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A HIGH-DEPENDENTLY-INDEXED ASSIGNMENT PROBLEM

by

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1. INTRODUCTION

The Brook Problem was originally presented by B.T. Bennett and R. B. Potts [1] as a three-dependently-indexed assignment problem. Let us formulate the problem in a general manner.

Find x_{i_l} satisfying (1.1)—(1.5)

(1.1) N: a natural number $\geqq 3$

(1.2) $\sum_{l=1}^{N} i_l = 0$

(1.3) $\sum_{i_l, l \neq L} x_{i_l} = 1$ for all $L \in \{1, \cdots, N\}$

(1.4) $i_l \in \{-m, -m+1, \cdots, 0, \cdots, m-1, m\}$
for all l and for a natural number m

(1.5) $x_{i_l} \varepsilon \{0, 1\}$

For $N=3$ the formulation above is evidently equivalent to the Bennett and Potts' problem in Figure 1 in which exactly one lattice point should be chosen on the same straight line. This is also interpreted as a problem of assigning a room to each person in a leaning tower so that no two rooms are occupied on the same floor, in the same row and in the same column where rooms are set up vertically and horizontally in the tower. (Figure 2). The latter interpretation shows us a close relation between the brook problem and the ordinary three-indexed assignment problem

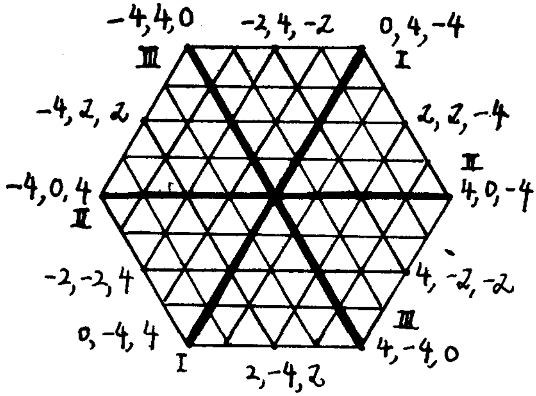
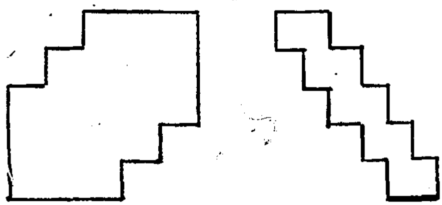


Fig. 1.



from the top diagonally

Fig. 2,

which is interpreted as an assignment problem in a cube instead of a leaning tower.

Potts showed that, once a feasible solution to the brook problem is given, other feasible solutions can be obtained by the proper transformations. Since there is no distinction among the indices, a permutation of the indices of a feasible solution brings about another feasible solution. Let us remember that we have to consider the sign of the indices because of (1.4). There are $N!=6$ permutations among three indices and for each permutation plus and minus sign occur. Potts interpreted each of twelve transformations as a product of rotations and reflection. Let r denote the rotation of the regular hexagon through $\frac{\pi}{3}$ around the center (0, 0, 0), h denote the reflection symmetry with respect to axis III joining points (−4, 4, 0) and (4, −4, 0) and e (Einheit) denote the identity.

The set of the twelve transformations is easily seen to constitute a group. Let B_3 denote this group. S_3, symmetric group of order 3, and D_{12}, dihedral group of order 12, are subgroups of B_3. The latter, particularly, is isomorphic to B_3.

2. DEVELOPMENT OF THE BROOK PROBLEM

h can also be interpreted as a rotation through π around axis III. Similarly hr^2 and hr^4 can also be interpreted as rotations through π around axis II and axis I respectively. hr, hr^3 and hr^5 can also be interpreted as rotations through π around line joining points (−4, 2, 2) and (4, −2, −2), line joining points (−2, −2, 4) and (2, 2, −4) and line joining points (−2, 4, −2) and (2, −4, 2) respectively.
Since $r^6=h^2=(hr)^2=(hr^2)^2=(hr^3)^2=(hr^4)^2=(hr^5)^2=e$, we have the following theorem.

Theorem 2.1

Group B_3 is decomposable into the seven rotation subgroups each of which is finite cyclic.

Let $[r]$ denote the cyclic group of r and we have another theorem which facilitates formal treatments of B_3.

