Journal of the
Operations Research
Society of Japan



VOLUME 11 *November 1968* NUMBER 1

# MULTI-STAGE REARRANGEMENT PROBLEM AND ITS APPLICATIONS TO MULTIPLE-SYSTEM RELIABILITY

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(Received October 11, 1967)

## 1. Introduction and Summary[1)]

We are given $m$ sets denoted by $A_1, A_2, \cdots, A_m$, where $m \geqslant 2$ and each set contains $n$ real numbers:

$$A_i = \{a_{i1}, a_{i2}, \cdots, a_{in}\}, \quad (i=1, \cdots, m).$$

It is convenient to express them as a matrix:

$$(1.1) \qquad \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

In this matrix, permuting the $i$-th row elements $a_{i1}, \cdots, a_{in}$, we get a

1) This paper is partly summarized in [1].

new arrangement $a'_{i1}, \cdots, a'_{in}$ for each $i$ and obtain a new matrix:

(1.2) $$\begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ a'_{21} & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ a'_{m1} & a'_{m2} & \cdots & a'_{mn} \end{bmatrix}$$

where it is assumed that

(1.3) $$A_i = \{a'_{i1}, \cdots, a'_{in}\}, \quad (i=1, \cdots, m)$$

and especially for $i=1$

(1.4) $$a_{1j} = a'_{1j}, \quad (j=1, \cdots, n).$$

For each matrix (1.2) we can easily compute the products of the column elements

(1.5) $$\prod_{i=1}^{m} a'_{ij}, \quad (j=1, \cdots, n)$$

and the sum of them

(1.6) $$\sum_{j=1}^{n} \prod_{i=1}^{m} a'_{ij}.$$

Now our $m$-stage rearrangement problem is how to find out such matrices that maximize or minimize the objective value (1.6) under the constraints (1.3) and (1.4). Let us call this "$m$-stage maximizing problem" or "$m$-stage minimizing problem," respectively. The matrices or arrangements of the elements which extremize the objective value (1.6) shall be called "optimal" and be denoted by

(1.7) $$\begin{bmatrix} a^0_{11} & a^0_{12} & \cdots & a^0_{1n} \\ a^0_{21} & a^0_{22} & \cdots & a^0_{2n} \\ \vdots & \vdots & & \vdots \\ a^0_{m1} & a^0_{m2} & \cdots & a^0_{mn} \end{bmatrix}$$

where it is assumed that

$$A_i=\{a^0_{i1}, a^0_{i2}, \cdots, a^0_{in}\}, \quad (i=1, \cdots, m)$$

and especially for $i=1$ that

$$a^0_{11}\leqslant a^0_{12}\leqslant\cdots\leqslant a^0_{1n}.$$

The products of the column elements

$$\prod_{i=1}^{m} a^0_{ij}, \quad (j=1, \cdots, n) \tag{1.8}$$

shall be called the "optimal products" or "optimal combination" and the products (1.5), "feasible products" or "feasible combination."

In the special case of $m=2$, our problem can be reduced to a special type of the so-called "assignment problem" and can be solved easily by the usual technique. Moreover, the problem is nothing but one of "the rearrangement of two sets" [4] and the optimal solution is obtained much more easily by the next propositions.[2]

**Proposition 1°.** "For our 2-stage maximizing problem, the necessary and sufficient condition for a feasible combination to be optimal is that the combination can be expressed by

$$a^0_{1j}\cdot a^0_{2j} \quad (a^0_{1j}\in A_1, a^0_{2j}\in A_2) \quad (j=1, \cdots, n) \tag{1.9}$$

after such change of notations that the elements satisfy

$$\begin{cases} a^0_{11}\leqslant a^0_{12}\leqslant\cdots\leqslant a^0_{1n}, \\ a^0_{21}\leqslant a^0_{22}\leqslant\cdots\leqslant a^0_{2n}." \end{cases} \tag{1.10}$$

**Proposition 2°.** "For our 2-stage minimizing problem, the necessary and sufficient condition for a feasible combination to be optimal is that the combination can be expressed by (1.9) after such change of notations that the elements satisfy

2) These results for $m=2$ are shown by two authors ([1] and [2]) independently. However, regrettable to them, the same results have already been proved in [4].

$$(1.11)\qquad \begin{cases} a_{11}^0 \leqslant a_{12}^0 \leqslant \cdots \leqslant a_{1n}^0 , \\ a_{21}^0 \geqslant a_{22}^0 \geqslant \cdots \geqslant a_{2n}^0 . \end{cases}$$"

Obviously, these propositions are equivalent to each other and assert that our 2-stage problem can be solved immediately whenever we can order the elements of $A_1$ and $A_2$, respectively.

In the cases of $m \geqslant 3$, we add the assumption that all elements of (1.1) are positive. Then, Proposition $1^0$ is easily extended to Theorem 1 and the max. problem can be solved right away using the theorem. However, the min. problem for $m \geqslant 3$ is not so easy to solve. It can also be transformed to the form of "multi-index problem" [3] or "integer programming problem," using $n^m$ unknown $0-1$ variables subject to $m \cdot n$ constraints. Perhaps there may be computational difficulties for large $m$ and $n$.

