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CONCENTRATED SERVICE QUEUE WITH LIMITED UNIT-SOURCE

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Abstract

The method of including supplementary variables has been used to obtain the Laplace transforms of the time-dependent distributions of the queue size and the virtual waiting time for the concentrated service queue with limited unit-source. Making use of these results stationary state of the queuing system has also been considered and explicit expressions for the mean number of units present in the service system and for the mean waiting time have been obtained.

1. Introduction

Many authors, Benson & Cox [5], Naor [6], Harison [7], Takâcs [8], et al have considered the usual queue with limited source in connection with machine interference problem. In this paper I will extend their considerations for the so-called concentrated service queue with limited unit-source.

Concentrated service queue with unlimited source has been treated in the references [1]~[4]. The behavior of the units considered in this paper is as follows: units departed from the limited source arrive at the service facility and after service completion they return immediately to

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the initial sorce. The feature of the concentrated service queue is in the service mechanism. The service for the unit arrived at in the idle period of the server is postponed until the number of units waiting for service amounts to the specified number, e. g., k, while in the usual queue it starts as soon as the unit arrives in that period. Accordingly the idle period of the server continues until queue size becomes the specified number, k, from the time point when no-queue state starts, and so the busy period starts as soon as the queue size amounts to k. The service of the units arrived at in the busy period of the server is the same as usual, and the units receive their services one by one without batches and obeys the queue discipline 'first-come, first-served.' Needless to say, the usual queue is the case where $k=1$. Number k is termed as the concentration-number.

The schema of the state transitions is given in Fig. 1, in which the digit in the parenthesis denotes the number of units present in the service system.

$$
\begin{array}{l}
(0) \rightarrow (1) \rightarrow \cdots\cdots \rightarrow (k-1) \searrow \\
\quad \nwarrow (1) \rightleftarrows \cdots\cdots \rightleftarrows (k-1) \rightleftarrows (k) \rightleftarrows \cdots\cdots \rightleftarrows (N)
\end{array}
$$

N: capacity of the limited source, k: concentration-number,
$\rightarrow$: transition by until-arrival, $\leftarrow$: tran. by service-completion.

Fig. 1 Schema of State Transitions.

In the following sections the Laplace transform representations of the time-dependent distributions of the queue size and the virtual waiting time for the above-mentioned queue characterized by M/G/1(N) are obtained by the methods of including supplementary variables, and the behavior of the stationary states will be discussed in detail.

2. Related Definitions

Let us suppose that the first order probability that if at time t a unit will call for service in the interval $(t, t+dt)$ is λdt. And let $\eta(u)du$ be the first order probability that a service completion occurs in the interval $(u, u+du)$, conditional on a unit served having reached the service age, u. Then the pdf $f(u)$ of the service time distribution is given as follows:

$$f(u)=\eta(u) \exp\left\{-\int_0^u \eta(\tau)d\tau\right\}.$$

Let us now define the following probabilities, in which (a)'s correspond to the states of the upper row and (b)'s, to those of the lower row, in fig. 1.

(a) $p_n(t)$ is the state probability that at time t there are n units in a service facility waiting for service starting. n can take integral values of $0\leq n\leq k-1$, where k is the concentration-number.

(b) $q_n(t, u)$ is the joint probability that at time t there are n units present in the service facility and the elapsed service time of the unit currently in service is u. n can take the integral values of $1\leq n\leq N$, where N is the capacity of the limited source.

3. Formulation of Equations

We can derive the difference-integro-differential equations for $p_n(t)$ and $q_n(t, u)$, refering to the schema (Fig. 1), and they are as follows:

$$\frac{d}{dt}p_0(t)= -N\lambda p_0(t)+\int_0^t \eta(u)q_1(t, u)du. \tag{3.1}$$

$$\frac{d}{dt}p_n(t)=-(N-n)\lambda p_n(t)+(N-n+1)\lambda p_{n-1}(t); \quad n=1, 2, \cdots, k-1. \tag{3.2}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial u}\right)q_n(t, u)=-\{(N-n)\lambda+\eta(u)\}q_n(t, u) + (1-\delta_{1,n})(N-n+1)\lambda q_{n-1}(t, u); \quad n=1, 2, \cdots, N. \tag{3.3}$$