Lemma 2.2.1

$[r]$ is a normal subgroup of B_3. I.e.,

$$\delta r^{\alpha} \delta^{-1} = r^{\beta} \qquad \text{for all } \delta \varepsilon B_3$$

Lemma 2.2.2

$$\delta r^{\alpha} \delta^{-1} = r^{-\alpha} \qquad \text{for all } \delta \varepsilon \{B_3 - [r]\}$$

Theorem 2.2

$$\rho\eta\rho = \eta \qquad \text{for all } \rho \varepsilon [r] \qquad \text{and all } \eta \varepsilon \{B_3 - [r]\}$$

Proof

From Lemmas,

$$\eta r^{\alpha} \eta^{-1} = r^{-\alpha} \qquad \text{for all } \eta \varepsilon \{B_3 - [r]\}$$

Let $\quad \eta = hr^{\gamma}, \quad$ then $(hr^{\gamma})r^{\alpha}(hr^{\gamma})^{-1} = r^{-\alpha}$

$$(hr^{\gamma})r^{\alpha} = r^{-\alpha}(hr^{\gamma})$$

$$r^{\alpha}(hr^{\gamma})r^{\alpha} = hr^{\gamma}$$

Q. E. D.

Theorem 2.3

The number of all variables of the brook problem, V_3, is expressible as a function in m:

$$V_3(m) = 3m(m+1) + 1$$

The number of the independent equations (1. 3), $E_3(m)$, is $2(3m+1)$.

Proof

V_3 is equal to the number of the lattice points in a regular hexagon each of whose edge is m long. All of the lattice points, except the center $(0, 0, 0)$, lie on edges of m regular hexagons whose edges are $1, 2, 3, \cdots, m$ long respectively. The number of all the lattice points on all the edges of length l is clearly $6l$. The center must count. Thus

$$V_3(m)=1+6(1+2+3+\cdots+m)$$
$$=1+3m(m+1)$$

The latter part of the theorem comes from a relation implied by equations (1. 3),

$$\sum_{i_1,i_2,i_3} x_{i_1,i_2i_2}=\sum_{i_2,i_3,i_1} x_{i_1,i_2,i_3}=\sum_{i_3,i_1,i_2} x_{i_1,i_2,i_3}=m$$

Thus we see an equation out of $3(2m+1)$ equations is dependent. Thus $E_3(m)=3(2m+1)-1=2(3m+1)$.

Q. E. D.

3. FOUR-INDEXED BROOK PROBLEM

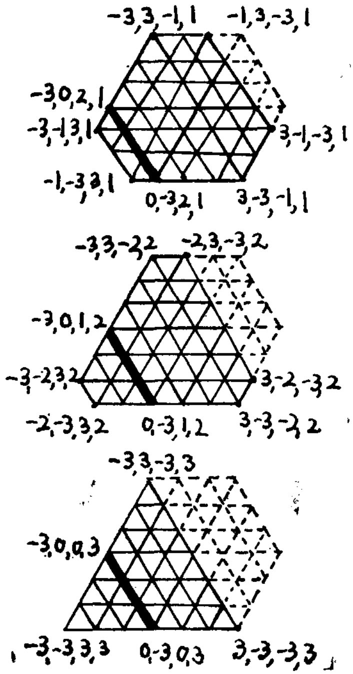
Because of a close relation of the four-indexed brook problem with the permutations of order 4 which is isomorphic to the octahedral group, it is seen that the indices satisfying (1. 2) and (1. 4) for $N=4$ form the lattice points of $(2m+1)$ convex polygons which are parallel hyperplanes of a regular octahedron. (Fig. 3)

Theorem. 3.1

Another description of the four-indexed brook problem is: choose $(2m+1)$ lattice points in a regular octahedron of size $(2m+1)$ so that no two lattice points lie on a line parallel to edges.

In terminology of geometry, an octahedron is dual to a cube (hexahedron) in the sense that a center of each plane of one polyhedron constitutes a vertex of the other polyhedron. (Fig. 4) It reflects a close relation of the three-indexed ordinary assignment problem to the four-indexed brook problem just three of whose indices are independent.