Of course, it is a "combinatorial problem," and there are $(n!)^{m-1}$ possible ways of combination. For example, in the case of $m=3$ and $n=10$, $(n!)^{m-1}=1.32 \times 10^{13}$. Direct enumeration method will be certainly inefficient for large $m$ and $n$.

Hence, we shall take a new approach to the problem. Theorem 2 states a simple method to obtain a nearly optimal solution for our min. problem with an efficient lower bound of the objective value. The theorem contains also a sufficient condition for the exact optimal solution of the problem. The usefulness of the theorem will be shown by several examples.

In a sense, Theorem 1 may be interpreted as a "policy of extremization" and Theorem 2, as a "policy of equalization." Propositions 3 and 4 clarify the correspondence of our problems with certain multiple-system reliability models. For example, the optimal construction of certain "2 out of $n$" system reduces to our $m$-stage min. problem. Summarizing these results briefly, our $m$-stage max. problem can be solved by the policy of extremization and is applicable to some reliability problems of series systems. In contrast, our $m$-stage min. problem can be approached by the policy of equalization and is applicable to some reliability problems of parallel systems.

Moreover, we shall treat our "modified problems," the modified max. problem in Example 5 and the modified min. problem in Example 6. It will be shown in Propositions 5 and 6 that the problem of constructing series-parallel systems which is a kind of resource allocation problem can be reduced to our modified problem.

## 2. $m$-Stage Problem ($m \geqq 3$)

### 2.1 $m$-Stage maximizing problem[3)]

**Theorem 1.** "For our $m$-stage max. problem ($m \geqq 3$), if all elements of (1.1) are positive, the necessary and sufficient condition for a feasible combination to be optimal is that the combination can be expressed by

$$\prod_{i=1}^{m} a_{ij}^{0}, \quad (j=1, \cdots, n) \tag{2.1}$$

after such renotations that the elements satisfy

$$a_{i1}^{0} \leqq a_{i2}^{0} \leqq \cdots \leqq a_{in}^{0}, \quad (i=1, \cdots, m) \tag{2.2}$$

where

$$\{a_{i1}^{0}, a_{i2}^{0}, \cdots, a_{in}^{0}\} = A_i, \quad (i=1, \cdots, m)." \tag{2.3}$$

*Proof.* The optimal combination of our $m$-stage max. problem ($m \geqq 3$) must be expressed, according to Proposition 1°, by the 2-stage products

$$a_{1j}^{0} \cdot b_{2j}^{0} \quad (j=1, \cdots, n)$$

whose elements satisfy

$$\begin{cases} a_{11}^{0} \leqq a_{12}^{0} \leqq \cdots \leqq a_{1n}^{0} \\ b_{21}^{0} \leqq b_{22}^{0} \leqq \cdots \leqq b_{2n}^{0} \end{cases} \tag{2.4}$$

and

3) Our max. problem for $m \geqq 3$ is different from the one treated in [4].

$$b_{2j}^0=\prod_{i=2}^{m} a_{ij}^0 \quad (j=1, \cdots, n), \tag{2.5}$$

where $a_{ij}^0$ $(i=1, \cdots, m; j=1, \cdots, n)$ must trivially be subject to (2.3). We shall prove that the elements in the combination (2.5) satisfy surely the inequalities

$$a_{i1}^0 \leqslant a_{i2}^0 \leqslant \cdots \leqslant a_{in}^0, \quad (i=2, \cdots, m). \tag{2.6}$$

If the contrary is hypothesized, there exists necessarily such a pair of elements, which we can assume to be $a_{2j}^0$ and $a_{2k}^0$ without loss of generality, that satisfy

$$a_{2j}^0 > a_{2k}^0, \quad 1 \leqslant j < k \leqslant n. \tag{2.7}$$

In this case, define $c_{3j}^0$ and $c_{3k}^0$ by

$$b_{2j}^0=a_{2j}^0 \cdot c_{3j}^0,\ b_{2k}^0=a_{2k}^0 \cdot c_{3k}^0 \tag{2.8}$$

and we get

$$c_{3j}^0 < c_{3k}^0 \tag{2.9}$$

from (2.4~5) and (2.7~8). Therefore, from these relations we obtain

$$\begin{aligned} &a_{1j}^0 \cdot a_{2k}^0 \cdot c_{3j}^0 + a_{1k}^0 \cdot a_{2j}^0 \cdot c_{3k}^0 - (a_{1j}^0 \cdot a_{2j}^0 \cdot c_{3j}^0 + a_{1k}^0 \cdot a_{2k}^0 \cdot c_{3k}^0) \\ &= a_{1j}^0 \cdot c_{3j}^0 \cdot (a_{2k}^0 - a_{2j}^0) + a_{1k}^0 \cdot c_{3k}^0 \cdot (a_{2j}^0 - a_{2k}^0) \\ &\geqslant a_{1j}^0 \cdot (a_{2j}^0 - a_{2k}^0) \cdot (c_{3k}^0 - c_{3j}^0) \\ &> 0. \end{aligned}$$

This contradicts the optimality of the combination $a_{1j}^0 \cdot b_{2j}^0$ $(j=1, \cdots, n)$ and therefore the inequalities (2.6) must be satisfied. Hence, every optimal combination has to be described by (2.1) and the elements are characterized by (2.2~3).