Initial and boundary conditions are as follows:

(3.4) $$p_0(0)=1,\ \ p_n(0)=0,\ \ q_n(0,u)=0\ ;\ \ u\neq 0\ ,$$

(3.5) $$q_n(t,0)=(1-\delta_{N,n})\int_0^t \eta(u)q_{n+1}(t,u)du+\delta_{k,n}(N-k+1)\lambda p_{k-1}(t)\ ;$$
$$n=1,2,\cdots,N\ .$$

4. Time-dependent Distribution of Queue Size

To solve the equations (3.1)~(3.5), the following transform has to be defined:

(4.1) $$A_m(t,u)=\sum_{i=m}^{N-1}\binom{i}{m}q_{N-i}(t,u)\ ;\ \ m=0,1,\cdots,N-1\ .$$

It is easily proved that the inversion formula for this transform is given by the following expression:

(4.2) $$q_n(t,u)=\sum_{m=N-n}^{N-1}(-1)^{m+n-N}\binom{m}{N-n}A_m(t,u).$$

These A_m's are called binomial moments of the set of functions $\{q_n(t,u)\}$. Once we obtain the A_m's, $q_n(t,n)$ can be obtained easy using the formula (4.2).

Changing n to $N-i$ in (3.3), multiplying throughout by $\binom{i}{m}$ and summing over i from m to $N-1$, we obtain

(4.3) $$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial u}\right)A_m(t,u)=-\{\lambda m+\eta(u)\}A_m(t,u).$$

The solution of the equation (4.3) is given by

(4.4) $$A_m(t,u)=A_m(t-u,0)\ \exp\left\{-\lambda mu-\int_0^u\eta(\tau)d\tau\right\}\ .$$

Following the similar steps, (3.5) is transformed into

$$A_m(t,0)=\int_0^t \eta(u)\,A_m(t,u)\,du+(1-\delta_{0,m})\int_0^t \eta(u)\,A_{m-1}(t,u)\,du$$
$$-\binom{N}{m}\int_0^t \eta(u)\,A_{N-1}(t,u)\,du+\binom{N-k}{m}(N-k+1)\lambda p_{k-1}(t). \tag{4.5}$$

Substituting (4.4) into (4.5), we obtain

$$\begin{aligned} A_m(t,0)&=\int_0^t e^{-\lambda m u}f(u)\,A_m(t-u,0)\,du\\ &+(1-\delta_{0,m})\int_0^t e^{-\lambda(m-1)u}f(u)\,A_{m-1}(t-u,0)\,du\\ &-\binom{N}{m}\int_0^t e^{-\lambda(N-1)u}f(u)\,A_{N-1}(t-u,0)\,du\\ &+\binom{N-k}{m}(N-k+1)\lambda p_{k-1}(t)\,; \qquad m=0,1,\cdots,N-1. \end{aligned} \tag{4.4'}$$

From (3.1) we obtain easily

$$\frac{d}{dt}p_0(t)=-N\lambda p_0(t)+\int_0^t e^{-\lambda(N-1)u}f(u)\,A_{N-1}(t-u,0)\,du. \tag{4.5'}$$

Laplace transforms are denoted by the sign $\wedge$ and thus

$$L[p_n(t)]\equiv\hat{p}_n(s), \quad L[A_m(t,u)]\equiv\hat{A}_m(s,u).$$

Equations (3.2), (4.5) and (4.4′) are transformed and they are as follows:

$$\hat{p}_n(s)=\frac{N!\lambda^n\hat{p}_0(s)}{(N-n)!\prod_{j=1}^{n}\{s+(N-j)\lambda\}}\,; \quad n=1,2,\cdots,k-1. \tag{4.6}$$

$$(s+N\lambda)\,\hat{p}_0(s)=1+\hat{f}\{s+(N-1)\lambda\}\,\hat{A}_{N-1}(s,0). \tag{4.7}$$

$$\begin{aligned} &\hat{A}_m(s,0)\,\hat{f}\{1-(s+m\lambda)\}-\hat{A}_{m-1}(s,0)\,\hat{f}\{s+(m-1)\lambda\}\\ &\qquad=\binom{N}{m}\{1-B_m(s)\,\hat{p}_0(s)\}, \end{aligned} \tag{4.8}$$

where

$$B_m(s)=(s+N\lambda)\left[1-\{1-E(m-N+k)\}\binom{N-m}{k}\prod_{j=0}^{k-1}\frac{(j+1)\lambda}{s+(N-j)\lambda}\right],$$

in which $E(\cdot)$ is the unit function.

