J. Operations Research Soc. of Japan
Vol. 11, No. 1, November 1968.

LIMIT THEOREMS FOR A QUEUEING PROCESS AND THEIR APPLICATIONS

TAKEJI SUZUKI and MASAO EBE

Defense Academy, Yokosuka
(Received January, 27, 1968)

1. Introduction

In the queueing theory up to now, various types of queueing models have been exploited and investigated. But we cannot find, so far as we know, any papers dealing with a certain type of queueing models which contain some "controlable factors" depending on the state of the process. In many queueing system, in practice, we must control the system to attain some purpose and we do fairly well. For instance, we often see at a booking office in a station or at a counter in a super market that the manager alters the number of servers as the number of waiting customers is increasing or decreasing. Now we consider the matter how to control if we must.

The factors to be controled are, in many cases, relating to the service mechanism and other factors of the system, but not to the arriving one. To alter the number of servers or service counters, to control the distribution of the service time and to reduce the waiting room are such of them. There are cases, though small, such as in automation systems, where we must control arriving units. In studying the above mentioned queueing systems, we will introduce the objective functions and we consider when the control actions should be done to optimize them.

In the present stage, it is difficult to deal with this type of queueing systems in general. And in studying this, we find, it is necessary to know asymptotic properties of the output process. Thus in this paper we primarily investigate the output process in case of a general queueing system having stationary inputs and arbitrary service mechanism. We have obtained that the mean inter-departure interval is the same as the mean inter-arrival interval and moreover we have obtained a central limit theorem on the output process.

In the section 3, using results on the output process we deal with the queueing system M/G/1 with service depending on queue-length, which was treated by one of authors [4], and consider when we alter the distribution functions of the service time, observing queue-length to maximize gains into the system.

2. Theorems

It is presumed that a queueing process is mathematically specified by three well-defined stochastic processes;

$\{\xi(t),\ -\infty<t<\infty\}$ = the number of units in the system at time t,

$\{w_k,\ -\infty<k<\infty\}$ = the time spent in the system by the k-th arriving unit,

$\{\tau_k,\ -\infty<k<\infty\}$ = the interarrival time between the k-th and the $(k+1)$st units.

These processes are defined on some space Ω and for any point ω, $\xi(t,\omega)$, $\{w_k(\omega)\}$ and $\{\tau_k(\omega)\}$ represent a specific realization of a queueing process over all time. The random variables $\xi(t)$, w_k and τ_k are non-negative.

The time of arrival of the k-th unit will be denoted by t_k, and is defined by $t_{k+1}(\omega)=t_k(\omega)+\tau_k(\omega)$. For convenience we choose $t_0(\omega)<0$ and $t_1(\omega)>0$.

And we put

$$u(x)=\begin{cases}1 & \text{for } x\geqq 0\,,\\ 0 & \text{for } x<0\,,\end{cases}$$

then, for any ω,

(1.1) $$\xi(t)=\sum_{-\infty}^{\infty} u(t-t_j)u(t_j+w_j-t)\ .$$

Moreover we denote the number of units having arrived to the system in $[0, t]$ by $N(t)$ and likewise by $M(t)$ we mean the number of units having departed or escaped from the system in $[0, t]$, which contains all the units who came to the system before time t and left in $[0, t]$, whether served or not. These are represented, for any ω, as follows;

(1.2) $$N(t)=\sum_{-\infty}^{\infty} u(t-t_j)u(t_j), \qquad \text{for } t\geqq 0$$

(1.3) $$M(t)=\sum_{-\infty}^{\infty} u(t-t_j)u(t_j+w_j)u(t-t_j-w_j), \qquad \text{for } t\geqq 0$$

and $N(t)=M(t)\equiv 0$, for $t<0$.

Thus we have the following evident lemma;

Lemma 1.

For any queueing system, if $\xi(t)$ is a proper random variable for any t, then

(1.4) $$M(t)=\xi(0)+N(t)-\xi(t)$$

for all ω and $t\geqq 0$.

From now on we assume the followings:

(a) the sequence $\{\tau_k\}$, not identically equal to zero, is a strictly stationary sequence of random variables with the first and the second finite valued moments,

(b) the sequence $\{\tau_k\}$ has a spectral density $f(\sigma)$, which intends that there are no periodicities in the sequence $\{\tau_k\}$, and

(c) $\xi(t)$ and w_x are proper random variables with the first and the second finite valued moments.

Let these assumptions be done through this paper, so we will not restate them even in theorems.

Using Lemma 1 we have immedidately

Theorem 1.