Conversely any combination (2.1) determined by the inequalities (2.2) is obviously optimal, as it determines the unique value of the objective. Thus, the theorem is proved.

*Note.* Theorem 1 is easily extended by a slight modification to the case where all elements of (1.1) are non-negative.

### 2.2 *m*-Stage minimizing problem

Our $m$-stage min. problem is not so easy as the max. problem mentioned above. Starting from some initial feasible combination:

$$\prod_{i=1}^{m} a_{ij}^{(0)}, \quad (j=1, \cdots, n) \tag{2.10}$$

we proceed to another feasible combination to decrease, successively, the value of the objective. For convenience of explanation, we show the procedure for $m=3$. Assume that the three sets $A_1, A_2$ and $A_3$ of (1.1) are given and that the initial feasible combination of the elements is

$$\prod_{i=1}^{3} a_{ij}^{(0)}, \quad (j=1, \cdots, n) \tag{2.11}$$

where

$$a_{ij}^{(0)} \in A_i, \quad (i=1, 2, 3; \; j=1, \cdots n).$$

The objective value is

$$z^{(0)} = \sum_{j=1}^{n} \prod_{i=1}^{3} a_{ij}^{(0)}.$$

First, we put

$$b_{2j}^{(0)} = a_{2j}^{(0)} \cdot a_{3j}^{(0)}, \quad (j=1, \cdots, n)$$

and consider the problem of 2-stage min. problem between the sets $A_1 = \{a_{11}^{(0)}, \cdots, a_{1n}^{(0)}\}$ and $\{b_{21}^{(0)}, \cdots, b_{2n}^{(0)}\}$.
From Proposition $2^0$ the optimal combination is given by

$$a_{1j}^{(1)} \cdot b_{2j}^{(1)}, \quad (j=1, \cdots, n)$$

where every $a_{1j}^{(1)} \in A_1$, every $b_{2j}^{(1)} \in \{b_{21}^{(0)}, \cdots, b_{2n}^{(0)}\}$ and

$$\begin{cases} a_{11}^{(1)} \leqslant a_{12}^{(1)} \leqslant \cdots \leqslant a_{1n}^{(1)} \\ b_{21}^{(1)} \geqslant b_{22}^{(1)} \geqslant \cdots \geqslant b_{2n}^{(1)}. \end{cases}$$

Rewriting

$$b_{2j}^{(1)}=a_{2j}^{(1)}\cdot a_{3j}^{(1)}$$

$$a_{1j}^{(1)}\cdot b_{2j}^{(1)}=\prod_{i=1}^{3} a_{ij}^{(1)}, \quad (j=1, \cdots, n), \tag{2.12}$$

we obtain the objective value $z^{(1)}$ of the new combination (2.12),

$$z^{(1)}=\sum_{j=1}^{n}\prod_{i=1}^{3} a_{ij}^{(1)}\leqslant z^{(0)}. \tag{2.13}$$

Thus, starting from the combination (2.11), we could obtain the new combination (2.12) to decrease the objective value as in (2.13). Let us call this the "fundamental procedure $A_1 : A_2A_3$." Similarly, we can define the fundamental procedures $A_2 : A_3A_1$ and $A_3 : A_1A_2$. And if we apply the procedure $A_2 : A_3A_1$ to the combination (2.12), we can obtain another feasible one

$$\prod_{i=1}^{3} a_{ij}^{(2)}, \quad (j=1, \cdots, n) \tag{2.14}$$

and the smaller objective value $z^{(2)}$:

$$z^{(2)}=\sum_{j=1}^{n}\prod_{i=1}^{3} a_{ij}^{(2)}\leqslant z^{(1)}.$$

Furthermore, applying the procedure $A_3 : A_1A_2$ to (2.14), we shall be able to obtain

$$\prod_{i=1}^{3} a_{ij}^{(3)}, \quad (j=1, \cdots, n)$$

$$z^{(3)}=\sum_{j=1}^{n}\prod_{i=1}^{3} a_{ij}^{(3)}\leqslant z^{(2)}.$$

Thus, applying the fundamental procedures successively and iteratively, we can rapidly decrease the objective value.

