J. Operations Research Soc. of Japan
Vol. 11, No. 3, May 1969.

ON SETS OF MOTION OF POINT PROCESSES

by

MASAO EBE

The Defense Academy

(Received January 10, 1969)

1. Introduction

In order to obtain informations on a road traffic flow, usually we analyze air photos of positions of cars taken at several fixed time points. From this stand point, in the case of low traffic density, many theoretical works to estimate a distribution of positions of cars at each time have been done by Breiman [1], Doob, [2], Maruyama [4], Thedéen [7, 8] and so on. But in this paper, in the same case as above, we study how much properties on a road traffic flow we can know by counting passing cars at a fixed point on the road. We will find soon, however, that this counting process does not give us enough informations about positions of cars on a road at a fixed time in the past. Concerning this counting process, there are only a few works by Suzuki [5, 6] and others.

Now we deal with a road traffic flow with low density as movement of a point process on R^1.

Let $\{x_n, n=1, 2, \cdots\}$, where $\{-x_n\}$ represents initial positions of cars on a road at the time $t=0$, be a discrete parameter stochastic process defined on a probability space $(\Omega_1, \mathfrak{F}_1, P_1)$ and each x_n be a finite valued non-negative random variable. And let $\{Y_n(t), t\geqq 0\}$ be a continuous parameter stochastic process for each n associated with a probability space $(\Omega_2, \mathfrak{F}_2, P_2)$. We require that $Y_n(t)$ $(n=1, 2, \cdots)$ are random variables with a common distribution, independent of each other and of the x_n. And we put $Y_n(0)=0$. $\{Y_n(t)\}$ describes a set of motion of cars, and we call $\{Y_n(t)\}$ with properties above mentioned an independent set of (random) motion. In the case of low traffic density, it seems that movements of cars are not much effected each other, so we may assume that cars are moving according to an independent set of motion.

We put

$$\varphi(x)=\begin{cases} 1 & (x\geqq 0) \\ 0 & (\text{otherwise}), \end{cases}$$

then the number of points contained in the interval $(x, x+h]$ at $t=0$, denoted by $M(x, h)$, is written as

$$(1) \qquad M(x, h)=\sum_n \{\varphi(x+h-x_n)-\varphi(x-x_n)\}.$$

We denote the position of points at the time t by

$$(2) \qquad x_n(t)=Y_n(t)-x_n.$$

And by T_n we intend the time just when each point will pass over the place $x=0$ at the first time, which is given by

$$(3) \qquad T_n=\min\{t\,;\, X_n(t)\geqq 0\}=\min\{\,;\, Y_n(t)\geqq x_n\}.$$

Further let $N(t, s)$ be the number of points passing over $x=0$ in the time interval $(t, t+s]$ for arbitrary $s\geqq 0$, then we can write

$$(4) \qquad N(t, s)=\sum_n \{\varphi(Y_n(t+s)-x_n)-\varphi(Y_n(t)-x_n)\}.$$

Now we will pay attention to this counting process $N(t, s)$. Suzuki [5, 6] has considered this counting process only in the case of an independent set of motion with constant velocities, that is the form $Y_n(t)=V_n\cdot t$. Here we are going to deal with an independent set of motion with general properties stated in the section 2. By this set of motion we consider a transformation from a point process on R^1 to an other point process on the time axis, and we study what conditions are required for this transformation and its reverse transformation to preserve the Poisson nature. Further we obtain that $N(t, s)$ has the Poisson tendency as $t\to\infty$ under very weak conditions on $\{x_n\}$. In the last section, more general motion will be considered.

2. Filtered Poisson Process

In this section, as for the initial distribution of the point process, let us assume that

(a) for any $x\geqq 0$, $M(0, x)$ is a Poisson process with parameter λ.

And we suppose that an independent set of random motion $\{Y_n(t)\}$ has the following properties;

(b)—(i) for every $\omega\in\Omega_2$, $Y_n(t)$ is a non-decreasing function of t and $Y_n(t)\uparrow\infty$ as $t\to\infty$.