There are $4!=24$ permutations for four indices for each of which plus and minus signs occur. Each of the twenty-four permutations can be interpreted as a rotation around an axis or its power. Thirteen axes are considered in a regular octahedron.



Planes for $i_4 \leq 0$ are omitted.

Fig. 3.

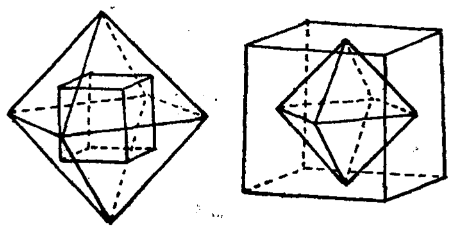


Fig. 4.

(3. 1) The identity e

(3. 2) Six rotations through π around axes joining midpoints of opposite edges of an octahedron. The axes correspond to six axes also joining midpoints of opposite edges in the dual hexahedron. The axes are all two fold. Accordingly the six associated transformations are all cyclic of order two. The general form of the associated permutation is $-(a_1\, a_2)$ with axis of $a_3=a_4=0$ implying $a_1+a_2=0$. The permutations are as follows.

$-(1\ 2)$, $-(1\ 3)$, $-(1\ 4)$, $-(3\ 4)$, $-(2\ 4)$, $-(2\ 3)$

(3. 3) Four rotations through $\frac{2}{3}\pi$ around axes joining centers of opposite faces of an octahedron and their squares. The axes correspond to four axes joining opposite vertices of the dual hexahedron. These axes are all three fold. Accordingly the four associated transformations are all cyclic of order three. The associated permutations have a form of $(a_1\, a_2\, a_3)$ with the axis joining centers of faces $a_4=-m$ and $a_4=m$. $(a_1\, a_2\, a_3)^2=(a_1\, a_3\, a_2)$ and vice versa. The four rotations and their squares are as follows.

$(1\ 3\ 2)$, $(1\ 2\ 4)$, $(1\ 4\ 3)$, $(2\ 3\ 4)$, $(1\ 2\ 3)$, $(1\ 4\ 2)$, $(1\ 3\ 4)$, $(2\ 4\ 3)$

(3. 4) Three rotations through $\frac{\pi}{2}$ around axes joining opposite vertices of an octahedron and their cubes. The axes correspond to three axes joining centers of opposite faces of the dual hexahedron. The axes are all four fold. Accordingly the associated transformations are all cyclic of order four. The associated permutations have form of $-(a_1\, a_2\, a_3\, a_4)$ having axes of $a_1+a_2=a_3+a_4=0$, $a_1=a_3$ and $a_2=a_4$.

$$\{-(a_1\, a_2\, a_3\, a_4)\}^2 \text{ will be discussed in (3. 5)}$$

$$\{-(a_1\, a_2\, a_3\, a_4)\}^3=-(a_1\, a_4\, a_3\, a_2) \text{ and vice versa.}$$

(3. 5) Three squares of rotations described in (3. 4). They are also interpreted as products of two rotations described in (3. 2) or also as products

of two reflections described later in (3.7). The general form of the associated permutations is $(a_1\ a_2)(a_3\ a_4)$. Being a square of a rotation cyclic of order four, $\{(a_1\ a_2)\ (a_3\ a_4)\}^2=e$.

(3.6) Another transformation has to be added to generate the opposite signs of transformations. It is $-e$ which is geometrically interpreted as the symmetry with respect to the center of a regular octahedron. Let i (inverse) denote this element of the group.

(3.7) Six reflections in planes cutting an octahedron at centers of four faces, i.e., planes passing through the centers of four faces. The planes correspond to planes bounded by two diagonals and two edges in the dual cube. (Fig. 5). Since the transformations are reflective, they are all cyclic of order two.

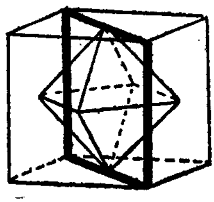


Fig. 5.

These reflections are formally expressible as $(a_1\ a_2)$ whose geometrical interpretation is a reflection in a plane cutting an octahedron at centers of four faces: $a_3=-m$, $a_3=m$, $a_4=-m$ and $a_4=m$.