For the general case of $m\geqslant 3$, the method is just the same as for $m=3$. There exist $(2^{m-1}-1)$ fundamental procedures which are well-defined. Applying the procedures $k$ times successively to (2.10), we shall get the feasible combination

$$\prod_{i=1}^{m} a_{ij}^{(k)}, \quad (j=1, \cdots, n) \tag{2.15}$$

and the objective value

$$z^{(k)}=\sum_{j=1}^{n} \prod_{i=1}^{m} a_{ij}^{(k)} \tag{2.16}$$

where, of course,

$$a_{ij}^{(k)} \in A_i, \quad (i=1, \cdots, m;\ j=1, \cdots, n).$$

**Lemma 1.** "Starting from some initial feasible combination, $z^{(0)}$, $z^{(1)}, \cdots$ defined above is a nonincreasing sequence, and there exists a finite number $k=K$ of iterations such that thereafter no fundamental procedure can decrease the objective value. That is, there exists $K$ such that

$$z^{(0)} \geqslant z^{(1)} \geqslant \cdots \geqslant z^{(K)}=z^{(K+1)}=\cdots=z^*$$

and the products (2.15) are not altered for $k=K, K+1, \cdots$, except the change of the notation or the exchange of the elements which does not vary the objective value. Denote the minimum of the objective value $z^0$ and, then,

$$z^0 \leqslant z^*."$$

*Note.* $z^*$ may depend on the choice of the initial combination (2.10) and the order of applying the fundamental procedures. It is an upper bound of $z^0$.

In order to find a lower bound for our $m$-stage min. problem, we consider the following auxiliary minimizing problem.

Minimize

$$\sum_{i=1}^{n} x_i \tag{2.17}$$

under the constraints

$$(2.18)\quad \begin{cases} \mu_i \leqslant x_i \leqslant \lambda_i, & (i=1, \cdots, n), \\ \prod_{i=1}^{n} x_i = \pi \end{cases}$$

where the constants $\lambda_i, \mu_i$ $(i=1, \cdots, n)$ and $\pi>0$ are assumed to satisfy

$$(2.19)\quad \begin{cases} 0 \leqslant \mu_i < \lambda_i \leqslant \infty, & (i=1, \cdots, n), \\ \prod_{i=1}^{n} \mu_i < \pi < \prod_{i=1}^{n} \lambda_i . \end{cases}$$

In this problem, if $k$ elements of $\{\lambda_1, \cdots, \lambda_n\}$ are smaller than $(n-l+1)$ elements of $\{\mu_1, \cdots, \mu_n\}$, they are denoted, without loss of generality, by

$$(2.20)\qquad 0=\lambda_0<\lambda_1\leqslant \cdots \leqslant \lambda_k \leqslant \mu_l \leqslant \cdots \leqslant \mu_n < \mu_{n+1}=\infty$$

and we define the new constant $\pi_{kl}$ by

$$(2.21)\qquad \pi_{kl}=\left[\frac{\pi}{\prod_{i=1}^{k} \lambda_i \prod_{j=l}^{n} \mu_j}\right]^{1/(l-k-1)}$$

where $0 \leqslant k < l-1 \leqslant n$ and in the special case of $k=0$ or $l=n+1$ we put conventionally

$$\prod_{i=1}^{0} \lambda_i = \prod_{j=n+1}^{n} \mu_i = 1 .$$

Now the next lemma holds.

**Lemma 2.** "In the problem of minimizing the objective function (2.17) under the constraints (2.18), if the constants are assumed to satisfy (2.19) and additionally (2.22):

$$(2.22)\quad \begin{cases} 0=\lambda_0<\lambda_1\leqslant \cdots \leqslant \lambda_k \leqslant \pi_{kl} \leqslant \lambda_i, & (i=k+1, \cdots, n) \\ \mu_j \leqslant \pi_{kl} \leqslant \mu_l \leqslant \cdots \leqslant \mu_n < \mu_{n+1}=\infty, & (j=1, \cdots, l-1) \end{cases}$$

$$(0 \leqslant k < l-1 \leqslant n),$$

then, the optimal solution is given by

$$x_i^* = \begin{cases} \lambda_i, & (i=1, \cdots, k) \\ \pi_{kl}, & (i=k+1, \cdots, l-1) \\ \mu_i, & (i=l, \cdots, n). \end{cases} \tag{2.23}$$

Hence, the minimum $z_*$ of the objective function (2.17) is

$$z_* = \sum_{i=1}^{k} \lambda_i + \sum_{i=l}^{n} \mu_i + (l-k-1) \cdot \pi_{kl} \tag{2.24}$$

$$\geqslant n \cdot \pi^{1/n}."$$

*Proof.* The solution (2.23) is obviously feasible and we can neglect those points $(x_1, \cdots, x_n)$ which have much larger objective values. Hence, it is sufficient for us to consider the points constrained to a compact set even if some $\lambda_i$'s are equal to infinity and it is clear that there exists an optimal solution $(x_1^0, \cdots, x_n^0)$. Now assume that the solution $(x_1^0, \cdots, x_n^0)$ is not equal to $(x_1^*, \cdots, x_n^*)$ given by (2.23), and there must exist such coordinates $x_i^0$ and $x_j^0$ that

$$x_i^0 < x_i^*, \quad 1 \leqslant i \leqslant l-1$$

$$x_j^0 > x_j^*, \quad k+1 \leqslant j \leqslant n.$$

Next, choosing $\varepsilon > 0$ so small that

$$x_i^0 + \varepsilon < x_i^* \quad \text{and} \quad x_j^0 - \delta \geqslant x_j^*$$

where $\delta > 0$ is defined by

$$(x_i^0 + \varepsilon) \cdot (x_j^0 - \delta) = x_i^0 \cdot x_j^0,$$

we can easily verify

$$\delta > \varepsilon.$$

From this we obtain the result:

$$\{(x_i^0 + \varepsilon) + (x_j^0 - \delta)\} - (x_i^0 + x_j^0) = \varepsilon - \delta < 0,$$

which contradicts the assumed optimality of $(x_1^0, \cdots, x_n^0)$. Therefore, ther exists no optimal solution but the solution given by (2.23). This mu be the optimal one because the existence is asserted above.

