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MEDIAN TWO-PERSON GAME THEORY FOR MEDIAN COMPETITIVE GAMES

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Abstract

A form of discrete two-person game theory based on median considerations is developed in [1]. Median game theory has very strong application advantages over expected value game theory [2]. In particular, the class of median competitive games, where both players can be simultaneously protective and vindictive, is huge compared to the corresponding class for expected value game theory. Moreover, the median approach is usable for games where the numbers in one or both payoff matrices do not satisfy the arithmetical operations (but can be ranked within each matrix). A subclass of the median competitive games is identified in [1]. The complete class is specified in this paper and a method is given for determining median optimum strategies. In addition, the class of games where a given player (but not necessarily the other one) can be simultaneously protective and vindictive is identified. Also, a way of finding a median optimum strategy for this player is developed. The evaluation methods given are oriented toward minimum application effort (and do not use preferred sequences).

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Introduction and Discussion

Only the case of two players with finite numbers of strategies is considered. Separately, each player selects one of his strategies. A specified pair of payoffs, one to each player, occurs for every combination of a strategy for each player. The payoffs that a player receives for the strategy combinations can be conveniently stated in matrix form, where the rows correspond to his strategies and the columns to the strategies of the other player.

A player is said to use a mixed strategy when the method of selecting a strategy is random. That is, the player randomly selects one of his possible strategies according to probabilities that he specifies. The concept of mixed strategies introduces probabilistic considerations into game theory. When at least one player uses a randomly chosen strategy, the payoff to each player is a random variable (with a distribution determined by the probabilities used). The distributions for these two random payoffs constitute the maximum information that is available.

A basic problem of game theory is determination of optimum mixed strategies for the players (given their payoff matrices). That is, the problem is to make an optimum choice for the probabilities that determine the mixed strategies (with unit probabilities possible). Unfortunately, such a choice has many complications when all the properties of probability distributions are considered. However, this determination can be greatly simplified by only considering some type of "representative value" for a distribution. The well established expected-value approach uses the distribution mean (expected payoff to the player) as the representative value. Another reasonable way is to represent a distribution by its median. This is the foundation for median game theory, whose basic properties are given in [1].

The concepts of a player acting protectively for himself, or vindictively toward the other player, are useful in determination of optimum strategies. That is, a protective player tries to maximize the payoff he receives, without consideration of the payoff received by the other player. A vindictive player tries to minimize the payoff received by the other player, without consideration of the payoff to himself. When a player has a strategy that allows him to simultaneously be protective and vindictive, this is an optimum strategy for him.

Let the players be designated as I and II. Median game theory has the properties: A largest value P_I (P_{II}) occurs in the payoff matrix for player I (II) such that, when acting protectively, he can assure himself at least this payoff with probability at least 1/2. Also, a smallest value P_I' (P_{II}') occurs in the matrix for player I (II) such that vindictive player II (I) can assure, with robability at least 1/2, that player I (II) receives at most this payoff. The inequalities $P_I' \leq P_I$ and $P_{II}' \leq P_{II}$ hold, with equality possible.

Payoff matrices occur such that each player can simultaneously be protective and vindictive (according to the median criterion). These situations are called median competitive. A subclass of the median competitive games is identified in [1]. The complete class is identified here and a method is given for determining strategies that are median optimum for this class.

Consider the pairs of payoffs that correspond to the strategy combinations for the players. Those pairs such that the payoff to player I is at least P_I and also the payoff to player II is at most P_{II}' constitute set I. Those pairs such that both the payoff to player II is at least P_{II} and the payoff to player I is at most P_I' constitute set II. *Median Competitive Class:* The payoff matrices for the players result in a median competitive game if and only if player I can assure, with probability at least 1/2 that a pair in set I occurs; also, player II can assure, with probability at leeast 1/2, that a pair in set II occurs. The combinations of payoff matrices that yield median competitive situations are extensive.

For expected-value game theory, the players can be simultaneously protective and vindictive when the payoff matrices satisfy a zero-sum condition (sum of payoffs is zero for every strategy combination) or one of some mild modifications of this condition. There "zero-sum" situations are a very small subclass of the median competitive situations that occur for the case where payoffs satisfy the arithmetical operations (are cardinal numbers). However, median competitive situations can also occur when the payoffs are not cardinal numbers. A sufficient condition for use of median game theory is that, separately for each matrix, the payoffs can be ranked.

Extensiveness of use is but one of the application advantages of median game theory in comparison with expected-value game theory. A discussion of the practical advantages of median game theory is given in [2].

A one-player form of the median competitive situation can occur. That is, player I (II) can assure, with probability at least 1/2, that a pair in set I (set II) is obtained, but player II (I) cannot necessarily assure that a pair in set II (set I) occurs with probability at least 1/2. Then player I (II) can be simultaneously protective and vindictive but this is not necessarily the case for player II (I). This one-player median competitive situation, called OPMC, seems to have no analogue in expected-value game theory and is a further application advantage of median game theory.