If we define that

$$(4.9)\quad \begin{cases} \phi_m(s)=\hat{f}(s)\prod_{r=1}^{m-1}\dfrac{\hat{f}(s+r\lambda)}{1-\hat{f}(s+r\lambda)}, \quad \phi'_m(s)=\prod_{r=1}^{m}\dfrac{\hat{f}\{s+(r-1)\lambda\}}{1-\hat{f}(s+r\lambda)}\ ; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m\neq 1, \\ \phi_0(s)=1-\hat{f}(s), \quad \phi'_0(s)=1\ , \end{cases}$$

equation (4.8) is rewritten as

$$(4.10)\qquad \frac{\hat{A}_m(s,0)}{\phi'_m(s)}-(1-\delta_{0,m})\frac{\hat{A}_{m-1}(s,0)}{\phi'_{m-1}(s)} =\frac{1}{\phi_m(s)}\binom{N}{m}\{1-B_m(s)\hat{p}_0(s)\}\ ;\quad m=0,1,\cdots,N.$$

Solving the equation (4.10) we have

$$(4.11)\qquad \hat{A}_m(s,0)=\phi'_m(s)\left[\sum_{j=0}^{m}\binom{N}{j}\phi_j^{-1}(s)\{1-B_j(s)\hat{p}_0(s)\}\right]\ ; \quad m=0,1,\cdots,N-1.$$

Substituting $\hat{A}_{N-1}(s,0)$ obtained from (4.11) into (4.7), the expression for $\hat{p}_0(s)$ is obtained as follows:

$$(4.12)\qquad \hat{p}_0(s)=\sum_{j=0}^{N}\binom{N}{j}\phi_j^{-1}(s)\Big/\sum_{j=0}^{N}\binom{N}{j}\phi_j^{-1}(s)B_j(s).$$

Substituting (4.12) into (4.6) and (4.11), we have

$$(4.13)\qquad \hat{p}_n(s)=\binom{N}{n}\prod_{j=1}^{n}\left(\frac{\lambda j}{s+(N-j)\lambda}\right)\sum_{j=0}^{N}\binom{N}{j}\phi_j^{-1}(s) \Big/\sum_{j=0}^{N}\binom{N}{j}B_j(s)\phi_j^{-1}(s)\ ;\quad n=1,2,\cdots,k-1$$

and

$$\hat{A}_m(s,0)=\phi'_m(s) \times\left[\sum_{j=0}^{m}\binom{N}{j}\phi_j^{-1}(s)-\frac{\sum_{j=0}^{m}\binom{N}{j}B_j(s)\phi_j^{-1}(s)\sum_{j=0}^{N}\binom{N}{j}\phi_j^{-1}(s)}{\sum_{j=0}^{N}\binom{N}{j}B_j(s)\phi_j^{-1}(s)}\right]; \quad m=0,1,\cdots,N-1. \tag{4.14}$$

Substituting (4.14) into the transformed equation of (4.4), we have

$$\hat{A}_m(s,u)=\exp\left\{-(s+\lambda m)u-\int_0^u\eta(\tau)\,d\tau\right\}\cdot \phi'_m(s)\left[\sum_{j=0}^{m}\binom{N}{j}\phi_j^{-1}(s)-\frac{\sum_{j=0}^{m}\binom{N}{j}B_j(s)\phi_j^{-1}(s)\sum_{j=0}^{N}\binom{N}{j}\phi_j^{-1}(s)}{\sum_{j=0}^{N}\binom{N}{j}B_j(s)\phi_j^{-1}(s)}\right]; \quad m=0,1,\cdots,N-1. \tag{4.15}$$