(ii) for any $t\geq 0$ and $s\geq 0$ and for all $\omega\in\Omega_2$, there is a constant $v_0>\infty$ such that $0\leqq Y_n(t+s)-Y_n(t)\leqq v_0 s$, and

(iii) $\boldsymbol{E}\{Y_n(t)\}=t\cdot\boldsymbol{E}\{Y(1)\}\equiv vt$.

The assumption (b)—(i) excludes counting each car two or more times at the place $x=0$. Thus $\{T_n\}$ satisfies

$$\{\omega\in\Omega_1\times\Omega_2;\ T_n>t\}=\{\omega\in\Omega_1\times\Omega_2;\ Y_n(t)<x_n\} \tag{5}$$

and clearly T_n is a finite random variable for any integer n. Now define $k(t)$ by

$$\{\omega\in\Omega_1;\ k(t)=k\}=\{\omega\in\Omega_1;\ x_{k+1}>v_0 t\geqq x_k\} \tag{6}$$

which means the maximal possible number of points expected to be counted at $x=0$ in the time interval $(0, t]$, while the initial positions of points have been $\{-x_n\}$. And we can easily see that $k(t)$ is a Poisson process with parameter λv_0, since the equation

$$\{\omega\in\Omega_1;\ k(t)=k\}=\{\omega\in\Omega_1;\ M(0, v_0t)=k\}$$

holds. By using $k(t)$ we can rewrite $N(0, t)$ as

$$N(0, t)=\sum_{n-1}^{k(t)}\varphi\{Y_n(t)-x_n\},$$

and we obtain the following theorem.

Theorem 1.

Under assumptions (a) and (b), $N(0, t)$ is a Poisson process with parameter λv for any $t\geqq 0$.

Proof.

The following proof will be done in a almost similar way as in [6].

For $0\leqq t_1<t_1+h\leqq t_2+h$, we consider the following characteristic function $\Phi(u_1, u_2)$:

$$\Phi(u_1, u_2)=E[\exp\{iu_1N(t_1, h_1)+iu_2N(t_2, h_2)\}]$$

We put

$$Z=u_1N(t_1, h_1)+u_2N(t_2, h_2),$$

then we have

$$\begin{aligned} Z = &\sum_{n=1}^{k(t_2+h_2)}[u_1\{\varphi(Y_n(t_1+h_1)-x_n)-\varphi(Y_n(t_1)-x_n)\} \\ &\qquad +u_2\{\varphi(Y_n(t_2+h_2)-x_n)-\varphi(Y_n(t_2)-x_n)\}] \\ \equiv &\sum_{n=1}^{k(t_2+h_2)} g(x_n, \overline{Y_n(t)}), \end{aligned}$$

where $\overline{Y_n(t)}=(Y_n(t_1), Y_n(t_1+h_1), Y_n(t_2), Y_n(t_2+h_2))$. Z is a filtered Poisson process by the Poisson nature of $k(t)$. Therefore under the condition of $k(t_2+h_2)=n$, the n points $x_1<x_2<\cdots<x_n$ in the interval $(0, v_0(t_2+h_2))$ are random variables with the same distributions as the order statistics

corresponding to n independent random variables $U_1, \cdots, U_n$, uniformly distributed on the interval $(0, v_0(t_2+h_2))$. Then we have

$$\Phi(u_1, u_2)=\sum_n E(e^{iZ}|k(t_2+h_2)=n)\cdot\frac{(\lambda v_0(t_2+h_2))^n}{n!}e^{-\lambda v_0(t_2+h_2)}$$

and since $Y_n(t)$ is independent of n, we can omit n and write

$$\begin{aligned}&E(e^{iZ}|k(t_2+h_2)=n)\\&=E(e^{i\Sigma g(x_n,\overline{Y_n(t)})}|k(t_2+h_2)=n)\\&=E(e^{ig(U,\overline{Y(t)})})^n\\&=\left\{\frac{1}{v_0(t_2+h_2)}\int_0^{v_0(t_2+h_2)}E(e^{ig(\tau,\overline{Y(t)})})d\tau\right\}^n\\&=\left\{\frac{1}{v_0(t_2+h_2)}\int_0^{v_0(t_2+h_2)}f(\tau)\,d\tau\right\}^n\end{aligned}$$

where

$$f(\tau)=E(e^{ig(\tau,\overline{Y(t)})})$$

Now we have

$$\Phi(u_1, u_2)=\exp\left\{\lambda\int_0^{v_0(t_2+h_2)}(f(\tau)-1)\,d\tau\right\}.$$