The next section provides simplified way to evaluate P_I, P_{II}, P_I', P_{II}' and a method (oriented toward minimum effort) of obtaining median optimum strategies for median competitive and OPMC situations. The final section presents justification for material stated previously, including the identification of median competitive and OPMC games.

Results

Determination of values for P_I, P_{II}, P_I', P_{II}' is considered first. This can be accomplished by a marking of some of the values in the payoff matrices. The method given here is a simplification of the procedure in [1] (and is not based on development and use of preferred sequences).

For player I (II) acting protectively, first mark the position(s) in his matrix of the largest payoff value. Then also mark the position(s) of the next to largest payoff value. Continue this marking, according to decreasing payoff value, until the first time that marks in all the columns can be obtained from two or fewer of the rows (then a marked value can be assured with probability at least 1/2, perhaps greater than 1/2). Now, remove the mark(s) for the smallest of the payoffs used and (by the following method) determine whether some one of the remaining marks can be assured with probability at least 1/2. This cannot occur unless there ore still marks in all the columns. When all columns are still marked, replace every marked payoff by the value unity and every unmarked payoff by zero. Consider the resulting matrix of ones and zeroes to be for a zero-sum game with an expected value basis and solve for the value of this game to player I (II). Some one of the remaining marks can be obtained with probility at least 1/2 if and only if this game value is at least 1/2.

When protective player I (II) cannot assure a remaining mark with probability at least 1/2, the value of P_I (P_{II}) is the payoff value in the matrix of player I (II) that had its marking(s) removed last. Otherwise (a game value of at least 1/2), those of the remaining marks that correspond to the smallest of the remaining marked payoffs are removed. Then, as just discussed, determine whether some one of the marks remaining now can be assured with probability at least 1/2. If not, P_I (P_{II}) equals the payoff in the matrix of player I (II) that had its marking(s) removed last. If a probability of at least 1/2 can be assured, continue in the same way (removing the mark(s) for the smallest of the remaining payoffs with marks) until the first time that some one of the remaining marks cannot be assured with probability at least 1/2. Then P_I (P_{II}) is the payoff in the matrix of player I (II) that had its marking(s) removed last. It is to be noted that P_I and P_{II} are often the payoffs which provided the first time that two or fewer rows contained marks in all columns of the respective matrices.

For player I (II) acting vindictively, first mark the position(s) in the matrix for player II (I) of the smallest payoff value. Then also mark the position(s) of the next to smallest payoff value. Continue this marking, according to increasing payoff value, until the first time that marks in all the rows can be obtained from two or fewer of the columns (assures that a marked value can be obtained with probability at least 1/2). Next remove the marks for the largest of the payoffs used and (by the following method) determine whether some one of the remaining marks can be assured with probability at least 1/2. This is not possible unless there are still marks in all the rows. When all rows still contain marks, replace every marked payoff by the value zero and every unmarked payoff by unity. The resulting matrix of ones and zeroes is considered to be for a zero-sum game with an expected value basis, with rows corresponding to strategies for player II (I). Solve this game for its value to player II (I). Some one of the remaining marks can be obtained with probability at least 1/2 by player I (II) if and only if this game value is at most 1/2.

When vindictive player I (II) cannot assure a remaining mark with probability at least 1/2, the value of P_{II}' (P_I') is the payoff value in the matrix of player II (I) that had its markings removed last. Otherwise (a game value of at most 1/2), those of the remaining marks that correspond to the largest of the remaining marks payoffs are removed. Then, as just discussed, determine whether some one of the marks still remaining can be assured by player I (II) with probability at least 1/2. If not, P_{II}' (P_I') equals the payoff in the matrix of player II (I) that had its marking(s) removed last. If a probability of at least 1/2 can be assured, continue in the same manner (removing the mark(s) for the largest of the remaining payoffs with marks) until the first time that some one of the remaining marks cannot be assured by player I (II) with probability at least 1/2. Then P_{II}' (P_I') is the payoff in the matrix of player II (I) that had its marking(s) removed last. Often, P_{II}' and P_I' are the payoffs which furnished the first time that two or fewer columns contained marks in all the rows of the respective matrices.

Statement of results for one-player median competitive (OPMC) games is sufficient. That is, a game is median competitive if and only if it is OPMC for each player.

To determine whether a OPMC situation occurs for player I (II), mark the positions of his matrix for his payoffs in the pairs of set I (II). If marks in all columns can be obtained from two or fewer rows, the situation is automatically OPMC for player I (II). If at least one column contains no marks, the situation is not OPMC for player I (II). Otherwise, replace every marked payoff by unity and every unmarked payoff by zero. Consider the resulting matrix to be for a zero-sum game with an expected value basis. The situation is OPMC for player I (II) if and only if the value of this game is at least 1/2.