Substituting (4.15) into the transformed (4.2) we have

$$\hat{q}_n(s,u)=\sum_{m=N-n}^{N-1}(-1)^{m+n-N}\binom{m}{N-n} \times\exp\left\{-(s+\lambda m)u-\int_0^u\eta(\tau)\,d\tau\right\}\cdot \phi'_m(s)\left[\sum_{j=0}^{m}\binom{N}{j}\phi_{j_i}^{-1}(s)-\frac{\sum_{j=0}^{m}\binom{N}{j}B_j(s)\phi_j^{-1}(s)\sum_{j=0}^{N}\binom{N}{j}\phi_j^{-1}(s)}{\sum_{i=0}^{N}\binom{N}{j}B_j(s)\phi_j^{-1}(s)}\right]; \quad n=1,2,\cdots,N. \tag{4.16}$$

A set of expressions (4.12), (4.13), and (4.16) is the time-dependent distribution of the queue size given by the Laplace transforms.

5. Limiting Distribution of Queue Size

If we assume that

$$p_n=\lim_{t\to\infty}p_n(t),\quad q_n(u)=\lim_{t\to\infty}q_n(t,u), \tag{5.1}$$

formulas for p_n and $q_n(u)$ are derived from (4. 12), (4. 13) and (4. 16) and they are given as follows:

(5. 2) $$p_0=1\Big/N\left[\sum_{j=0}^{k-1}\frac{1}{N-j}+\rho\left\{\sum_{j=1}^{N}\psi_j^{-1}(0)\left(\binom{N}{j}-\binom{N-k}{j}\right)\right\}\right],$$

where

$$\psi_j(0)=\prod_{i=1}^{j-1}\hat{f}(\lambda i)/\{1-\hat{f}(\lambda i)\},\quad \text{for } j\neq 1;$$

$\psi_1(0)=1$ and $\rho=\lambda/\mu$, in which $\mu^{-1}=\int_0^\infty tf(t)\,dt$. In the special case where the queuing system belongs to the type of M/M/1 (N), $k=1$, we have the following well-known result:

(5. 2′) $$p_0=1\Big/\sum_{j=0}^{N}\binom{N}{j}j!\ \rho^j.$$

From (4. 13) and (4. 16) we have

(5. 3) $$p_n=\frac{N}{N-n}p_0;\quad n=1,2,\cdots,k-1,$$

and

(5. 4) $$q_n(u)=\sum_{m=N-n}^{N}(-1)^{m+n-N}\binom{m}{N-n}\times\exp\left\{-\lambda mu-\int_0^u\eta(\tau)\,d\tau\right\}A_m(0),$$

where $A_m(0)$'s are given as follows:

(5. 5) $$\begin{cases} A_0(0)=\mu\left\{1-Np_0\displaystyle\sum_{j=0}^{k-1}\frac{1}{N-j}\right\}, \\ A_m(0)=\psi'_m(0)\left\{A_0(0)-p_0\displaystyle\sum_{i=1}^{m}\binom{N}{j}\psi_i^{-1}(0)B_j(0)\right\}, \quad m\neq 1. \end{cases}$$

The special case where $m=1$ in (5. 5) will be used in the following section and so the explicit expression will be given after brief calculation in the following

$$A_1(0)=\frac{1}{1-f(\lambda)}\left[\mu-Np_0\left\{k\lambda+\mu\sum_{j=0}^{k-1}\frac{1}{N-j}\right\}\right]. \tag{5.6}$$

6. Mean Queue Length

Let $L(t)$ denote the mean number of units present in the service system at time t, and it may be defined as

$$L(t)=\sum_{n=0}^{k-1}np_n(t)+\int_0^t du\sum_{n=1}^{N}nq_n(t,u). \tag{6.1}$$

Denoting the first, the second term in the r.h.s. of (6.1) by $L_1(t)$, $L_2(t)$ respectively, we have them in the Laplace transforms as follows:

$$\begin{aligned}\hat{L}_1(s)&=LL_1(t)]\\&=N\sum_{n=0}^{k-2}\binom{N-1}{n}\prod_{j=1}^{n+1}\left(\frac{\lambda j}{s+(N-j)\lambda}\right)\hat{p}_0(s).\end{aligned} \tag{6.2}$$

