also obeys $F_i(t)$, which is independent of the standby time.

## 3. Derivation of the LS Transform

In this note we shall apply the signal flow graph method [6] of analyzing the model just mentioned above. Our concern is the first time distribution to system down, where we assume that initially at $t=0$ unit 1 begins to be operative and at that time unit 2 begins to be standby. For such a problem Osaki [6] has given the relationship between the system reliability and the signal flow graph. We shall simply describe his results: We define the states of the system, and construct the signal flow graph of the system. The LS transform of the first time distribution from a state $s_0$ to a state $s_a$ can be obtained by using Mason's gain formula [1], where we define that the state $s_0$ is a source and the state $s_a$ is a sink, and each branch gain from a state $s_i$ to a state $s_j$ corresponds to the LS transform $q_{ij}(s)$ of the (one step) time distribution (which may be an "improper one [3]") from the state $s_i$ to the state $s_j$.

In our model, we define the following four states, where a state is the instant (or the epoch) of the system.

State $s_0$; unit 1 begins to be operative and unit 2 begins to be standby.

State $s_1$; unit 1 begins to be standby and unit 2 begins to be operative.

State $s_2$; unit 1 begins to be repaired and unit 2 begins to be operative.

State $s_3$; unit 1 begins to be operative and unit 2 begins to be repaired.

State $s_4$; two units are under repair or failure, which denotes the system down.

Using the above states, we obtain the signal flow graph of the system in Fig. 1. We shall give each branch gain.

In state $s_0$ we can consider the following two (exclusive and exhaustive) cases:

(i) The operative unit 1 fails before the standby unit 2 fails.

(ii) The standby unit 2 fails before the operative unit 1 fails.

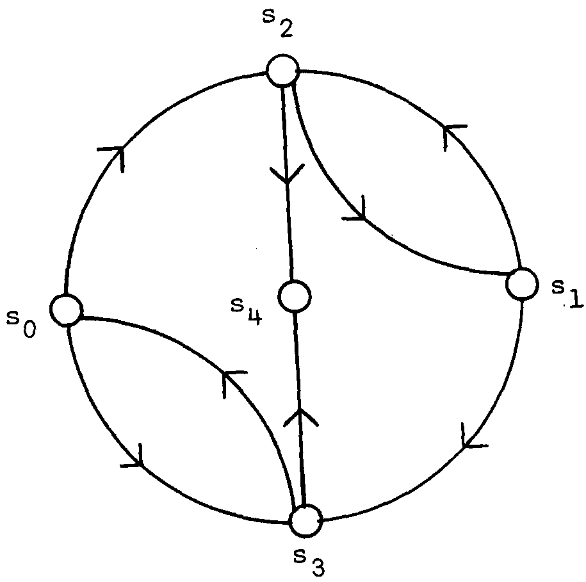


Fig. 1

In the first case the system goes to state $s_2$. Its branch gain is

$$(1) \qquad q_{02}(s) = \int_0^\infty e^{-st}\bar{H}_2(t)\,dF_1(t) = \frac{\lambda_1}{s+\lambda_1}[1-h_2(s+\lambda_1)],$$

where we denote the LS transforms of $G_i(t)$ and $H_i(t)$ by the corresponding small letters $g_i(s)$ and $h_i(s)$, respectively, throughout this note. In the second case, noting that the failure time of unit 1 has the momoryless property, we obtain

$$(2) \qquad q_{03}(s) = \int_0^\infty e^{-st}\bar{F}_1(t)\,dH_2(t) = h_2(s+\lambda_1).$$

In state $s_1$ we obtain each branch gain from the similar discussion of state $s_0$ as follows:

$$(3) \qquad q_{13}(s) = \int_0^\infty e^{-st}\bar{H}_1(t)\,dF_2(t) = \frac{\lambda_2}{s+\lambda_2}[1-h_1(s+\lambda_2)],$$

$$q_{12}(s)=\int_0^\infty e^{-st}\bar{F}_2(t)\,dH_1(t)=h_1(s+\lambda_2). \tag{4}$$

In state $s_2$ we can consider the following two (exclusive and exhaustive) cases:

(i) The operative unit 2 fails before the repair completion of unit 1.