We remark here that if the constants are

$$\mu_i=0, \quad \lambda_i=\infty \quad (i=1, \cdots, n),$$

then, the conditions (2.22) are satisfied by $k=0$ and $l=n+1$ and we obtain the result

$$(2.25) \qquad \begin{cases} x_i^*=\pi^{1/n}, & (i=1, \cdots, n) \\ z_*=n\cdot\pi^{1/n}, \end{cases}$$

which is well known as the relation between the arithmetic and the geometric means. This fact proves the inequality in (2.24).

*Note.* The uniqueness of the pair $(k, l)$ in Lemma 2 is easily verified if $\leqslant\pi_{kl}\leqslant$ is replaced by $<\pi_{kl}<$ in (2.22).

From Lemmas 1 and 2 the next theorem is easily derived.

**Theorem 2.** "Assume that all elements of (1.1) are positive and let the feasible combination of our $m$-stage problem be denoted by $x_j=\Pi_{i=1}^m a_{ij}$ $(j=1, \cdots, n)$. If there exist such constants $\lambda_j$ and $\mu_j$ that

$$0\leqslant\mu_j\leqslant x_j\leqslant\lambda_j, \quad (j=1, \cdots, n)$$

and if the condition (2.22) is satisfied for these constants putting $\pi=\Pi_{i=1}^m \Pi_{j=1}^n a_{ij}$, then, the minimum $z^0$ of the objective value of our $m$-stage minimizing problem is evaluated by

$$n\cdot\pi^{1/n}\leqslant z_*\leqslant z^0\leqslant z^*$$

where $z^*$ and $z_*$ are obtained from Lemmas 1 and 2.

If, moreover, $a_{i1}, \cdots, a_{in}$ are integral multiples of the greatest possible common factor $\delta_i$ $(i=1, \cdots, m)$ and if we put $\delta=\Pi_{i=1}^m \delta_i$, then, there exists an integer $N$ such that $(N-1)\cdot\delta<z_*\leqslant N\cdot\delta$ and

$$N\cdot\delta\leqslant z^0\leqslant z^*.$$

Therefore, if

$$z^*-z_*<\delta, \quad \text{or} \quad z^*=N\cdot\delta$$

holds, then,

$$z^*=z^0."$$

**Remark 1.** This theorem and Lemma 2 indicate that we should equalize the $n$ products as far as we can in our min. problem. Let us call this the "policy of equalization." In contrast, Theorem 1 shows that we should extremize the $n$ products as far as we can in our max. problem. We shall call this the "policy of extremization." As for the numerical examples of Theorem 2, see Examples 1~3 and Remark 2 in the next section.

## 2.3 Numerical examples

**Example 1.** Three sets $A_1$, $A_2$, $A_3$ are given as follows:

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 5 & 7 & 9 & 10 \\ 2 & 3 & 7 & 8 \end{bmatrix}.$$

The maximum of our objective is given by

$$\max z = 1\cdot5\cdot2+2\cdot7\cdot3+3\cdot9\cdot7+5\cdot10\cdot8=641$$

according to Theorem 1. To find the minimum of our objective, using the "policy of equalization," we take the initial feasible combination

$$\begin{bmatrix} a_{11}^{(0)} & \cdots & a_{14}^{(0)} \\ a_{21}^{(0)} & \cdots & a_{24}^{(0)} \\ a_{31}^{(0)} & \cdots & a_{34}^{(0)} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 10 & 9 & 7 & 5 \\ 8 & 7 & 3 & 2 \end{bmatrix}. \quad (2.26)$$

Applying the fundamental procedures $A_1 : A_2A_3$ and $A_2 : A_3A_1$, successively, to (2.26) we obtain

$$\begin{bmatrix} a_{11}^{(2)} & \cdots & a_{14}^{(2)} \\ a_{21}^{(2)} & \cdots & a_{24}^{(2)} \\ a_{31}^{(2)} & \cdots & a_{34}^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 10 & 5 & 9 & 7 \\ 8 & 7 & 3 & 2 \end{bmatrix} \quad (2.27)$$

and no fundamental procedure can decrease the objective. That is,

$$z^* = z^{(2)} = 1\cdot 10\cdot 8 + 2\cdot 5\cdot 7 + 3\cdot 9\cdot 3 + 5\cdot 7\cdot 2$$

$$= 80 + 70 + 81 + 70 = 301.$$

Here we can interchange the elements 8 and 7 of 3rd row of (2.27) without changing the objective value. According to Theorem 2, taking $k=0$ and $l=n+1=5$, we obtain $z_* = 4\cdot\pi^{1/4} \doteqdot 300.28$, $\delta=1$, $N=301$, $z^* - N\cdot\delta = 0$ and, hence,

$$z^* = z^0.$$

The combination of the column elements of (2.27) produces the optimal solution of our min. problem.