Let $Q(t,u;\ w)$ denote the moment-generating function of the set of functions, $\left\{\int_0^t q_n(t,u)\,du\right\}$ and we have

$$\begin{aligned}Q(t,u;\ w)&=\int_0^t du\sum_{n=1}^{N}q_n(t,u)w^n\\&=w^N\sum_{j=0}^{N-1}\left(\frac{1-w}{w}\right)^j\int_0^t du\exp\left\{-\lambda ju-\int_0^u\eta(\tau)\,d\tau\right\}A_j(t-u,0),\end{aligned} \tag{6.3}$$

and thus

$$\begin{aligned}L_2(t)&=\int_0^t du\sum_{n=1}^{N}nq_n(t,u)=\left[\frac{\partial}{\partial w}Q(t,u;\ w)\right]_{w=1}\\&=N\int_0^t du\,A_0(t-u,0)\exp\left\{-\int_0^u\eta(\tau)\,d\tau\right\}\\&\quad-\int_0^t du\,A_1(t-u,0)\exp\left\{-\lambda u-\int_0^u\eta(\tau)\,d\tau\right\}.\end{aligned} \tag{6.4}$$

Transforming both sides of (6.4), we have

(6.5) $$\hat{L}_2(s)=N\hat{A}_0(s,0)\frac{1-\hat{f}(s)}{s}-\hat{A}_1(s,0)\frac{1-\hat{f}(s+\lambda)}{s+\lambda}.$$

Using the above results, Laplace transform of $L(t)$, $L(s)$, is obtained as follows:

(6.6) $$\begin{aligned}\hat{L}(s)&=\hat{L}_1(s)+\hat{L}_2(s)\\&=N\sum_{n=0}^{k-2}\binom{N-1}{n}\prod_{j=1}^{n+1}\left(\frac{\lambda j}{s+(N-j)\lambda}\right)\hat{p}_0(s)\\&\quad+\frac{N\{1-\hat{f}(s)\}}{s}\hat{A}_0(s,0)-\frac{1-\hat{f}(s+\lambda)}{s+\lambda}\hat{A}_1(s,0),\end{aligned}$$

where $\hat{p}_0(s)$ and $\hat{A}_m(s,0)$ for $m=0,1$ are given by the expressions (4.13) and (4.14) respectively.

We shall next obtain L, the mean number of units present in the service system in the stationary state and the formula for L is given as follows:

(6.7) $$L=\lim_{s\to 0}s\hat{L}_1(s)+\lim_{s\to 0}s\hat{L}_2(s)\equiv L_1+L_2,$$

where

$$L_1=Np_0\left\{\sum_{j=0}^{k-1}\frac{N}{N-j}-k\right\}$$

(6.8) $$\begin{aligned}L_2&=\frac{N}{\mu}A_0(0)-\frac{1-\hat{f}(\lambda)}{\lambda}A_1(0)\\&=\left(N-\frac{1}{\rho}\right)\left\{1-Np_0\sum_{j=0}^{k-1}\frac{1}{N-j}\right\}+Nkp_0,\end{aligned}$$

where p_0 is given by (5.2).

Substituting (5.8) into (5.7) we have

(6.9) $$L=N-\frac{1}{\rho}\left\{1-Np_0\sum_{j=0}^{k-1}\frac{1}{N-j}\right\}.$$

Setting $k=1$ in (6.9) we have the well-known formula:

(6.9) $$L=N-\frac{1}{\rho}(1-p_0).$$

7. Distribution of Waiting Time

We introduce the following notations:

$B(t)\equiv$event, in which server is busy at time t.

$B^c(t)\equiv$complementary event of event $B(t)$.

$\xi(t)\equiv$number of units present in the service system at time t.

$\tau(t)\equiv$virtual waiting time; the time that a unit would wait if it joined the queue at time t.

$\varepsilon(t)\equiv$elapsed service time of the unit currently in service at time t.

If we define that

(7.1) $$W(t,x)=p_r\{\tau(t)\leq x\},$$

$W(t,x)$ is expresses as follows:

(7.2) $$\begin{aligned}W(t,x)=&\sum_{n=0}^{k-1}p_n(t)p_r\{(t)\leq x|(\xi(t)=n)\cap B^c(t)\}\\&+\int_0^t du\sum_{n=1}^{N}q_n(t,u)p_r\{\tau(t)\leq x|(\xi(t)=n)\cap(\varepsilon(t)=u)\cap B(t)\}.\end{aligned}$$

Denoting the first and the second term by $W_1(t,x)$ and $W_2(t,x)$ respectively and defining that