Now, consider determination of an optimum strategy for player I (II) when the situation is OPMC for him. Use the same marking as for the preceding paragraph and replace marked values by unity and unmarked values by zero. Again treat the resulting matrix as a zero-sum game with an expected value basis. An optimum strategy for player I (II) in this zero-sum game is a median optimum OPMC strategy for that player. The probability that player I (II) receives at least P_{I} (P_{II}) and simultaneously player II (I) receives at most P_{II}' (P_{I}') is at least equal to the game value. This method of choosing a median optimum OPMC strategy tends to maximize the game value and also tends to minimize the application effort. Other methods based on choice of a preferred sequence order for the pairs of payoffs [1] can be developed in a straightforward manner, but only this method is considered here.

Incidentally, a similar method, in which all payoffs at least equal to P_{I} (P_{II}) are marked in the matrix for protective player I (II), could be used in determining protective median optimum strategies (rather than the method in [1] that uses preferred sequences). This tends to maximize the probability of player I (II) receiving at least P_{I} (P_{II}), also to minimize the application effort (since a preferred sequence is not developed and the determination of P_I (P_{II}) requires evaluation of at most one zero-sum game). Likewise, the method in which all payoffs at most equal to P_I' (P_{II}') are marked in the matrix for player I (II) could be used in determining a vindictive median optimum strategy for player II (I).

Justification of Material

First, consider identification of median competitive and OPMC games. Player I (II) is simultaneously protective and vindictive if and only if he can simultaneously assure, with probability at least 1/2, that he receives at least P_I (P_{II}) and that player II (I) receives at most P_{II}' (P_I'). Evidently, this occurs if and only if player I (II) can assure the occurrence of a pair in set I (set II) with probability at least 1/2.

The assertions about the probability properties when two or fewer rows contain marks in all columns follow from:

Theorem 1. *When the marked payoffs in a player's matrix are such that marks in all columns can be obtained from two or fewer rows, the player can assure occurrence of a marked value with probability at least 1/2.*

Proof. When one row is fully marked, the probability is unity that some one of its values can be assured by the player.

Suppose that two rows are needed to provide marks in all the columns. Let $p_1, \cdots, p_r$ and $q_1, \cdots, q_s$ be the mixed strategies used (where the matrix has r rows and s columns), with a unit probability value being possible. The probability of obtaining a marked payoff is

$$\sum_{i=1}^{r} p_i Q_i ,$$

where Q_i is the sum of the q's for the columns that have marked payoffs in the i-th row. The largest value of this probability that the player can assure, by choice of $p_1, \cdots, p_r$, is

$$G = \min_{q_1, \cdots, q_s} (\max_i Q_i).$$

Let $i(1)$ and $i(2)$ be two rows that together contain marked values in all columns. For any minimizing $q_1, \cdots, q_s$, both $Q_{i(1)}$ and $Q_{i(2)}$ are at most G. Hence

$$2G \geq Q_{i(1)} + Q_{i(2)} \geq 1$$

and a probability of at least 1/2 can be assured. In fact, use of $p_{i(1)} = p_{i(2)} = 1/2$ guarantees that a marked value in one of rows $i(1)$ and $i(2)$ can be assured with probability at least 1/2. However, the value of G may exceed the probability assured by use of this mixed strategy.

The value of G is exactly 1/2 when the marking is such that two columns together contain unmarked payoffs in all rows (then, analogously, some one of the unmarked payoffs can be assured with probability at least 1/2). The probability is also 1/2 when there are two columns that have an unmarked payoff in row $i(1)$ or $i(2)$, and are such that no row of the matrix has marks in both of these columns.

The assertion about the probability properties when two or fewer columns of the other player's matrix contain marks in all rows is verified in a similar fashion. Specifically, a vindictive player can assure a marked value with probability at least 1/2 if and only if the other player can assure some one of the unmarked values (in his matrix) with probability at most 1/2. This happens if and only if the game value (to the other player) is at most 1/2.

Now consider probability statements based on expected-value solution of zero-sum games whose payoff matrices contain only ones and zeroes.

Theorem 2. *A lower bound on the probability that a player can assure some payoff of a specified subset of the payoffs in his matrix, and corresponding optimum strategies, can be determined by solution of a zero-sum game with an expected value basis. The matrix for this game has ones at all payoffs in the specified subset and zeroes elsewhere.*

Proof. Let each player use an arbitrary mixed strategy (with a unit probability possible). The expression for the expected payoff with these strategies is also the expression for the probability that some one of the payoffs in the specified subset occurs. This theorem can also be applied to the vindictive case when the other player's matrix is considered and the unmarked values are replaced by unity (see the discussion following Theorem 1).

REFERENCES

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