**Example 2.** We are given

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 7 & 5 & 8 & 1 & 6 \\ 8 & 10 & 5 & 5 & 9 \\ 7 & 7 & 6 & 5 & 5 \end{bmatrix}.$$

We rearrange the elements of each set in the reverse orders and get the initial combination

$$\text{(2.28)} \qquad \begin{bmatrix} 1 & 5 & 6 & 7 & 8 \\ 10 & 9 & 8 & 5 & 5 \\ 5 & 5 & 6 & 7 & 7 \end{bmatrix}.$$

Applying the fundamental procedures $A_3 : A_1A_2$, $A_1 : A_2A_3$ and $A_2 : A_3A_1$ successively to (2.28), we obtain

$$\text{(2.29)} \qquad \begin{bmatrix} 1 & 5 & 6 & 7 & 8 \\ 10 & 9 & 8 & 5 & 5 \\ 7 & 5 & 5 & 7 & 6 \end{bmatrix}$$

and no fundamental procedure can alter the combination any more. That is, according to Theorem 2

$$z^*=1\cdot 10\cdot 7+5\cdot 9\cdot 5+6\cdot 8\cdot 5+7\cdot 5\cdot 7+8\cdot 5\cdot 6$$
$$=70+225+240+245+240=1020\,.$$

If the product including the element 1 of $A_1$ is denoted by $x_1$, it is obviously bounded:

$$0\leqslant x_1\leqslant 70\,,$$

since the elements 10 and 7 in the first column of (2.29) are the largest ones in the respective rows. Now applying Theorem 2 to this case of

$$\lambda_1=70\,,\quad \lambda_2=\cdots=\lambda_5=\infty\,,\quad \mu_1=\cdots=\mu_5=0\,,$$

we can get

$$\pi_{kl}=\pi_{16}\fallingdotseq 237.38\,,\quad z_*=\lambda_1+4\cdot\pi_{16}\fallingdotseq 1019.6$$

and

$$N\cdot\delta=1020=z^*.$$

This asserts that the matrix (2.29) is optimal for our min. problem.

**Example 3.** We are given the following matrix:

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}=\begin{bmatrix} 4 & 3 & 9 & 10 & 5 & 9 & 4 \\ 4 & 9 & 10 & 3 & 10 & 7 & 4 \\ 7 & 3 & 8 & 5 & 8 & 3 & 10 \end{bmatrix}.$$

Our aim is to find the optimal solution of the min. problem. We rearrange the elements of each set in the reverse orders to obtain the initial combination:

$$\text{(2.30)}\qquad \begin{bmatrix} 3 & 4 & 4 & 5 & 9 & 9 & 10 \\ 10 & 10 & 9 & 7 & 4 & 4 & 3 \\ 3 & 3 & 5 & 7 & 8 & 8 & 10 \end{bmatrix}.$$

Apply the procedure $A_1: A_2A_3$, $A_2: A_3A_1$, $A_3: A_1A_2$ and $A_1: A_2A_3$ successively to (2.30), and we can get

$$(2.31) \qquad \begin{bmatrix} 3 & 4 & 4 & 5 & 9 & 9 & 10 \\ 10 & 10 & 4 & 4 & 9 & 3 & 7 \\ 7 & 5 & 10 & 8 & 3 & 8 & 3 \end{bmatrix}$$

No fundamental procedure can alter the combination (2.31) any more, and

$$\begin{aligned} z^* &= 3\cdot 10\cdot 7+4\cdot 10\cdot 5+4\cdot 4\cdot 10+5\cdot 4\cdot 8+9\cdot 9\cdot 3+9\cdot 3\cdot 8+10\cdot 7\cdot 3 \\ &= 210+200+160+160+243+216+210 \\ &= 1399\,. \end{aligned}$$

In this problem,

$$n\cdot \pi^{1/n}=7\cdot \pi^{1/7}=1384.67\cdots$$

and owing to Theorem 2 the bounds of $z^0$ are derived immediately:

$$1385\leqslant z^0 \leqslant 1399=z^*.$$

This result may be unsatisfactory. In fact, we can improve the lower bound and can prove in the sequal that the combination (2.31) is optimal. If we assume that $0\leqslant x_1\leqslant 130$, $\lambda_2=\cdots=\lambda_7=\infty$, and $\mu_1=\cdots=\mu_7=0$, we obtain $\pi_{18}=212.14\cdots$. Therefore, we can apply Theorem 2 to our problem to obtain

$$z_*=130+6\cdot 212.14=1402.84>z^*.$$

This implies that none of the products $3\cdot 3\cdot *$, $3\cdot *\cdot 3$, $*\cdot 3\cdot 3$, $4\cdot *\cdot 3$, $4\cdot 3\cdot *$, $*\cdot 4\cdot 3$, can occur in the optimal solution, where the notation $a\cdot b\cdot c$ means the product of the elements $a\in A_1$, $b\in A_2$ and $c\in A_3$.