And

$$\begin{aligned}f(\tau)-1&=E(e^{ig(\tau,\overline{Y(t)})})-1\\&=(e^{iu_1}-1)\cdot P\{Y(t_1+h_1)\geqq\tau>Y(t_1)\}\\&\quad+(e^{iu_2}-1)\cdot P\{Y(t_2+h_2)\geqq\tau>Y(t_2)\}\\&=(e^{iu_1}-1)\{P(Y(t_1+h_1)\geqq\tau)-P(Y(t_1)\geqq\tau)\}\\&\quad+(e^{iu_2}-1)\{P(Y(t_2+h_2)\geqq\tau)-P(Y(t_2)\geqq\tau)\}\end{aligned}$$

where the last equation derived by using the assumption (b)—(i). And since

$$\int_0^{v_0(t_2+h_2)}P\{Y(t)\geqq\tau\}d\tau=E(Y(t))\qquad\text{for } 0\leqq t\leqq(t_2+h_2),$$

we have

$$\Phi(u_1, u_2)=\exp[(e^{iu_1}-1)\cdot\lambda E\{Y(t_1+h_1)-Y(t_1)\} + (e^{iu_2}-1)\cdot\lambda E\{Y(t_2+h_2)-Y(t_2)\}] \tag{7}$$

$$=\exp[(e^{iu_1}-1)\lambda h_1\cdot v+(e^{iu_2}-1)\lambda h_2\cdot v]$$

$$=\Phi(u_1 : h_1)\cdot\Phi(u_2 : h_2) \tag{8}$$

where $\Phi(u:t)=\boldsymbol{E}[\exp iuN(0,t)]$, thus we complete the proof of the theorem.

From this proof we easily see that if $M(0,x)$ is a weighted Poisson process with parameter distribution $W(\lambda)$, then $N(0,t)$ is also a weighted Poisson process with parameter distribution $W(\lambda/v)$.

Further from the proof of the theorem, if we replace the assumption (b)—(iii) by

(b)—$\widetilde{\text{(iii)}}$. $\frac{d}{dt}E(Y_n(t))=\lambda(t)$ exists for any $t>0$.

Then the counting process $N(0,t)$ is a non-homogeneous Poisson process with time dependent parameter $v\cdot\lambda(t)$. Also we easily find the following:

Corollary

Under the conditions (a) and (b)—(i), (ii), $N(0,t)$ is a homogeneous Poisson processes if and only if (b)—(iii) holds.

Proof.

Through the above proof, it is enough to show that (b)—(iii) is neccessary to derive the equation (8) from (7). $E\{Y(t)\}$ is a continuous function of t under the assumption (b)—(ii), since

$$\lim_{\Delta\to 0}\{\boldsymbol{E}(Y(t+\Delta)-\boldsymbol{E}(Y(t))\}\leqq\boldsymbol{E}\{\overline{\lim_{\Delta\to 0}}(Y(t+\Delta)-Y(t)\}$$

$$\leqq\lim_{\Delta\to 0} v_0\cdot\Delta=0.$$

We put $\boldsymbol{E}(Y(t))=a(t)$ and we show that $a(t)=a\cdot t$ for some $a>0$. In order that the equation (8) would be true,

$$a(t+h)=a(t)+a(h) \tag{9}$$

must hold for any $t \geqq 0$ and $h \geqq 0$. Now we know $a(0)=0$. Put $a(1)=a$. From (9) it is easily seen that $a(n)=a \cdot n$ for any positive integer n and further that $a(\gamma)=a \cdot \gamma$ for any positive rational number γ. Since $a(t)$ is a continuous function, thus $a(t)=a \cdot t$ for any positive real number t.

