J. Operations Research Soc. of Japan
Vol. 11, No. 3, May 1969.

APPROXIMATIONS TO LARGE PROBABILITIES OF ALL SUCCESSES FOR GENERAL CASE AND SOME OPERATIONS RESEARCH IMPLICATIONS

JOHN E. WALSH
*Southern Methodist University**
Dallas, Texas, U.S.A.

GRACE J. KELLEHER
Institute for Defense Analyses
Arlington, Virginia, U.S.A.

(Received March 17, 1969)

ABSTRACT

There are many cases where an overall effort is successful if and only if the efforts (or events) of a sequence are all successful. Often, the principal interest is in cases where overall success has a large probability (say, at least .8). Suppose there are n efforts in the sequence and that $p_i(s_{i-1})$ is the probability that the i-th effort is a success given that preceding efforts $1, \cdots, i-1$ are successes ($i=1, \cdots, n$), where s_0 denotes no conditions. An approximate value, also sharp upper and lower bounds, are developed for the probability that all n events are successes. This is done for various levels of generality, including a form of complete generality. These results depend only on n, the generality level, and the arithmetic average of the $p_i(s_{i-1})$. They are useful when the probability of all successes is at least .8; then the approximate value is near both bounds. The necessity of only considering the arithmetic average of the $p_i(s_{i-1})$, rather than their product, sometimes can be useful in analyses

* Research partially supported by ONR Contract N00014-68-A-0515 and by Mobil Research and Development Corp. Also associated with NASA Grant NGR 44-007-028.

of an operations research nature (including reliability situations). Consider optimum allotment of resources to obtain a stated high probability of all successes. This can be obtained by minimizing the resource use subject to the arithmetic average of the $p_i(s_{i-1})$ equaling a determined value. This minimization is often less complicated than minimization subject to the product of the $p_i(s_{i-1})$ having the stated value. Also, statistical estimation of the probability of all successes is simplified when the expression using the arithmetic average of the $p_i(s_{i-1})$ is considered.

INTRODUCTION AND RESULTS

Consider n binomial events (not necessarily independent) where success and failure are the possible outcomes for an event. Often, there is interest in whether all the events are successes. That is, there is overall success if and only if every event results in success. In fact, many kinds of overall efforts can be considered to occur as a sequence of steps (or events) with overall success occurring if and only if all of the steps result in success. This is often the case for reliability situations and for accomplishment of missions (for example, military missions).

The principal interest is frequently in high probabilities of overall success (say, at least .8). Only cases with high probabilities are considered here.

Let the n events be numbered according to the sequence in which they can be considered to occur (event 1 is resolved first, etc.). Use $p_i(s_{i-1})$ to donote the probability that the i-th event results in success given that events $1, \cdots, i-1$ are successes ($i=2, \cdots, n$), while $p_1(s_0)=p_1$ is the unconditional probability that the first event is a success. Then,

$$P(\text{all successes})=\prod_{i=1}^{n} p_i(s_{i-1})$$

is the probability that all the events are successes.

If the approximation is sufficiently accurate, there are advantages in expressing P(all successes) in terms of the arithmetic average (denoted by p) of the $p_i(s_{i-1})$. First, consider operations research type situations where a stated large value of P(all successes) is required and this is to be obtained with an optimum allotment of resources. This can be accomplished (approximately) by minimizing the resource use subject to setting p equal to a determined value. Experience indicates that this minimization is usually less complicated than minimization subject to the product of the $p_i(s_{i-1})$ having the stated value.

A second advantage is in performing statistical investigation when the $p_i(s_{i-1})$ must be estimated. Probabilistic properties of an arithmetic average of estimates are usually much more easily determined than those of a product of estimates.