Similarly, if we apply Theorem 2, assuming that $\lambda_1=150$, $\lambda_2=160$, $\lambda_3=\cdots=\lambda_7=\infty$, and $\mu_1=\cdots=\mu_7=0$, we obtain $\pi_{28}=218.12$ and

$$z_*=150+160+5\cdot 218.12=1400.60>z^*.$$

This implies that none of the pairs of the products $4\cdot 4\cdot *$ and $4\cdot 4\cdot *$, $4\cdot 4\cdot *$ and $5\cdot 3\cdot *$, $4\cdot 4\cdot *$ and $5\cdot *\cdot 3$, can occur in the optimal solution. Hence, the optimal combination must be either the form

$$(2.32)\qquad \begin{bmatrix} ③ & ④ & ④ & ⑤ & 9 & 9 & 10 \\ * & * & ④ & ④ & ③ & a_{26} & a_{27} \\ * & * & a_{33} & a_{34} & * & ③ & ③ \end{bmatrix}$$

or

$$(2.33)\qquad \begin{bmatrix} ③ & ④ & ④ & ⑤ & 9 & 9 & 10 \\ * & * & ④ & ④ & a_{25} & a_{26} & ③ \\ * & * & a_{33} & a_{34} & ③ & ③ & a_{37} \end{bmatrix}.$$

In either case, we must have essentially

$$4\cdot 4\cdot a_{33}\leqslant 160\,,\quad 5\cdot 4\cdot a_{34}\leqslant 160\,,\quad 9\cdot a_{26}\cdot 3\geqslant 243\,,$$

because $\{a_{33}, a_{34}\}\subset A_3$ and because $\{a_{26}, a_{27}\}$ in (2.32) or $\{a_{25}, a_{26}\}$ in (2.33) are included in $\{7, 9, 10, 10\}\subset A_2$. Now we apply Theorem 2 to our case taking $\lambda_1=\lambda_2=160$, $\lambda_3=\cdots=\lambda_7=\infty$, $\mu_1=\cdots=\mu_6=0$ and $\mu_7=243$. We obtain $\pi_{27}=208.91$, $z_*=160+160+243+4\cdot 208.91$ and $N\cdot\delta=1399$. This implies that

$$N\cdot\delta=1399=z^0=z^*$$

and hence the combination (2.31) is itself optimal.

**Remark 2.** For the initial feasible combination of our min. problem we recommend one of those which are obtained by rearranging the elements of the respective sets in the reverse orders, as exemplified in the above examples. See the combinations (2.26), (2.28), and (2.30). It should be remarked here that the limiting value $z^*$ may depend on the initial combination and the order of applying the fundamental procedures. Therefore, if $z^*-z_*$ is not sufficiently small, then, first, try the other candidates of the initial combination that are characterized above. In the case of $m=3$, there are two more such candidates in a problem. And if $z^*$ can not be decreased any more by these trials, then, second, proceed to improve the lower bound by further applications of Theorem 2 as shown in Example 3, although in many cases the simplest applications may give effective bounds as in Examples 1 and 2.

## 3. Applications to Multiple-System Reliability

### 3.1 Relation of the problem to multiple-system reliability

We are given the $m\times n$ matrix of reliabilities:

$$(3.1)\qquad \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

where $A_i$ is the set of $n$ reliabilities $a_{i1}, \cdots, a_{in}$ of $i$-th kind elements $(i=1, \cdots, m)$.

Suppose that we are going to construct $n$ series or parallel systems of the same type, each of which is composed by $m$ different kinds of elements. If we connect the $m$ elements on each column of (3.1) in series [parallel], we have $n$ series systems denoted by $(S)_1, \cdots, (S)_n$ [parallel systems $(P)_1, \cdots, (P)_n$]. Series [parallel] system here means that the system failure occurs whenever any one of its elements fails [only when all of its elements fail simultaneously]. In these cases there arise the reliability problems of how to find the optimal combination of the elements in the construction of the $n$ systems. The next propositions hold under the assumption of independent failures of the elements.

**Proposition 3.** "Constructing the series systems $(S)_1, \cdots, (S)_n$ [parallel systems $(P)_1, \cdots, (P)_n$] from given $m\cdot n$ elements, whose reliabilities [unreliabilities] are assumed to be positive, the problem of finding the optimal combination of elements that maximizes [minimizes] the probability for $n-1$ or more systems to survive [to fail] can be reduced to our $m$-stage max. [min.] problem."

*Note.* In the above proposition, the probability for $n-1$ or more systems to survive is nothing but the reliability for the so-called "$n-1$ out of $n$" structure, and minimizing the probability for $n-1$ or more systems to fail is equivalent to maximizing the reliability of "2 out of $n$" structure. Therefore, taking $n=3$, the proposition is reduced to the reliability problem of "2 out of 3" structures.