3. Reversibility

In the section 2, we have considered, so to speak, the following random transformation which is characterised by the independent set of motion $\{Y_n(t)\}$ with properties (b)—(i), (ii) and (iii). The transformation, which is denoted by A_Y, has the form

$$T_n(\omega)=A_Y(\omega_2)x_n(\omega_1)$$

for each n where $\omega_1 \in \Omega, \omega_2 \in \Omega_2$ and $\omega=(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$. That is, A_Y is the transformation from the space, i.e. the x axis, to the time axis for each $\omega_2 \in \Omega_2$. And we have proved that under this transformation A_Y the Poisson nature is invariant. Now in this section we consider its reverse transformation, which is a mapping from the time axis to the spatial one. We examine under what conditions for A_Y, which are equivalent to conditions for the independent set of motion $\{Y_n(t)\}$, its reverse transformation also preserves Poisson natures. In order to define its reverse transformation clearly, we modify the assumptions on the independent set of motion $\{Y_n(t)\}$ as follows,

(b)—($\hat{\text{i}}$). adding (b)—(i), $Y_n(t)$ is a one to one mapping from $(0, \infty)$ to $(0, \infty)$ for each $\omega \in \Omega_2$.

Now we define $Z_n(x)$ by $Z_n(x)=Y_n^{-1}(x)$. We denote by $\{\hat{T}_1<\hat{T}_2<\hat{T}_3<\cdots<\hat{T}_n<\cdots\}$ a point process on the time axis associated with a probability space $(\hat{\Omega}_1, \hat{\mathfrak{F}}_1, \hat{P}_1)$. Further we assume ; (â). $\{T_n\}$ forms a Poisson process with parameter λv. Let us give a new point process on the spatial axis by

$$\hat{x}_n=\min\{x\,;\ \hat{T}_n-Z_n(x) \leqq 0\} \tag{10}$$

which is the same type as (3), or equivalently we can write $\hat{x}_n = Y_n(\hat{T}_n)$ for each $\omega \in \hat{\Omega}_1 \times \Omega_2$. And the number of points $\{\hat{x}_n\}$ in the interval $(x, x+h]$ is denoted by $\hat{M}(x, h)$, which is defined on a probability space $(\hat{\Omega}_1 \times \Omega_2, \hat{\mathfrak{F}}_1 \times \mathfrak{F}_2, \hat{P}_1 \times P_2)$. In order that $\hat{M}(x, h)$ should be a finite random variable, we require the following assumption in stead of (b)—(ii);

(b)—$(\hat{\text{ii}})$. for any $t \geqq 0$ and $s \geqq 0$ and for all $\omega \in \Omega_2$, there are positive constants ε and v_0 such that $\varepsilon s \leqq Y_n(t+s) - Y_n(t) \leqq v_0 s$.

This assumption will play the same role for $Z_n(x)$ as (b)—(ii) has done for $Y_n(t)$ in the section 2. Thus $\{\hat{x}_n\}$ is a filtered Poisson process.

Let us name the transformation from $\{x_n\}$ to $\{T_n\}$, generated by the independent set of motion $\{Y_n(t)\}$ with properties (b)—(i), (ii) and (iii), as $\boldsymbol{A}_0$. And it will be found that Poisson processes turn out to be invariant under $\boldsymbol{A}_0$ excluding parameters, since the modification to assumptions makes no effect for the proof of Theorem 1. And its reverse transformation $\boldsymbol{A}_0{}^r$ is defined as the transformation from $\{\hat{T}_n\}$ to $\{\hat{x}_n\}$ by the equation (10). In other words, $\boldsymbol{A}_0{}^r$ is the reverse motion of $\boldsymbol{A}_0$ through the equation (10).

Now we check whether $\hat{M}(0, x)$ is a Poisson process or not. From above, the new independent set of motion $Z_n(x)$, permitting to call this 'motion,' is also a monotone increasing function of x, $Z_n(x) \uparrow \infty$ as $x \to \infty$ and further it satisfies Lipschitz's condition. So $Z_n(x)$ has the same property as $Y_n(t)$ which satisfies (b)—$(\hat{\text{i}})$ and $(\hat{\text{ii}})$. Remark the corollary in the section 2, and it is enough only to test whether $\boldsymbol{E}(Z_n(x)) = x \cdot \boldsymbol{E}(Z_n(1))$ holds or not. But we are disappointed that generally the above equation does not hold only with the assumption (b)—(iii) for $Y_n(t)$. Thus even if $\{\hat{T}_n\}$ is a Poisson point process, we can not say generally that $\{\hat{x}_n\}$ forms a Poisson point process only under the conditions (b)—$(\hat{\text{i}})$, $(\hat{\text{ii}})$ and (iii) for $Y_n(t)$.