(ii) The repair of unit 1 is completed before the operative unit 2 fails.

In the first case the system goes to state $s_4$ (*i.e.*, the system down). Its branch gain is

$$q_{24}(s)=\int_0^\infty e^{-st}\bar{G}_1(t)\,dF_2(t)=\frac{\lambda_2}{s+\lambda_2}[1-g_1(s+\lambda_2)]. \tag{5}$$

In the second case, noting that the failure time of unit 2 has the memoryless property, we obtain

$$q_{21}(s)=\int_0^\infty e^{-st}\bar{F}_2(t)\,dG_1(t)=g_1(s+\lambda_2). \tag{6}$$

From the similar discussion we obtain two branch gains in state $s_3$:

$$q_{34}(s)=\int_0^\infty e^{-st}\bar{G}_2(t)\,dF_1(t)=\frac{\lambda_1}{s+\lambda_1}[1-g_2(s+\lambda_1)], \tag{7}$$

$$q_{30}(s)=\int_0^\infty e^{-st}\bar{F}_1(t)\,dG_2(t)=g_2(s+\lambda_1). \tag{8}$$

We define $\varphi_0(s)$, the LS transform of the first time distibution to system down starting from state $s_0$ at $t=0$. Using Osaki's results [6], and defining that state $s_0$ is a source and state $s_4$ is a sink, we obtain immediately from Mason's gain formula

$$\varphi_0(s)=\frac{q_{02}(s)q_{24}(s)+q_{02}(s)q_{21}(s)q_{13}(s)q_{34}(s)+q_{03}(s)q_{34}(s)(1-q_{21}(s)q_{12}(s))}{1-q_{21}(s)q_{12}(s)-q_{03}(s)q_{30}(s)-q_{02}(s)q_{21}(s)q_{13}(s)q_{30}(s)+q_{21}(s)q_{12}(s)q_{03}(s)q_{30}(s)}, \tag{9}$$

where each branch gain is given in (1)–(8). The mean time is given by

$$T=-\left.\frac{d\varphi_0(s)}{ds}\right|_{s=0}. \tag{10}$$

## 4. Identical Unit Case

In this section we shall consider the same system of identical units. We assume that the failure time distribution of each operative unit is $F(t)=1-\exp(-\lambda t)$, the repair time distribution is $G(t)$, and the failure time distsibution of each standby unit is $H(t)$.

The states of the system are defined as follows:

State $s_0$; a unit begins to be operative and the other remaining unit begins to be standby.

State $s_1$; a unit begins to be operative and the other remaining unit begins to be repaired.

State $s_2$; two units are under repair or failure, which denotes the system down.

The signal flow graph of the system is given in Fig. 2. Each branch gain is given by

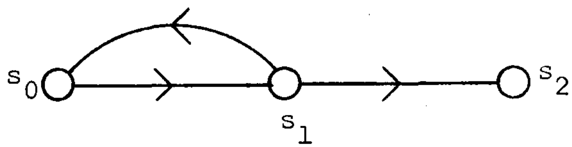


Fig. 2. Signal flow graph of the system of identical units.

$$(11)\qquad q_{01}(s)=\int_0^\infty e^{-st}\bar{H}(t)\,dF(t)+\int_0^\infty e^{-st}\bar{F}(t)\,dH(t)$$

$$=\frac{\lambda}{s+\lambda}[1-h(s+\lambda)]+h(s+\lambda),$$

$$(12)\qquad q_{10}(s)=\int_0^\infty e^{-st}\bar{F}(t)\,dG(t)=g(s+\lambda),$$

$$(13)\qquad q_{12}(s)=\int_0^\infty e^{-st}\bar{G}(t)\,dF(t)=\frac{\lambda}{s+\lambda}[1-g(s+\lambda)].$$

Defining that state $s_0$ is a source and state $s_2$ is a sink, we obtain immediately from Mason's gain formula

$$\varphi_0(s) = q_{01}(s) q_{12}(s) / [1 - q_{01}(s) q_{10}(s)]. \tag{14}$$

The mean time is given by

$$T = \frac{1}{\lambda} + \frac{1-h(\lambda)}{\lambda[1-g(\lambda)]}. \tag{15}$$

These results can be also obtained by assuming that two units are identical in (9) and (10).