If in a queueing process, with assumptions (a) and (c), the τ_k process is metrically transitive with mean $T=1/\lambda$, then we have

$$\lim_{t\to\infty}\frac{M(t)}{t}=\lambda \qquad \text{(a.s.)} \tag{1.5}$$

Proof.

Consider a specific $\omega\in\Omega$. And note that $N(t)$ has another representation

$$N(t)=\text{no. of } t_j\text{'s in } [0,t]\,. \tag{1.6}$$

From (a), we see easily that $N(t)\to\infty$ as $t\to\infty$ (a.s.).
Now Lemma 1 and the assumption (c) imply that

$$\lim_{t\to\infty}\frac{M(t)}{t}=\lim_{t\to\infty}\frac{N(t)}{t}=\lim_{t\to\infty}\frac{N(t)}{t_{N(t)}}\frac{t_{N(t)}}{t} \qquad \text{(a.s.)}\,. \tag{1.7}$$

And $1\geqq\frac{t_{N(t)}}{t}\geqq\frac{t-\tau_{N(t)+1}}{t}\to 1$ as $t\to\infty$ w.p. 1. Therefore the equation (1.7) and the metric transitivity mean that

$$\lim_{t\to\infty}\frac{M(t)}{t}=\lim_{t\to\infty}\frac{N(t)}{t}=\lim_{n\to\infty}\frac{n}{t_n}=\frac{1}{T} \qquad \text{(a.s.)}\,.$$

In the above, the last equation has been derived by J.D.C. Little [2].

Here we put the covariance function of the $\{\tau_k\}$ process

$$r(n)=E(\tau_{k+n}\,\tau_k)-\frac{1}{\lambda^2} \tag{1.8}$$

whose spectral representation is

$$r(n)=\int_0^\pi \cos u\sigma\cdot f(\sigma)d\sigma\,. \tag{1.9}$$

Now we consider asymptotic properties of the distribution and the variance of $M(t)$.

Lemma 2.

In a queueing process with assumption (c), we have for any real x

$$\lim_{t\to\infty} P\left\{\frac{M(t)-t}{\sqrt{t}} \leqq x\right\} = \lim_{t\to\infty} P\left\{\frac{N(t)-t}{\sqrt{t}} \leqq x\right\}, \tag{1.10}$$

If either side of the above exists.

Proof.

Since $\xi(t)$ is a proper random variable, $\dfrac{\xi(0)-\xi(t)}{\sqrt{t}} \to 0\ (t\to\infty)$ a.s..

And from (1.4) we have (1.10) evidently.

In order to show a central limit theorem, we introduce other notations and further we will describe the concept of "complete regularity." Let

$$H(n) = \sum_{j=1}^{n} \tau_j^*,$$

$$\eta(n) = \frac{H(n)-E(H(n))}{\sqrt{n}} = \frac{H(n)-n/\lambda}{\sqrt{n}}$$

and $F_{\eta(n)}(x)$ be the distribution function of $\eta(n)$. Further we will call a stationary process $\{\tau_k\}$ "completely regular," if

$$\lim_{n\to\infty} \sup_{\substack{A\in x_{-\infty}^{k} \\ B\in x_{k+n}^{\infty}}} |P(AB)-P(A)P(B)| = 0 \tag{1.11}$$

where x_l^m denotes the σ-algebra of ω-sets generated by the variables τ_k, $l \leqq k \leqq m$.

In other words, complete regularity means that random variables τ_k and τ_{k+n} become asymptotically independent as n increases.

Now we describe the central limiting property of a stationary process $\{\tau_k\}$ obtained by Rozanov [3, pp. 190-198].

Lemma 3 (Rozanov).

Suppose that a stationary process $\{\tau_k\}$ is

(i) completely regular and (ii) has a bounded spectral density $f(\sigma)$ which is continuous at $\sigma=0$ and $f(0)>0$. If for any $\varepsilon>0$ one can find N_ε and T_ε such that

(iii) $\int_{|x|>N_\varepsilon} x^2\, dF_{\eta(n)}(x) \leqq \varepsilon$

when $n>T_\varepsilon$, then

$$\lim_{n\to\infty} P\{\eta(n)<x\} = \frac{1}{\sqrt{2\pi b}}\int_{-\infty}^{x} e^{-y^2/2b}\, dy \tag{1.12}$$

where

$$b=\pi f(0).$$

Using the above lemmas we obtain

Theorem 2.

If in a queueing process, with assumptions (a), (b) and (c), the process $\{\tau_k\}$ satisfies the three conditions in Lemma 3, then we have

$$\lim_{t\to\infty} P\left\{\frac{M(t)-\lambda t}{\sqrt{t\lambda^3}} \leqq x\right\} = \frac{1}{\sqrt{2\pi b}}\int_{-\infty}^{x} e^{-y^2/2b}\, dy \tag{1.13}$$

Proof.