(3.8) Four products of a rotation with a reflection which are equivalent to the products of a reflection with a rotation in the opposite order with different combinations out of the same rotations and the same reflections and the fifth powers of the four. A rotation is one though π around an axis joining centers of opposite faces and a reflection is one in a plane cutting an octahedron at centers of four faces. Formally

$$\{-(a_1\ a_3)\}\ (a_1\ a_2)=(a_1\ a_3)\ \{-(a_1\ a_2)\}=-(a_1\ a_2\ a_3).$$

Since $(a_1\,a_2\,a_3)$, a cycle of length three, is cyclic of order three, $-(a_1\,a_2\,a_3)$ is cyclic of order six.

$$\{-(a_1\,a_2\,a_3)\}^2=(a_1\,a_3\,a_2)=(a_1\,a_2\,a_3)^2$$

$$\{-(a_1\,a_2\,a_3)\}^3=-e=i$$

$$\{-(a_1\,a_2\,a_3)\}^4=(a_1\,a_2\,a_3)$$

$$\{-(a_1\,a_2\,a_3)\}^5=-(a_1\,a_3\,a_2)$$

$$\{-(a_1\,a_2\,a_3)\}^6=e$$

Thus it is seen that only $\{-(a_1\,a_2\,a_3)\}^5$ generates a new one and that the others are already contained in (3. 3), (3. 6) and (3. 3) respectively. It is noted that $\{-(a_1\,a_3\,a_2)\}^5=-(a_1\,a_2\,a_3)$.

(3. 9) Three products of a rotation with a reflection which are commutative, and their products with the square of the rotation. The general form of the associated permutation is $(a_1\,a_2\,a_3\,a_4)$ which is resolved in the following way.

$$\begin{aligned}(a_1\,a_2\,a_3\,a_4)&=\{-(a_1\,a_4\,a_3\,a_2)\}\{-(a_1\,a_3)\,(a_2\,a_4)\}\\&=\{-(a_1\,a_3)\,(a_2\,a_4)\}\{-(a_1\,a_4\,a_3\,a_2)\}\end{aligned}$$

It is noted that

$$(a_1\,a_4\,a_3\,a_2)=\{-(a_1\,a_4\,a_3\,a_2)\}^3\{-(a_1\,a_3)\,(a_2\,a_4)\}$$

(3. 10) Three reflections in planes cutting an octahedron along edges. The general form of the associated permutation is

$$-(a_1\,a_2)\,(a_3\,a_4)$$

i.e., a product of two transpositions with the negative sign. The plane of reflection cuts an octahedron at such points that $a_1+a_2=a_3+a_4=0$, i.e., along a plane which contains four edges and the center of an octahedron. As a reflection, it is cyclic of order two.

As is easily seen, the set of forty-eight elements of (3. 1)—(3. 10) forms a group. Let the group be denoted by B_4. Since it is known that each of forty-eight transformations has a one to one correspondence with a permutation having either sign and that every permutation is expressible as a product of transpositions, every transformation is also expressible as a product of rotations through π of (3.2) with reflections of (3.7). A product is composed at most of three transpositions because a cycle of length four can be resolved at most to three transpositions.

Other ways of describing the transformations are to express each of them as a product of rotations of (3.2) with i of (3.6) or as a product of reflections of (3.10) with i of (3.6). In either case a productis composed at most of three rotations or three reflections possibly plus an i.

Cycles of length three or four can be resolved to a product of transpositions in the following way.

$$(a_1\, a_2\, a_3)=(a_1\, a_3)\,(a_1\, a_2)$$

$$(a_1\, a_2\, a_3\, a_4)=(a_1\, a_4)\,(a_1\, a_3)\,(a_1\, a_2)$$

It can be easily seen that each of the sets of transformations (3.1)—(3.10) constitutes a conjugate class. Thus B_4 has exactly ten conjugate classes.

Theorem 3.2

The number of all variables of the four-indexed brook problem, V_4, is expressible as a function in m:

$$V_4(m)=\frac{1}{3}(2m+1)(8m^2+8m+3)$$

The number of the independent equations of (3.2) for $N=4$, $E_4(m)$, is:

$$E_4(m)=8m+3$$

Proof

Each parallel hyperplane having an integer index, say $i_4=0, \pm 1, \cdots, \pm m$ as in Figure 3, is a regular hexagon lacking a trapezoid (shown by broken lines) and gaining another trapezoid (shown by real lines outside the bold line).