Now we discuss in the case of a set of motion with the form $Y_n(t) = V_n \cdot t$, where V_n is a random variable. In this case $\hat{M}(0, x)$ is surely a Poisson process, but its parameter is different from λ, because

$E\left(\frac{1}{V_n}\right)=\frac{1}{E(V_n)}$ is not always true. Thus even in this constant speed case, we will find that $M(0, x)$ and $\hat{M}(0, x)$ are Poisson processes with the same parameter λ if and only if V_n is a constant with probability one by the following lemma.

Lemma 1.

Suppose that $g(x)$ is a convex function and that it has the first and the second order derivatives. And suppose $g''(x)$ is strictly positive for any finite x. Let X be a random variable. Then in Jensen's inequality

$$\boldsymbol{E}(g(X))\geqq g(\boldsymbol{E})(X)),$$

equality holds if and only if X is a constant with probability one.

Proof.

We put $\boldsymbol{E}(X)=\mu$ and expand $g(x)$ around $x=\mu$, then we have

$$g(x)=g(\mu)+(x-\mu)g'(\mu)+\frac{1}{2}(x-\mu)^2 g''(\xi)$$

for some ξ in $[x, \mu]$ or in $[\mu, x]$. So we can write

$$g(X)=g(\mu)+(X-\mu)g'(\mu)+\frac{1}{2}(X-\mu)^2 g''(h(X, \mu))$$

for some function $h(\cdot, \mu)$, where $h(X, \mu)$ is a finite random variable. Thus $\boldsymbol{E}(g(X))=g(\mu)$ if and only if the variance of X is zero.

Now the assertion stated above this lemma will be assured by putting $g(x)=\frac{1}{x}$ in this proof.

4. Asymptotic Poisson property

In the section 2, we deal with the case that $\{x_n\}$ forms a Poisson process, but from now on under general assumptions on $\{x_n\}$ we investigate asymptotic properties of the counting process $N(t, s)$. Suzuki [5] has proved that under fairly weak conditions for $\{x_n\}$ the limiting process of $N(t, s)$ as $t\to\infty$ turns out to be a Poisson process, if the set of motion is of the form $Y_n(t)=V_n\cdot t$. Now we also show that the above statement is true even in the case of a set of fluctuating motion.

In proving the above, we refer the following theorem obtained by Goldman [3], in which he considers point processes on R^n. But in this paper we restate it as the results on R' for convienience. Before introducing the theorem, we state a property to be possessed by $\{x_n\}$.

Definition. $\{x_n\}$ will be called a G-process with parameter distribution $W(\lambda)$, if for every $\omega\in\Omega_1$ we have

$$\lim_{x\to\infty}\frac{M(0,x)(\omega)}{x}=\mu(\omega)$$

where μ is a random variable with distribution $W(\lambda)$. And we also call a sample point of a G-process a G-set.

Note that a G-set permits the existence of clusters of points by this definition.

Theorem (Goldman).

Let us assume an independent set of motion $\{Y_n(t)\}$ satisfies

(A) $\sup_y P\{Y_n(t)\in I+y\}\to 0$ as $t\to\infty$ for any interval $I\in R^1$.

Then for any G-process with parameter distribution $W(\lambda)$, the limiting process of

$$M^t(I)=no.\text{ of } X_n(t)\text{'s in } I$$

exists and it forms a weighted Poisson process with mixture distribution $W(\lambda)$ if and only if

(B) for any G-set $\{y_n\}$ with parameter λ and for any $I\in R^1$

$$\sum_n P\{Y_n(t)\in I+y\}\to\lambda|I| \qquad (t\to\infty)$$

holds, where $|I|$ denote the length of the interval I.

Now we study what conditions we must require concretely for a set of motion $Y_n(t)$ to satisfy assumptions (A) and (B), which intend a set of spread out motion. The matter is not so simple, but we can propose the following lemma, which answers to the question that what kind of an independent set of motion will satisfy the requirements of (A) and (B).