From Lemma 2, it is enough to show that under the three assumptions in Lemma 3 we have

$$\lim_{t\to\infty} P\left\{\frac{N(t)-\lambda t}{\sqrt{t\lambda^3}} \leqq x\right\} = \frac{1}{\sqrt{2\pi b}}\int_{-\infty}^{x} e^{-y^2/2b}\, dy\ .$$

Let $n\to\infty$ and $t\to\infty$ in such a way that for any fixed x

$$\frac{t-n/\lambda}{\sqrt{n}} = -x\ . \tag{1.14}$$

From Lemma 3 we have

$$P\{H(n)>t\} = P\left\{H(n)>\frac{n}{\lambda}-\sqrt{n}x\right\} = P\left\{\frac{H(n)-n/\lambda}{\sqrt{n}} > -x\right\}$$

$$\to \frac{1}{\sqrt{2\pi b}}\int_{-x}^{\infty} e^{-y^2/2b}\, dy = \frac{1}{\sqrt{2\pi b}}\int_{-\infty}^{x} e^{-y^2/2b}\, dy$$

as $n, t\to\infty$ in the manner of (1.14). Furthermore, since $\frac{t_1}{\sqrt{n}}\leqq\frac{\tau_0}{\sqrt{n}}\to 0$ $(n\to\infty)$ a.s., we have

(1.15)

$$P\{N(t)\leqq n\} = P\{H(n)>t-t_1\}\to \frac{1}{\sqrt{2\pi b}}\int_{-\infty}^{x} e^{-y^2/2b}\,dy\,, \quad (n, t\to\infty)\,.$$

Now

$$P\{N(t)\leqq n\}=P\left\{\frac{N(t)-\lambda t}{\sqrt{t\lambda^3}}\leqq\frac{n-\lambda t}{\sqrt{t\lambda^3}}\right\} \tag{1.16}$$

and the equation (1.14) reduces

$$\frac{n-\lambda t}{\sqrt{t\lambda^3}}=x\sqrt{\frac{n}{\lambda t}}=x\left(1+\frac{x\sqrt{n}}{t}\right)^{1/2}. \tag{1.17}$$

Also from (1.14)

$$n-\lambda x\sqrt{n}-\lambda t=0$$

which implies

$$\sqrt{n}=\frac{\lambda x+\sqrt{\lambda^2 x^2+4\lambda t}}{2}.$$

Therefore $\frac{\sqrt{n}}{t}\to 0$ as $t\to\infty$. Thus from (1.16), (1.17) and (1.15) we have

$$\lim_{n,t\to\infty} P\{N(t)\leqq n\}=\lim_{t\to\infty} P\left\{\frac{N(t)-\lambda t}{\sqrt{t\lambda^3}}\leqq x\right\}=\frac{1}{\sqrt{2\pi b}}\int_{-\infty}^{x} e^{-y^2/2b}\,dy\,.$$

Generally the convergence in law does not imply the convergence of moments. So we want to know of an asymptotic property of $\frac{\mathrm{Var}(M(t))}{t}$.

Theorem 3.

In a queueing process with assumptions (a), (b) and (c) we have

$$\lim_{t\to\infty}\frac{\mathrm{Var}(M(t))}{t}=\lim_{t\to\infty}\frac{\mathrm{Var}(N(t))}{t}=\frac{\pi f(0+)}{E(\tau_k)^3} \tag{1.18}$$

Proof.

In (1.18) the last equation was obtained by Cox and Lewis [1, pp. 72-78]. So it is enough to show that the first equation holds. From (1.4) we have

$$\mathrm{Var}(M(t))=\mathrm{Var}(N(t))+\mathrm{Var}(\xi(0)-\xi(t))+\mathrm{Cov}(\xi(0), N(t))-\mathrm{Cov}(\xi(t), N(t)).$$

Since the last equation in (1.18) is true and $\xi(s)$ has the finite valued second moment, we have $\lim_{t\to\infty}\frac{|\mathrm{Cov}(\xi(s), N(t))|}{t}\leqq\lim_{t\to\infty}\sqrt{\frac{\mathrm{Var}(\xi(s))}{t}\cdot\frac{\mathrm{Var}(N(t))}{t}}$ $=0$ for any s and $\lim_{t\to\infty}\frac{\mathrm{Var}(\xi(0)-\xi(t))}{t}=0$. Thus (1.18) has been proved.