(7.3) $$\begin{cases}w(t,\zeta)=\displaystyle\int_0^\infty e^{-\zeta x}\,dW(t,x),\\ w_i(t,\zeta)=\displaystyle\int_0^\infty e^{-\zeta x}\,dW_i(t,x), \quad i=1,2,\end{cases}$$

we have

(7.4) $$w(t,\zeta)=w_1(t,\zeta)+w_2(t,\zeta).$$

Then $w_1(t,\zeta)$ is obtained as follows:

(7.5)
$$w_1(t,\zeta)=p_{k-1}(t)\hat{f}^{k-1}(\zeta)+\sum_{j=0}^{k-2}p_{k-2-j}(t)\hat{f}^{k-2-j}(\zeta)(N-k+1)\lambda\binom{N-k+1+j}{j}L_j(\zeta),$$

where

(7.6)
$$L_j(\zeta)=\int_0^\infty e^{-\zeta x}\{e^{-\lambda x(N-k+1)}(1-e^{-\lambda x})^j\}dx=\sum_{i=0}^{j}(-1)^i\binom{j}{i}\{\zeta+\lambda(N-k+i+1)\}^{-1}.$$

Denoting the Laplace transform of $w_1(t,\zeta)$ by $\hat{w}_1(s,\zeta)$ and transforming (7.4), we have

(7.7)
$$\hat{w}_1(s,\zeta)=\hat{p}_{k-1}(s)\hat{f}^{k-1}(\zeta)+(N-k+1)\lambda\sum_{j=0}^{k-2}\binom{N-k+1+j}{j}\hat{p}_{k-2-j}(s)\hat{f}^{k-2-j}(\zeta)L_j(\zeta).$$

Substituting (4.6) into (7.7) we have

(7.7')
$$\hat{w}_1(s,\zeta)=\hat{p}_0(s)\left[\frac{N!\ \lambda^{k-1}\hat{f}^{k-1}(\zeta)}{(N-k+1)!\ \prod_{i=1}^{k-1}\{s+(N-i)\lambda\}}+\sum_{j=0}^{k-2}\frac{N!\ \lambda^{k-1-j}\hat{f}^{k-2-j}(\zeta)L_j(\zeta)}{(N-k+j+2)(N-k)!\ j!\prod_{i=1}^{k-2-j}\{s+(N-i)\lambda\}}\right].$$

Following the similar steps, $w_2(t,\zeta)$ and $\hat{w}_2(s,\zeta)$, the Laplace transform of $w_2(t,\zeta)$, are obtained as follows:

(7.8)
$$w_2(t,\zeta)=\int_0^t du\sum_{n=1}^{N}q_n(t,u)\hat{f}^{n-1}(\zeta)\times\exp\left\{\int_0^u\eta(\tau)d\tau\right\}\int_0^\infty e^{-\zeta x}f(x+u)dx,$$

where

$$\int_0^\infty e^{-\zeta x}f(x+u)dx=e^{\zeta u}\left\{\hat{f}(\zeta)-\int_0^u e^{-\zeta x}f(x)dx\right\}.$$

Making use of (6.3), (7.8) is simplified as follows:

$$(7.9)\qquad w_2(t,\zeta)=\frac{1}{\hat{f}(\zeta)}\int_0^t du\,\hat{f}^N(\zeta)\sum_{j=0}^{N-1}\left\{\frac{1-\hat{f}(\zeta)}{\hat{f}(\zeta)}\right\}^j A_j(t,u)$$

$$\cdot\frac{e^{\zeta u}}{\exp\left\{-\int_0^u \eta(\tau)\,d\tau\right\}}\left\{\hat{f}(\zeta)-\int_0^u e^{-\zeta x}f(x)\,dx\right\}$$

$$=\hat{f}^{N-1}(\zeta)\int_0^t du\sum_{j=0}^{N-1}\left\{\frac{1-\hat{f}(\zeta)}{\hat{f}(\zeta)}\right\}^j A_j(t-u,0)$$

$$\cdot\, e^{-u(\lambda j-\zeta)}\left\{\hat{f}(\zeta)-\int_0^u e^{-\zeta x}f(x)\,dx\right\}.$$

Transforming (7.9) concerning t, we have

$$(7.10)\qquad \hat{w}_2(s,\zeta)=\hat{f}^{N-1}(\zeta)\sum_{j=0}^{N-1}\left\{\frac{1-\hat{f}(\zeta)}{\hat{f}(\zeta)}\right\}^j \hat{A}_j(s,0)\left\{\frac{\hat{f}(\zeta)-\hat{f}(s+\lambda j)}{s+\lambda j-\zeta}\right\},$$

where $\hat{A}_j(s,0)$'s are given in (4.11) or (4.14).