Lemma 2.

The conditions of (A) and (B) will be satisfied if

(c) $P\left\{\frac{Y_n(t)}{t}<x\right\}$ converges to a distribution function $H(x)$ as $t\to\infty$, where $H(x)$ is absolutely continuous and its density function $h(x)$ is continuous a.e. with respect to Lebesgue measure.
(It is reasonable to call this $\{Y_n(t)\}$, which satisfies (c), a set of spread out motion.)

Proof.

We first show that (c) implies (A). For any y and any $I\in R^1$ we have

$$P\{Y_n(t)\in I+ty\}=P\left\{\frac{Y_n(t)}{t}\in y+\frac{I}{t}\right\}.$$

Now from (c), we can choose $\varepsilon(J)$ small enough for any interval $J\in R^1$, and there exists T such that

$$\left|P\left\{\frac{Y_n(t)}{t}\in J\right\}-\int_{z\in J}h(z)\,dz\right|\leqq\varepsilon(J)$$

for any $t>T$. Putting $J=y+\frac{I}{t}$, we can make $\int_{z\in y+(I/t)}h(z)\,dz$ small enough, thus (A) holds. Next we reduce (B) from (c). Let us fix $\omega\in\Omega_1$ and put $M_t(y)=M(0,ty)$, where $\lim_{x\to\infty}\frac{M(0,x)}{x}=\lambda$ for this ω, so we have for any $I\in R^1$

$$S_t(I) \equiv \sum_n P\{X_n(t) \in I \mid x_1, x_2, \cdots\}$$

$$= \sum_n P\{Y_n(t) \in I + x_n\} = \int_0^\infty P\{Y(t) \in I + ty\} dM_t(y),$$

since $Y_n(t)$ is independent of n and of $\{x_n\}$. Place $m_t(y) = \frac{M_t(y)}{t}$, and we have

$$S_t(I) = \int_0^\infty t \cdot P\{Y(t) \in I + ty\} dm_t(y)$$

$$= |I| \int_0^\infty \frac{P\{Y(t) \in I + ty\}}{|I|/t} dm_t(y).$$

Therefore, the condition (c) implies

$$\lim_{t \to \infty} S_t(I) = |I| \int_0^\infty h(y) \cdot \lambda dy = \lambda |I|$$

As for details of this proof we can discuss similarly in [1] (pp. 310—311), so we omit them.

Thus using this lemma, from the theorem of Goldman and from Theorem 1 we can expect the following result.

Theorem 2.

If $\{x_n\}$ is a G-process with parameter distribution $W(\lambda)$, then under the conditions (b)—(i), (ii) and (c) the limiting process of $N(t, s)$ exists as $t \to \infty$ and it is a weighted Poisson process with mixture distribution $W(\lambda/v_1)$, where $v_1 = \int_0^\infty uh(u) du$.

Proof.

From the fundamental lemma in [3], it will be sufficient only to show that

(11) $$\lim_{t \to \infty} \sup_k P\{t \leqq T_k \leqq t + T \mid x_1, x_2, \cdots\} = 0$$

and

$$(12)\qquad \lim_{t\to\infty}\sum_k P\{t\leq T_k\leq t+T|x_1, x_2, \cdots\}=\lambda v_1 T.$$

for fixed T and for any G-set $\{x_n\}$ with parameter λ.

Since $Y_k(t)$ is independent of k and of $\{x_k\}$, by omitting k of $Y_k(t)$ we have

$$(13)\qquad \begin{aligned} P\{t\leq T_k\leq t+T|x_1, x_2, \cdots\}&=P\{Y(t)\leq x_k\leq Y(t+T)\}\\ &=P\{Y(t+T)>x_k\}-P\{Y(t)>x_k\}. \end{aligned}$$

where the last equation is derived from the non-decreasing property of $Y(t)$. Then

$$(14)\qquad \begin{aligned} &P\{Y(t+T)>x_k\}-P\{Y(t)>x_k\}\\ &\leqq\left|P\{Y(t+T)x_k\}-\int_{x_k/t+T}^{\infty}h(u)du\right|+\left|P\{Y(t)>x_k\}\right.\\ &\qquad\left.-\int_{x_k/T}^{\infty}h(u)du\right|+\left|\int_{x_k/t+T}^{x_k/t}h(u)du\right| \end{aligned}$$

From the assumption (b)—(ii), the equation (13) has meaning only if $x_k<(t+T)v_0$, so the last term of the above equation is smaller than $\frac{T}{t}v_0\cdot\max h(u)$. And considering (c), we can obtain (11).