3. An application.

3.1. The system.

We deal with the following system: The interarrival times $\{\tau_k\}$ are independent random variables with the same distribution function (d.f.) $F(x)=1-e^{-\lambda x}$ $(x\geqq 0)$; $=0$ $(x<0)$. We consider the case where there is only one server at the counter. Let χ_n denote the service time of the n-th customer. The service times are assumed to be independent positive random variables and also they are independent of the sequence $\{\tau_k\}$. Let $\xi(t)$ denote the number of customers in the system at time t. Let $\{s_n\}$ denote the instants of the successive departures. Further define $\xi_n=\xi(s_n+0)$. And as before denote by $M(t)$ the number of departures in $[0, t]$.

Let N be a non-negative integer and suppose that if $0\leqq\xi_{n-1}\leqq N$, χ_n is a random variable with d.f. $H_1(x)$ and if $N<\xi_{n-1}$, χ_n is a random variable with d.f. $H_2(x)$. We assume that each d.f. $H_i(x)$ has the finite mean value μ_i.

In particular, we define $N=-1$ for the case where all χ_n's have a common d.f. $H_2(x)$ and $N=\infty$ for the case where all χ_n's have a common distribution function $H_1(x)$.

If a customer arrives at the counter at an instant when the server is idle, then his service starts immediately. If he arrives at an instant when the server is busy, then his service starts immediately after the departure of the preceding customer in the queue.

In this model, $\xi(t)$, $M(t)$ and so on are random variables depending on N for each λ, $H_1(x)$ and $H_2(x)$.

Let ν_n denote the number of customers arriving at the counter during the n-th service time. Then it is easy to see that

$$\xi_{n+1}=[\xi_n-1]^+ +\nu_{n+1},$$

where $[a]^+=\max(a, 0)$. Hence the sequence $\{\xi_n\}$ of random variables forms a homogeneous Markov chain. This chain is aperiodic and irreducible since $p_{ij}>0$ for all pairs (i, j) such that $j\geqq\max(i-1, 0)$, $i=0, 1, 2, \cdots$, where

$$p_{ij}=P[\xi_{n+1}=j|\xi_n=i].$$

Now define $\rho_i=\lambda\mu_i$ and let us introduce the generating functions:

$$P_i(z)=\sum_{n=0}^{\infty} k_n^{(i)} z^n \qquad \text{for} \quad |z|\leqq 1,$$

where

$$k_n^{(i)}=\int_0^\infty e^{-\lambda t}\,[(\lambda t)^j/j!]\,dH_i(t), \quad j=0, 1, 2, \cdots, \quad i=1, 2.$$

and

$$P(z)=\sum_{n=0}^{\infty} \pi_n z^n \qquad \text{for} \quad |z|<1,$$

where $\{\pi_n\}$ is a solution of the equations

$$\sum_{i=0}^{\infty} \pi_i p_{ij}=\pi_j, \qquad (j=0, 1, 2, \cdots)$$

and further

$$Q_1(z)=\sum_{n=0}^{N} \pi_n z^n \qquad \text{for} \quad |z|\leqq 1,$$

$$Q_2(z)=\sum_{N+1}^{\infty}\pi_n z^n \qquad \text{for} \quad |z|\leqq 1 .$$

Then the system is ergodic if and only if $\rho_2<1$. Therefore $\{\pi_n\}$ with $\sum_{n=0}^{\infty}\pi_n=1$ is a stationary distribution of the queue-length and we have

$$P(z)=\pi_0\left(1-\frac{1}{z}\right)P_1(z)+\frac{1}{z}[Q_1(z)P_1(z)+Q_2(z)P_2(z)] . \tag{2.1}$$

Putting $z\to 1-0$, we have

$$1-\rho_1=\pi_0+(\rho_2-\rho_1)Q_2(1) \tag{2.2}$$

and

$$1-\rho_2=\pi_0+(\rho_1-\rho_2)Q_1(1) \tag{2.3}$$

Now if we set $Q_i^*=Q_i(1)/\pi_0$, Q_i^* is a positive constant satisfying the equation

$$Q_1^*+Q_2^*=\frac{1}{\pi_0} \tag{2.4}$$

and Q_i^* is determined only by $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$. Using Q_i^* from (2.2) and (2.3) we have

$$1-\rho_1=[1+Q_2^*(\rho_2-\rho_1)]\,\pi_0 \tag{2.5}$$

and

$$1-\rho_2=[1+Q_1^*(\rho_1-\rho_2)]\,\pi_0 . \tag{2.6}$$

Combining (1.5) and (1.6) we get the following equation

$$(1-\rho_1)Q_1^*+(1-\rho_2)Q_2^*=1 . \tag{2.7}$$

Further, if

$$\theta_i=\int_0^{\infty}t^2\,dH_i(t)<\infty \qquad (i=1,2),$$

the expectation L of the stationary distribution $\{\pi_n\}$ is given by

(2.8)
$$L=\frac{1}{1-\rho_1}\left[\pi_0+(\rho_2-\rho_1)Q_2'(1)+\frac{\lambda^2}{2}(Q_1(1)\theta_1+Q_2(1)\theta_2)\right]$$
$$=\frac{1}{1-\rho_2}\left[\pi_0+(\rho_1-\rho_2)Q_1'(1)+\frac{\lambda^2}{2}(Q_1(1)\theta_1+Q_2(1)\theta_2)\right].$$