For $i_4=\pm m'(1\leqq m'\leqq m)$, the number of lattice points in a dropped trapezoid is $(m+1)+(m+2)+\cdots+(m+m')=mm'+\frac{1}{2}m'(m'+1)$ and the number of lattice points in an added trapezoid is

$$m+(m-1)+\cdots+(m-m'+1)=m'(m+1)-\frac{1}{2}m'(m'+1).$$

Thus the ballance is

$$m'(m+1)-\frac{1}{2}m'(m'+1)-mm'-\frac{1}{2}m'(m'+1)=-m'^2.$$

From the theorem 2.3,

$$\begin{aligned} V_4(m)&=(2m+1)(3m^2+3m+1)-2\sum_{m'=1}^{m} m'^2 \\ &=(2m+1)(3m^2+3m+1)-\frac{1}{3}m(m+1)(2m+1) \\ &=\frac{1}{3}(2m+1)(8m^2+8m+3) \end{aligned}$$

The latter part of the theorem comes from the same reasoning as in the proof to the theorem 2.3 that an equation out of $4(2m+1)$ equations is dependent.

Q.E.D.

4. REMARKS ON HIGH-INDEXED BROOK PROBLEMS

Several properties can be claimed for the high-indexed brook problems.

Theorem 4.1

Any high-indexed brook problem is solvable.

Proof

Any natural number larger than one can be decomposed as a sum of 2's and 3's. Since the two-indexed and the three-indexed brook problems are solvable, any high-indexed brook problem has solutions which are composed of solutions to the two-indexed and the three-indexed brook problems. It may have other solutions undecomposable to any lower-indexed brook problems.

Q. E. D.

Theorem 4.2

The ratio of the number of variables taking on value one to the number of all variables decreases very quickly in N. Its proof will be easily seen since just $(2m+1)$ variables take on value one irrespective of N, the number of indices, which makes $V_N(m)$ tremendously large. If an algorithm for integer type problems is available which disregards the variables taking on zero values, it is not very much affected by increase of the number of indices, N, because the number of the variables taking on value one remains unchanged. A few steps towards it have been made by F. Hillier [3], [4] and H. Eto [2] but they are still quite insufficient for it.

5. ECONOMIC INTERPRETATIONS OF THE BROOK PROBLEMS

Let us consider a n-person integer cooperative game G1 consisting of $2m+1$ rounds (for a natural number m) in each of which each player chooses an integer from $1, 2, \cdots, 2m+1$ so that the sum of n-players' choices is equal to $2(m+1)$ and that a player is prohibited to make the same choice as he made before.

Rule 1: each player chooses in each round an integer from $1, 2, \cdots, 2m+1$ exactly once.

Rule 2: the sum of the n integers chosen by the n-players is equal to $2(m+1)$ in every round.

Let goods be defined to be indifferent with respect to their utilities for the whole society when the sum of the preference ranks given by all consumers are equal. Then G1 represents a problems as to what kinds of rankings make goods indiferent for the society of n-consumers by substituting goods for rounds.

One may doubt the theoretical foundation of this definition. It is, however, just an extension of the ordinal definition with which it coincides when every consumer gives the same preference ranking. In the case where each consumer gives a different preference rankings, no appropriate definition has been given so far, or more appropriately, it has been given up. In practice, however, we are sometimes faced with such a case.

Let us consider a slightly modified game G2 which is equivalent to G1.

Rule 1′: each player chooses in each round (or for each good) an integer from $-m, -m+1, \cdots, 0, \cdots, m-1, m$ exactly once.

Rule 2′: the sum of the n integers chosen by the n-players is equal to zero in every round (or for every good).

G2 is applicable to a model of a cooperating complex of n-firms in which every product is consumed inside the complex, every material is supplied from inside it and every firm enjoys the same amount of profit which is naturally zero. Let us denote by a row a firm forming the complex, by a column a good and by an entry of row i and column j an amount of good j firm i supplys. Since a row is interpretable as a trade with the outside, the complex need not be a closed one. G2 is also applicable to a model of n-cooperating economic regions.

G2 is clearly the brook problem itself.

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