As for (12), we will do the proof in a similar way with in the proof of Lemma 2.

$$\begin{aligned} S_t'&=\sum_k P\{t\leqq T_k\leqq t+T|x_1, x_2, \cdots\}\\ &=\sum P\{Y(t)\leqq x_k\leqq Y(t+T)\}\\ &=\int_0^{v_0(t+T/t)}tP\{Y(t)\leqq ty\leqq Y(t+T)\}dm_t(y) \end{aligned}$$

where the range of the integral is restricted by (b)—(ii). Since $\{x_n\}$ is a G-set, for any $\varepsilon>0$ there exists S such that for any $t>S$,

$$|dm_t(y)-\lambda dy|\leqq\varepsilon dy$$

uniformly in y. Thus we have, for any $t>S$,

$$\left|S_t'-\lambda\int_0^{v_0(t+T)/t} P\{Y(t)\leqq ty\leqq Y(t+T)\}tdy\right|$$
$$\leqq\varepsilon\int_0^{v_0(t+T)/t} P\{Y(t)\leqq ty\leqq Y(t+T)\}\cdot tdy$$

which we can rewrite as

$$|S_t'-\lambda\boldsymbol{E}\{Y(t+T)-Y(t)\}|\leqq\varepsilon\cdot\boldsymbol{E}\{Y(t+T)-Y(t)\}.$$

From (c) we can do $E\{Y(t+T)-Y(t)\}\to v_1T$ as $t\to\infty$ where $v_1=\int_0^\infty uh(u)du$. Thus we complete the proof.

(In this asymptotic case, the assumption (b)—(iii) is abbreviated, since (c) has played the same role as (b)—(iii) if $t\to\infty$.)

5. More General Motion

In this section we consider an independent set of motion $\{Y_n(t)\}$ with more general properties than in the former sections. We study what kinds of properties are required for $\{Y_n(t)\}$ only in order that $\{T_k\}$ should be also a G-process while $\{x_n\}$ has been a G-process. Since we are concerned in a counting process $N(0, t)$, we only consider non-decreasing $Y_n(t)$.

Now we state the assumptions for $\{x_n\}$ and $\{Y_n(t)\}$:

(a)′ $\{x_n\}$ is a G-process with parameter distribution $W(\lambda)$,

(b)′—(i) for every $\omega\in\Omega_2$, $Y_n(t)$ is a non-decreasing function of t,

(ii) $Y(t)\uparrow\infty$ as $t\to\infty$ for almost all $\omega\in\Omega_1$, and

(iii) $E(Y_n(t))=v(t)<\infty$ for any $t\geqq0$.

First we easily see that from the equation (5), under (a)′ and (b)′—(i) T_n is a proper random variable for all n if and only if (b)′—(ii) holds. And we have the following:

Lemma 3.

$N(0, t)$ is a proper random variable for any finite t under (a)′ and (b)′.

Proof.

Fix $\omega\in\Omega_1$ such that $\{x_n\}$ forms a G-set with parameter λ. Now it is sufficient only to show $E\{N(0, t) \mid x_1, x_2\cdots\}=n(t)$ is finite, since in this case by Markov's inequality

$$P\{N(0, t)>x \mid x_1, x_2\cdots\}\leqq \frac{n(t)}{x}$$

holds for any $x\geqq n(t)$.

By using (4) we have

$$n(t)=E\{N(0, t) \mid x_1, x_2, \cdots\}=\sum_n P\{Y(t)\geqq x_n\}$$

$$=\int_0^\infty P\{Y(t)\geqq ty\}dM(0, ty)$$

For any $\varepsilon>0$, there is $V<\infty$ such that for any $y>V$ we have

$$\left|\frac{dM(0, ty)}{ty}-\frac{\lambda dy}{y}\right|\leqq \varepsilon\frac{dy}{y}.$$

Then by Markov's inequality we have

$$n(t) \leqq\int_0^V P\{Y_n(t)\geqq ty\}dM(0, ty)+\int_V^\infty \frac{v(t)}{ty}dM(0, ty)$$

$$\leqq M(0, tV)+v(t) \cdot\int_V^\infty (\lambda+\varepsilon)\frac{dy}{y}<\infty.$$

Lemma 4.