Then the expectation W of the stationary waiting-time distribution is given by

(2.9)
$$W=\frac{L-1+\pi_0}{\lambda}.$$

In the special case where

$$H_i(t)=\begin{cases}1-e^{-t/\mu_i} & (t\geqq 0),\\ 0 & (t<0),\end{cases}$$

then

(2.10), (2.11)
$$\begin{cases}P_i(z)=1/(1+\rho_i-\rho_i z),\\ \pi_n=\rho_1^n\pi_0, \qquad (n=0,1,\cdots,N),\end{cases}$$

which lead to the following relations

(2.12)
$$\pi_0=\begin{cases}\dfrac{1-\rho_2}{1+(1-\rho_2)(1+N)}, & (\rho_1=1)\\[2ex] \dfrac{(1-\rho_1)(1-\rho_2)}{1-\rho_2-(\rho_1-\rho_2)\rho_1^{N+1}}, & (\rho_1\neq 1),\end{cases}$$

$$L=\begin{cases}\dfrac{(1-\rho_2)[-N(N+1)\rho_2+N^2+3N+4]+2\rho_2^2}{2(1-\rho_2)[1+(1-\rho_2)(N+1)]} & (\rho_1=1)\\[2ex] \{\rho_1(1-\rho_2)^2+(1-\rho_2)\rho_1^{N+4}+[\rho_2^2+(1-\rho_2)(N-1)]\rho_1^{N+3}+[(N-2)\rho_2^2+\rho_2-(N \\ \quad +1)]\rho_1^{N+2}+[-N\rho_2^2+(N+1)\rho_2]\rho_1^{N+1}\}/(1-\rho_1)(1-\rho_2)[1-\rho_2-(\rho_1-\rho_2)\rho_1^{N+1}] \\ \qquad (\rho_1\neq 1)\end{cases}$$

$$W=\begin{cases}\dfrac{N^2+N+2-2(N^2+1)\rho_2+(N^2-N+2)\rho_2^2}{2\lambda(1-\rho_2)[1+(1-\rho_2)(N+1)]} & (\rho_1=1)\\ \{(1-\rho_2)^2\rho_1^2+(1-\rho_2)\rho_1^{N+4}+[\rho_2^2+(1-\rho_2)N]\rho_1^{N+3}+[(N-3)\rho_2^2+\rho_2-N]\rho_1^{N+2} & \\ \quad +[(-N+1)\rho_2^2+N\rho_2]\rho_1^{N+1}\}/\lambda(1-\rho_1)(1-\rho_2)[1-\rho_2-(\rho_1-\rho_2)\rho_1^{N+1}] & (\rho_1\neq 1)\end{cases}$$

3.2. The objective functions

In this section we will introduce some objective functions. In general two types of losses or gains are distinguished. That they change continuously in the course of time is one type, and the other is that they have in- or de-crements at discrete time points.

As an objective function we take

$$A(N)=\lim_{T\to\infty}\frac{1}{T}E\{A_1(M(T))+A_2(T)\} \tag{2.13}$$

where $A_1(x)$ and $A_2(x)$ are functions representing losses or gains. Then our decision problem is to find N maximizing the above function N. Thus the problem is reduced to obtain the distribution of $M(T)$. But in special case where $A_1(x)=a_1x$ and $A_2(x)=a_2x$, $A(N)$ turns out to be constant, $a_1\lambda+a_2$, from the result of Theorem 1. Further in the case where $E\{A_1(M(T))\}=a_1\mathrm{Var}(M(T))$, from Theorem 3 $A(N)$ is also constant. In these cases, so far as the objective function $A(N)$ is taken up, we can choose an arbitrary N as an optimal rule if the ergodicity conditions are satisfied.