Estimation of a $p_i(s_{i-1})$ is often not difficult. That is, $p_i(s_{i-1})$ is ordinarily the probability that the i-th event is a success given that nothing very unusual has happened (something very unusual happens only if at least one of events $1, \cdots, i-1$ is a failure). For example, a failure in a reliability situation can result in a large stress (perhaps an explosion) in the overall system. No unusual stress occurs if all preceding events are successes. Thus, effectively, $p_i(s_{i-1})$ can be estimated by observations on the i-th event when it receives no unusual stress. Fortunately, this is the case in which observations are most easily and inexpensively obtained. In fact, it is often possible to separately consider the i-th event (perhaps removed from the overall system) if suitable environmental conditions (temperature, pressure, etc.) are maintained.

For the cases considered, the level of generality is defined in terms of the number r of the $p_i(s_{i-1})$ are required to be at most equal to p. Complete generality occurs for $r=1$ and the generality level decreases as r increases. However, r values as large as, say, $n/4$ would still seem to represent a moderate degree of generality.

Only cases where n, p, r are such that $(n-r+1)(1-p)$ is less than unity are considered. Then, for any meaningful n and r,

$$[1-(n-r+1)(1-p)]p^{r-1} \leq P(\text{all successes}) \leq p^n.$$

The lower bound is at least .8 if $n(1-p)\leq .2$ and is a monotonic increasing function of r. The bounds are close together and P(all successes) is very nearly equal to

$$(1/2)p^{r-1}[p^{n-r+1}+1-(n-r+1)(1-p)]$$

when $[1-(n-r+1)(1-p)]p^{r-1}$ is at least .8.

The final section contains derivations of these probability results. When the $p_i(s_{i-1})$ are all very large, n is large, and dependence is of a restricted nature, the more easily applied approximate results of (Walsh, 1955) are usable.

VERIFICATION

For notational simplicity, let $p_i=p_i(s_{i-1})$. Since the geometric mean of the p_i is at most equal to their arithmetic mean, P(all successes) $\leq p^n$, with equality possible (when the p_i are equal). This upper bound and its basis are applicable for all r.

Now, consider derivation of the sharp lower bound. The value of P(all successes) is

$$\prod_{i=1}^{n} p_i=\exp\left\{\sum_{i=1}^{n}\log_e[1-(1-p_i)]\right\}$$

$$=\exp\left[-\sum_{i=1}^{n}\sum_{j=1}^{\infty}(1-p_i)^j/j\right]$$

$$=\exp\left[-\sum_{j=1}^{\infty}j^{-1}\sum_{k=0}^{j}\binom{j}{k}(1-p)^{j-k}\sum_{i=1}^{n}(p-p_i)^k\right]$$

For $(n-r+1)(1-p)<1$ and $k\geq 2$, $\sum_{i=1}^{n}(p-p_i)^k$ is largest when all but r of the p_i are unity, $r-1$ of them equal p, and the other one is such that their arithmetic average is p (easily verified by considering the cases of $k=2, 3$ and the relationships for larger k). This implies that this other p_i is $1-(n-r+1)(1-p)$, so that the maximum is

$$\sum_{i=1}^{n}(p-p_i)^k=(n-r)^k(1-p)^k+(-1)^k(n-r)(1-p)^k.$$

Thus, since $r-1$ of the p_i are equal to p and also

$$\sum_{k=0}^{j}\binom{j}{k}[(n-r)^k+(-1)^k(n-r)]=[(n-r)+1]^j+(n-r)(1-1)^j,$$

which equals $(n-r+1)^j$, P(all successes) is at least equal to p^{r-1} times

$$\exp\left\{-\sum_{j=1}^{\infty}j^{-1}(1-p)^j\sum_{k=0}^{j}\binom{j}{k}[(n-r)^k+(-1)^k(n-r)]\right\}$$

$$=\exp\left\{-\sum_{j=1}^{\infty}j^{-1}[(n-r+1)(1-p)]^j\right\}$$

$$=\exp\{\log_e[1-(n-r+1)(1-p)]\}=1-(n-r+1)(1-p),$$

with equality possible.

REFERENCE

[1] Walsh, John E., "The Poisson distribution as a limit for dependent binomial events with unequal probabilities," *Operations Research*, Vol. 3 (1955), pp. 198–209.