## 5. Special Cases

We shall consider in this section two special cases: One is a two-unit standby redundant system in no consideration of the standby failure. Another is a two-unit paralleled redundant system.

The first special case is a two-unit standby redundant system, where we assume that the failure of the standby unit never occurs in the standby interval. For the model, we should set $H_i(t) \equiv 0 (\bar{H}_i(t) \equiv 1)$ in (1)–(8). In this case we obtain

$$\varphi_0(s) = \tag{16}$$

$$\frac{\frac{\lambda_1}{s+\lambda_1} \cdot \frac{\lambda_2}{s+\lambda_2} \cdot \frac{\lambda_1}{s+\lambda_1} g_1(s+\lambda_2)[1-g_2(s+\lambda_1)] + \frac{\lambda_1}{s+\lambda_1} \cdot \frac{\lambda_2}{s+\lambda_2}[1-g_1(s+\lambda_2)]}{1 - \frac{\lambda_1}{s+\lambda_1} \cdot \frac{\lambda_2}{s+\lambda_2} g_1(s+\lambda_2) g_2(s+\lambda_1)}$$

and

$$T = \frac{1}{\lambda} + \frac{1/\lambda_2 + g_1(\lambda_2)/\lambda_1}{1 - g_1(\lambda_2) g_2(\lambda_1)}. \tag{17}$$

These results have been given by Gaver [4] under the same model of identical units. However, in the model we need not the memoryless property of the operative unit, *i.e.*, the failure time distribution of the operative unit $i (i=1, 2)$ is an arbitrary one $F_i(t)$. In the most generalized model, Srinivasan [7] gave the LS transform of the first time distribution to sytem down, and Osaki [6] also gave the same LS transform by using the signal flow graph method.

The second case is a two-unit paralleled redundant system. In the results of Section 3, we assume that the failure time of the standby unit $i(i=1,2)$ has also the same distribution $H_i(t)=F_i(t)=1-\exp(-\lambda_i t)$ as unit $i$ is operative. For the model we obtain

$$\varphi_0(s)=\frac{\lambda_1\lambda_2\left[\dfrac{1-g_1(s+\lambda_2)}{s+\lambda_2}+\dfrac{1-g_2(s+\lambda_1)}{s+\lambda_1}\right]}{s+\lambda_1[1-g_1(s+\lambda_2)]+\lambda_2[1-g_2(s+\lambda_1)]}, \tag{18}$$

and

$$T=\frac{1+\dfrac{\lambda_1}{\lambda_2}[1-g_1(\lambda_2)]+\dfrac{\lambda_2}{\lambda_1}[1-g_2(\lambda_1)]}{\lambda_1[1-g_1(\lambda_2)]+\lambda_2[1-g(\lambda_1)]}, \tag{19}$$

which have been given by Gaver [5]. Gaver [4] also gave the same results under the same model of identical units. These results given by Gaver [4] can be immediately obtained by setting $h(s)=\lambda/(s+\lambda)$ in (14) and (15).

*Addendum*: In the refereeing process, a referee pointed out an interesting paper by J. Fukuta†. He has considered a system of $N+M$ units where $N$ units are operative and $M$ units are in standby. His assumptions are that the failure time distributions of an operative unit and a standby unit are exponential, and the repair time distribution of each failed unit is arbitrary. For the system he has derived the transition probabilities, the first passage distribution to system failure and the mean time to system failure by using the supplementary variable techniques. In this paper, we have derived the first passage time distribution to system failure and the mean time to system failure by the signal flow graph method. Our system is a two-unit standby redundant system, and our assumptions are that the failure time distribution of an

† Fukuta, J., "Reliability Considerations for a Complex Redundant System," *Seminar Reports of the Osaka Statist. Assoc.*, 11, 1, (1967), and also presented at the *Operations Research Around the World Meetings*, Japan Meetings, Kyoto and Tokyo, Aug. 14–18, 1967.

operative unit is exponential and the failure time distribution of a stand-by unit and the repair time distribution of each failed unit are both arbitrary.

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