Next, we will consider the variance of the inter-departure interval and want to find N on which the variance is minimum. By virtue of the similar method used by L. Takacs [6], we have

$$\lim_{n\to\infty}P\{s_{n+1}-s_n\leqq x\}=\pi_0\,F*H_1(x)+H_1(x)\sum_{n=1}^{N}\pi_n+H_2(x)\sum_{n=N+1}^{\infty}\pi_n\,, \tag{2.14}$$

and its Laplace-Stieltjes transform

$$h(t)=Q_1(1)\psi_1(t)+Q_2(1)\psi_2(t)-\frac{t}{t+\lambda}\pi_0\psi_1(t)\,,$$

where

$$\psi_i(t)=\int_0^\infty e^{-tx}\,dH_i(x)\,,\qquad Re(t)\geqq 0\,.$$

Therefore

$$h''(0)=Q_1(1)\theta_1+Q_2(1)\theta_2+\frac{2\pi_0}{\lambda^2}(1+\rho_1)\,.$$

If $\theta_1=\theta_2=\theta$, $h''(0)=\theta^2+\frac{2\pi_0}{\lambda^2}(1+\rho_1)$, which implies that the variance desired is determined by π_0. Since π_0 is a monotone function of N as in Fig. 1, then we will take as an optimal decision either $N=\infty$ or $N=-1$ according to $\mu_1>\mu_2$ or $\mu_1<\mu_2$ respectively. However the means of the queue-length and the waiting time are maximum by these decisions.

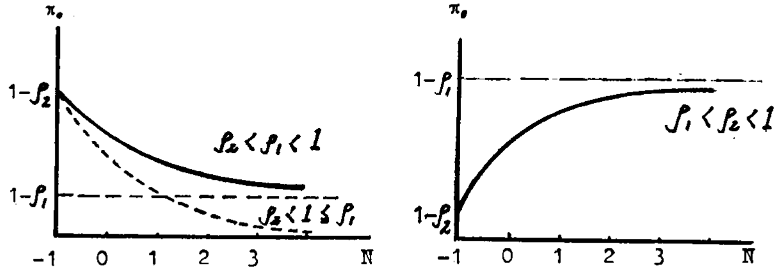


Fig. 1.

Therefore if losses caused by queue-length or waiting-time are considered we must take an other decision. For such a case we will introduce an other objective function and have a statement in the following.

Now let us denote the loss function depending on the queue-length ξ_n by $u_n(\xi_n)$. If u_n is independent of n, it is described by $u(\xi_n)$. The limiting expectation of $\frac{1}{T}\sum_{1}^{M(T)} u(\xi_n)$ is given by $\sum_{n=1}^{\infty} \pi_n u(n)$. Then our problem is to find N maximizing the following objective function;

$$B(N)=A(N)-\sum_{0}^{\infty} \pi_n u(n). \tag{2.15}$$

Analysis of this problem and numerical calculations will be given in the following section 3.3.

Also in the above statement we can take the expectation of the waiting-time in place of the queue-length. In this case we will use the equation (2.9).

3.3. Numerical examples

There are indeed various types of objective functions $B(N)$. And it is not possible to obtain optimal rules widely useful for general objective functions. Here we consider optimal rules only for the following special one.

If $A(N)$ is constant for all N, as is the case considered in the section 3.2, it is enough to minimize $\sum_{0}^{\infty} \pi_n u(n)$ in place of maximizing $B(N)$. As an example of the loss function we deal with

$$u(n)=C(n)+D[n-M]^{+} \qquad (n=0, 1, 2, \cdots) \tag{2.16}$$

where

$$C(n)=\begin{cases} C_1 & \text{if} \quad n\leq N \\ C_2 & \text{if} \quad n\geq N+1 \end{cases}$$

and C_1, C_2, D and M are non-negative constant.

Hence our problem is reduced to find N minimizing the following objective function:

$$B^*(N)=\sum_{0}^{\infty}\pi_n u(n)=C_1Q_1(1)+C_2Q_2(1)+DL_M \tag{2.17}$$

where

$$L_M=\sum_{0}^{\infty}[n-M]^{+}\pi_n .$$

Now in order to discuss optimal rules, we show numerical tables for $M/M/1$. In this case from (2.10) and (2.11) we have

$$Q_2(z)=\frac{\pi_0(1+\rho_2-\rho_2 z)}{(1+\rho_1-\rho_1 z)(1-\rho_2 z)}\rho_1^{N+1}z^{N+1}$$

and then

$$\pi_{N+j}=\frac{\pi_0\rho_1^{N+1}}{\rho_2-\rho_1+\rho_1\rho_2}\left\{\rho_2^{j+1}-\frac{\rho_1-\rho_2}{\rho_1}\left(\frac{\rho_1}{1+\rho_1}\right)^{j}\right\}\qquad (j=1,2,3,\cdots)$$

Therefore we have

$$L_M=\begin{cases} L-M+\pi_0 M\dfrac{1-\rho_1^M}{1-\rho_1}-\pi_0\dfrac{\rho_1-M\rho_1^M+(M-1)\rho_1^{M+1}}{(1-\rho_1)^2} & (M-1\leqq N)\\[2ex] \dfrac{\pi_0\rho_1^{N+1}}{\rho_2-\rho_1+\rho_1\rho_2}\left\{\dfrac{\rho_2^{M-N+2}}{(1-\rho_2)^2}-(\rho_1-\rho_2)(1+\rho_1)\left(\dfrac{\rho_1}{1+\rho_1}\right)^{M-N}\right. & (0\leqq N\leqq M-2)\\[2ex] \dfrac{\rho_2^{M+1}}{1-\rho_2} & (N=-1)\,. \end{cases}$$

Table 1.