Substituting (7.7') and (7.10) into the transform of (7.4), we have the distribution function of the virtual waiting time in the form of Laplace transform:

$$(7.11)\qquad \hat{w}(s,\zeta)=\hat{w}_1(s,\zeta)+\hat{w}_2(s,\zeta)$$

$$=\left[\frac{N!\,\lambda^{k-1}\hat{f}^{k-1}(\zeta)}{(N-k+1)!\prod_{i=1}^{k-1}\{s+(N-i)\lambda\}}\right.$$

$$\left.+\sum_{j=0}^{k-2}\frac{N!\,\lambda^{k-1-j}\hat{f}^{k-2-j}(\zeta)L_j(\zeta)}{(N-k+j+2)(N-j)!\,j!\prod_{i=1}^{k-2-j}\{s+(N-i)\lambda\}}\right]\hat{p}_0(s)$$

$$+\hat{f}^{N-1}(\zeta)\sum_{j=0}^{N-1}\left\{\frac{1-\hat{f}(\zeta)}{\hat{f}(\zeta)}\right\}^j\frac{\hat{f}(\zeta)\hat{f}-(s+\lambda j)}{s+\lambda j-\zeta}\hat{A}_j(s,0).$$

Making use of the formula (7.11), we will obtain the distribution function of the waiting time in the stationary state in the form of the Laplace transform:

$$\hat{w}(\zeta)=\lim_{s\to 0} s\hat{w}(s,\zeta)$$
$$=\Big\{\frac{N}{N-k+1}\hat{f}^{k-1}(\zeta)+N\lambda\sum_{j=0}^{k-2}\frac{N-k+1}{N-k+j+2}$$
$$\times\binom{N-k+j+1}{j}L_j(\zeta)\hat{f}^{k-2-j}(\zeta)\Big\}p_0$$
$$+\hat{f}^{N-1}(\zeta)\sum_{j=0}^{N-1}\Big\{\frac{1-\hat{f}(\zeta)}{\hat{f}(\zeta)}\Big\}^j\frac{\hat{f}(\zeta)-\hat{f}(\lambda j)}{\lambda j-\zeta}A_j(0). \tag{7.12}$$

Denoting the mean waiting time in the stationary state as $\bar{w}$, we have

$$\bar{w}=-\Big[\frac{\partial}{\partial\zeta}\hat{w}(\zeta)\Big]_{\zeta=0}=-\Big[\frac{\partial}{\partial\zeta}\hat{w}_1(\zeta)+\frac{\partial}{\partial\zeta}\hat{w}_2(\zeta)\Big]_{\zeta=0}, \tag{7.13}$$

where $\hat{w}_1(\zeta)$ and $\hat{w}_2(\zeta)$ are the first and the second term in (7.12) respectively. Each terms of the r.h.s. of (7.13) are obtained as follows:

$$-\Big[\frac{\partial}{\partial\zeta}\hat{w}_1(\zeta)\Big]_{\zeta=0}$$
$$=\frac{Np_0}{\lambda}\Big[\frac{k-1}{N-k+1}\rho+\sum_{j=0}^{k-2}\frac{N-k+1}{N-k+j+2}\binom{N-k+j+1}{j}$$
$$\cdot\Big\{\sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{1}{N-k+i+1}\Big((k-j-2)\rho+\frac{1}{N-k+i+1}\Big)\Big\}\Big]. \tag{7.14}$$

$$-\Big[\frac{\partial}{\partial\zeta}\hat{w}_2(\zeta)\Big]_{\zeta=0}$$
$$=\frac{1}{\lambda}\Big[Nk\rho p_0+\Big(1-Np_0\sum_{j=0}^{k-1}\frac{1}{N-j}\Big)\Big\{\frac{\rho\mu^2}{2}\hat{f}''(0)+\rho(N-1)-1\Big\}\Big]. \tag{7.15}$$