Under (b)′, for any G-set $\{x_n\}$ with parameter λ, we have

$$\lim_{t\to\infty}\frac{1}{t}E\{N(0, t) \mid x_1, x_2, \cdots\}=\lambda\cdot\lim_{t\to\infty}\frac{v(t)}{t} \tag{15}$$

if the right side exists.

Proof.

For any $t>0$ we have

$$Q_t = \frac{1}{t} E\{N(0, t) \mid x_1, x_2, \cdots\}$$

$$= \frac{1}{t} \int_0^\infty P(Y(t) \geqq yt) dM(0, yt).$$

For any $\varepsilon>0$, we can choose S such that for any $t>S$ we have

$$\left| Q_t - \frac{1}{t} \int_0^\infty P(Y(t) \geqq yt) \cdot \lambda t dy \right| \leqq \varepsilon \frac{v(t)}{t}$$

Thus we have (15).

Theorem.

Under (a)′ and (b)′, further if we assume that $\lim_{t\to\infty} \frac{v(t)}{t} = v_2$ exists, then $\{T_n\}$ forms a G-process with parameter distribution $W(\lambda/v_2)$.

Proof.

Let us define

$N_V(t) = no.$ of T_k's in $(0, t)$ such that their corresponding x_k is smaller than $V \cdot t$.

For any $\varepsilon<0$, we can choose V such that

(16) $$E(N(0, t) - N_V(t) \mid x_1, x_2, \cdots) < \varepsilon t$$

for any G-set $\{x_n\}$ with parameter λ by the following reason. By choosing $V \geqq v(t)$, we have

$$E\{N(0, t) - N_V(t) \mid x_1, x_2, \cdots\}$$

$$= \sum_{x_n \geqq V \cdot t} P\{Y_n(t) \geqq x_n\} = \int_V^\infty P\{Y_n(t) \geqq yt\} dM(0, ty)$$

$$\leqq \int_V^\infty \frac{v(t)}{yt} dM(0, ty)$$

where the last equation comes from Markov's inequality. And considering that $\{x_n\}$ forms a G-set and taking V large enough, we have (16). In order to say that $\{T_n\}$ forms a G-set, it will be sufficiently only to show $\frac{N_{\gamma}(t)}{t} \to \lambda v_2$ as $t \to \infty$, thus the problem is reduced to the bounded speed case. So we can do the proof in almost similar way as in [5], now we omit it.

Acknowledgements

I thank Dr. T. Suzuki for his inspiring guidance in the development of this paper.

From Theorem 2, as long as we consider queueing problems concerning road traffic flow with low density, it is reasonable that we assume Poisson process as the input process to the queue. In case of fluctuating motion, an input process to a queueing system is to be observed in the time axis, but not in the spatial one.

References

[1] Breiman, L. (1963). The Poisson tendency in traffic distridution. Ann. Math. Statist. 34, 308–311.

[2] Doob, J. (1953). "*Stochastic processes.*" Wily, New York.

[3] Goldman, J.R. (1967). Stochastic point processes: limit theorems. Ann. Math. Statist. 38, 771–779.

[4] Maruyama, G. (1955). On the Poisson distribution derived from independent random walks. Natural Science Report, Ochanomizu Univ. 6, 1–6.

[5] Suzuki, T. (1966). Some results on road traffic distribution. J. Opns. Res. Soc. Japan. 9, 16–25.

[6] Suzuki, T. (1967). A filtered Poisson process on road traffic flow. The Memoris of the Defense Academy. 6, 501–510.

[7] Thedéen, J. (1964). A note on the Poisson tendency in traffic distribution. Ann. Math. Statist. 35, 1823–1824.

[8] Thedéen, J. (1968). On point systems in R under random motion. Arkiv för Matematik.