$\rho_1=1.5,\quad \rho_2=0.5$

N	π_0	$Q_2(1)$	L	L_2	L_3
−1	0.500	1.000	1.000	0.250	0.031
0	0.250	0.750	3.000	0.975	0.245
1	0.143	0.643	3.429	1.286	0.336
2	0.087	0.587	4.043	1.825	0.487
3	0.055	0.557	4.438	2.300	0.721
4	0.035	0.532	5.352	3.265	1.036
5	0.023	0.524	6.274	4.216	1.724
6	0.015	0.515	7.210	5.172	2.503
7	0.010	0.510	8.160	6.135	3.356
8	0.006	0.506	9.160	7.147	4.277
9	0.004	0.503	10.106	8.097	5.184
∞	0	0	∞	∞	∞

$\rho_1=0.9,\quad \rho_2=0.6$

N	π_0	$Q_2(1)$	L	L_1	L_2	L_3
−1	0.400	1.000	1.500	0.900	0.540	0.117
0	0.308	0.693	1.938	1.246	0.787	0.186
1	0.255	0.516	2.190	1.445	0.905	0.226
2	0.221	0.403	2.479	1.700	1.098	1.285
3	0.197	0.323	2.799	1.996	1.351	0.367
4	0.176	0.264	3.106	2.285	1.667	0.459
5	0.163	0.218	3.361	3.524	1.317	0.503
6	0.156	0.186	3.741	2.897	2.151	0.791
7	0.147	0.158	4.042	3.189	2.454	0.974
8	0.141	0.136	4.304	3.445	2.700	1.157
9	0.135	0.118	4.609	3.744	2.987	1.383
∞	0.100	0	9.000	8.100	7.290	5.314

$\rho_1=0.8,\quad \rho_2=0.4$

N	π_0	$Q_2(1)$	L	L_1	L_2
−1	0.600	1.000	0.667	0.267	0.107
0	0.429	0.572	1.195	0.610	0.305
1	0.349	0.372	1.423	0.772	0.396
2	0.303	0.259	1.682	0.985	0.530
3	0.275	0.188	1.942	1.217	0.712
4	0.254	0.139	2.174	1.428	0.886
5	0.242	0.106	2.413	1.655	1.090
6	0.232	0.081	2.620	1.852	1.270
7	0.223	0.062	2.775	1.998	1.399
8	0.219	0.049	2.964	2.183	1.577
9	0.215	0.038	3.111	2.326	1.713
∞	0.200	0	4.000	3.200	2.560

Table 2.

Numerical values of the objective function; $C_1Q_1(1)+C_2Q_2(1)+DL_M$, where $C_1=1$, $D=0.5$ but C_2 changes.

$\rho_1=1.5,\quad \rho_2=0.5$

$M=5$

N \ C_2	2	3	5	10
−1	2.02	3.02	5.02	10.02
0	1.87	2.62	4.12	7.87
1	*1.81	2.45	3.74	6.95
2	1.83	*2.42	3.59	5.53
3	1.92	2.47	*3.58	6.37
4	2.05	2.58	3.64	*6.30
5	2.39	2.91	3.96	6.58

$M=2$

N \ C_2	2	3	5
−1	*2.13	3.13	5.13
0	2.24	2.99	4.49
1	2.29	*2.93	*4.22
2	2.50	3.09	4.26
3	2.71	3.27	4.38
4	3.16	3.69	4.75
5	3.63	4.15	5.20

$\rho_1=0.9, \quad \rho_2=0.6$

$M=0$

N \ C_2	2	3	5
−1	2.75	3.75	5.75
0	2.66	3.35	4.74
1	*2.61	3.13	4.16
2	2.64	*3.04	3.85
3	2.72	3.05	3.69
4	2.82	3.08	3.61
5	2.90	3.11	3.55
6	3.00	3.16	*3.53
7	3.19	3.33	3.65
∞	5.50	5.50	5.50