Substituting (7.14) and (7.15) into (7.13), we have

$$\bar{w}=\frac{N}{\lambda}p_0\left[\frac{k-1}{N-k+1}\rho+\sum_{j=0}^{k-2}\frac{N-k+1}{N-k+j+2}\binom{N-k+j+1}{j}\right.$$
$$\left.\cdot\sum_{i=0}^{j}(-1)^i\binom{j}{i}\left\{\frac{k-2-j}{N-k+i+1}\rho+\frac{1}{(N-k+i+1)^2}\right\}\right]$$
$$+\frac{1}{\lambda}\left[Nk\rho p_0+\left(1-Np_0\sum_{j=0}^{k-1}\frac{1}{N-j}\right)\right.$$
$$\left.\times\left\{\frac{\rho\mu^2}{2}f''(0)+\rho(N-1)-1\right\}\right]. \tag{7.16}$$

The special case where $k=1$ in (7.16) is as follows:

$$\bar{w}=\frac{1}{\lambda}\left[N\rho p_0+(1-p_0)\left\{\frac{\rho\mu^2}{2}f''(0)+\rho(N-1)-1\right\}\right]. \tag{7.16'}$$

The formula (7.16′) coincides with that of Takâcs.

If the queuing system belongs to M/M/1(N), the formula (7.16′) is simplified in the following

$$\bar{w}=\frac{1}{\mu}\left\{N-\frac{1}{\rho}(1-p_0)\right\}=\frac{1}{\mu}L. \tag{7.17}$$

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Abstracts

Analysis of Multiqueue

by ON HASHIDA and GISAKU NAKAMURA
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A queueing model is studied in which a single server serves N queues. The server attends N queues in cyclic order and at each counter he serves all those customers who are present when the serving starts and also all the newcomers as long as there are customers at the counter. A walking time is required by the server to go from one counter to the next. Counters are labeled $1, 2, \cdots, N$, respectively. It is supposed that customers arrive at the counter $i (i=1, 2, \cdots, N)$ in accordance with a Poisson process of density λ_i. The service times at the counter i have a general distribution $H_i(x)$, and the walking times from the counter i to the next also have a general distribution $U_i(x)$.

The model is first analyzed in the case $N=2$, and then the analysis is extended to a general case. The number of waiting customers, waiting times and busy periods for each counter are analyzed in statistical equilibrium, and the equilibrium condition of the system is considered. The results obtained from the analysis of this model are generating functions for the probability distribution of the number of customers, Laplace-Stieltjes transforms for the probability distributions of both the waiting times and busy periods, and expressions for their means. In the case that $N=2$ and the walking times are neglected, the results are reduced to Takács' formulas.

Optimal Location of Mass Transportation System within a Metropolitan Area

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The increasing process of the gross population in a metropolitan

area is based on a economic, sociologic, political conditions and etc. relatively to other metropolitans. But the variation of the population distribution on each zone is, much more effected by the mass transportation system. The zonal population is the occurrence source of the commuter traffic demand, so the interaction between them should be analysed first, in order to establish the mass transportation system planning. In this theses I have analysed the interaction by applying the information theory and consequently I can say that the variation of the zonal population distribution maximizes entropy per unit characteristic value, which involves land value and time distance through mass transportation facilities as informations. And next I have proposed a formula to estimate the generating volume of commuting passengers. That formula is an opportunity model which is defined by commuting time from residential zone to commercial zone and employee in commercial zones.

O D traffic volume of commuter passengers can be estimated generally by Detroit Method or Frater Method. And for each origin and destination of urban commuters, generally there are multi-traffic-routes. Each commuter choices one of them respectively comparing with some criteria. The relation between characteristics of each route and commuter traffic assignment have been expressed as a linear formula by multi regression analysis.

By these formulations between transportation improvement and urban development pattern, effect of transportation planning can be measured and urban planner can decide a location pattern of transportation facilities in order to guide land use development towards some more desirable pattern.

At last I showed the applying process of planning in the flow diagram and this planning process has been applicated to mass transportation planning in Tokyo Metropolis and Osaka Metropolis by Japanese National Railways and Osaka City Authorities.