$M=2$

N \ C_2	1.5	2.0	3.0	5.0
−1	1.77	2.27	3.27	5.27
0	1.74	2.09	2.78	4.17
1	*1.71	1.97	2.48	3.51
2	1.75	*1.95	2.35	3.17
3	1.84	2.00	*2.32	2.97
4	1.97	2.10	2.36	2.88
5	2.03	2.15	2.37	*2.81
6	2.16	2.26	2.44	2.81
7	2.30	2.38	2.53	2.84
∞	4.65	4.65	4.65	4.65

$M=5$

N \ C_2	1.5	2.0	3.0
−1	1.56	2.06	3.06
0	1.44	1.79	2.48
1	1.37	1.63	2.15
2	*1.34	1.55	1.95
3	1.35	1.51	1.83
4	1.36	1.49	1.75
5	1.37	*1.47	*1.69
6	1.49	1.58	1.77
7	1.57	1.65	1.81
∞	3.66	3.66	3.66

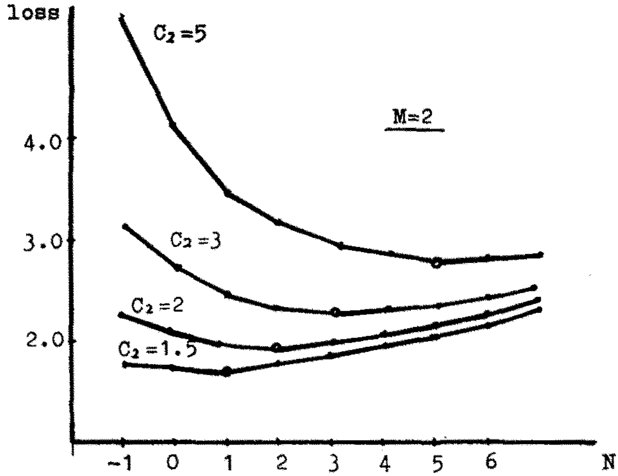
(*: optimum)

$\rho_1=0.8, \quad \rho_2=0.4$

$M=2$

N \ C_2	1.5	2.0	3.0	5.0
−1	1.55	2.05	3.05	5.05
0	1.44	1.72	2.30	3.44
1	*1.39	1.57	1.94	2.68
2	1.40	*1.52	1.78	2.30
3	1.45	1.54	1.73	2.10
4	1.51	1.58	*1.72	2.00
5	1.60	1.65	1.76	1.97
6	1.68	1.72	1.80	1.96
7	1.73	1.76	1.82	*1.94
8	1.81	1.83	1.88	1.98
∞	2.28	2.28	2.28	2.28

$\rho_1=0.9$, $\rho_2=0.6$; $C_1=1.0$, $D=0.5$



$\rho_1=1.5$, $\rho_2=0.5$; $C_1=1.0$, $D=0.5$

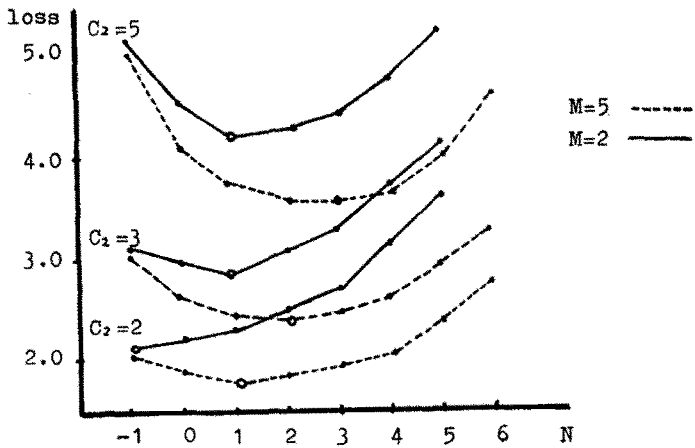


Fig. 2.

References

[1] D. R. Cox and P. A. Lewis; *The Statistical Analysis of Series of Events,* London: Methuen, 1966.

[2] J. D. C. Little; "A proof for the queueing formula: $L=\lambda W$," *Opens. Res.*, Vol. 9, No. 3, 1961.

[3] Y. A. Rozanov; *Stationary Random Processes,* Holden-dey, San Francisco, 1967.

[4] T. Suzuki; "On a queueing process with service depending on queue-length," *Seorsum Impresum Ex Tom X Commentariorum Mathematicorum Universitatis Sancti Pauli,* Japan, 1961.

[5] T. Suzuki and M. Ebe; "Decision Rules for the Queueing System M/G/1 with Service depending on Queue-length" *The Memoirs of Defense Academy,* Vol. VII, No. 3, 1967.

[6] L. Takacs; *Introduction to the Theory of Queues,* New York, 1962.