*Proof* of Proposition 3. Let $p_j$ be the probability that the series system $(S)_j$ can survive ($j=1, \cdots, n$). Then, the probability for $n-1$ or more systems among $(S)_1, \cdots, (S)_n$ to survive is

$$\prod_{j=1}^{n} p_j \cdot \left[ \sum_{j=1}^{n} p_j^{-1} - (n-1) \right],$$

where $\Pi_{j=1}^{n} p_j$ is equal to the product of given $m \cdot n$ reliabilities which does not depend on the combination of the elements. Therefore, we have only to maximize $\sum_{j=1}^{n} p_j^{-1}$. By the definition, $p_j$ is given by the product of $m$ reliabilities of these elements which compose the system $(S)_j$ ($j=1, \cdots, n$). Hence, the problem is reduced to our $m$-stage max. problem concerning the $m \times n$ matrix $(b_{ij})$, where $b_{ij}=a_{ij}^{-1}$ for all $i$ and $j$, since

$$\sum_{j=1}^{n} p_j^{-1} = \sum_{j=1}^{n} \left( \prod_{i=1}^{m} a_{ij} \right)^{-1} = \sum_{j=1}^{n} \prod_{i=1}^{m} b_{ij} .$$

On the other hand, according to Theorem 1, the present problem is equivalent to the one concerning the original matrix $(a_{ij})$ and it is easily solved.

In the case of parallel systems $(P)_1, \cdots, (P)_n$ the problem is obviously reduced to our $m$-stage min. problem concerning the $m \times n$ matrix $(c_{ij})$, where $c_{ij}$ is defined by the reciprocal of the given unreliability $a_{ij}$; that is $c_{ij}=a_{ij}^{-1}$ for every $i$ and $j$.

**Proposition 4.** "The $n$ devices $(S)_1, \cdots, (S)_n$ or $(P)_1, \cdots, (P)_n$ constructed from given $m \cdot n$ elements whose reliabilities are (3.1), are assigned, one by one, to $n$ stations. Assume that each of the stations has its own intensity, or expected frequency, $w_i$ of usage for the device assigned there ($i=1, \cdots, n$), and that an accident occurs in the station if and only if the device is in failure when it is used. Then, the problem of finding the optimal construction and assignment of the devices that minimize the total expected number of accidents can be reduced to our $(m+1)$-stage max. [min.] problem in the case of series systems $(S)_1, \cdots, (S)_n$ [parallel systems $(P)_1, \cdots, (P)_n$]."

*Proof.* Let $a'_{i1}, \cdots, a'_{in}$ be a rearrangement of the elements of $A_i$ $(i=1, \cdots, m)$ and we obtain

$$(3.2)\qquad \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ a'_{21} & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ a'_{m1} & a'_{m2} & \cdots & a'_{m1} \end{bmatrix}.$$

If the device $(S)_j$ constructed by the series connection of the $j$-th column elements of (3.2) is assigned to the station with expected value $w_j$ $(j=1, \cdots, n)$, then, the total expected number of accidents is evaluated by

$$(3.3)\qquad z=\sum_{j=1}^{n} w_j\left(1-\prod_{i=1}^{m} a'_{ij}\right).$$

In order to minimize the objective value (3.3), we have only to maximize

$$\sum_{j=1}^{n} \prod_{i=1}^{m+1} a'_{ij},$$

where $a'_{m+1,j}=w_j$ $(j=1, \cdots, n)$. This is obviously our $(m+1)$-stage max. problem concerning the $(m+1)\times n$ matrix $(a_{ij})$.
In the case of parallel systems, if the device $(P)_j$ obtained by the parallel connection of $j$-th column elements of (3.2) is assigned to the station with value $w_j$ $(j=1, \cdots, n)$, then, the objective value to be minimized is

$$(3.4)\qquad \sum_{j=1}^{n} \prod_{i=1}^{m+1} \bar{a}'_{ij},$$

where $a'_{ij}=1-\bar{a}'_{ij}$ $(i=1, \cdots, m)$ and $\bar{a}'_{m+1,j}=w_j$ for $j=1, \cdots, n$. This is apparently our $(m+1)$-stage min. problem concerning $(m+1)\times n$ matrix $(\bar{a}_{ij})$.

***Note.*** In Proposition 4, if $w_j=1$ for $j=1, \cdots, n$, both of the objective (3.3) and (3.4) can be interpreted as the expected numbers of the failed devices. Moreover, if $w_j$'s are assumed to be probabilities for the exclusive use of the devices in their respective stations, both of the objective (3.3) and (3.4) are the probabilities that an accident occurs in one of the $n$ stations.

**Example 4.** There are two types ($U$ and $V$) of error detecting devices and there are five devices for each type. The probabilities for each device to detect one error are (in %)

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 96.5 & 96.0 & 94.5 & 94.0 & 93.0 \\ 88.0 & 82.0 & 70.0 & 70.0 & 64.0 \end{bmatrix}.$$

Errors arise at five places and the expected numbers of errors per unit time are

$$W = \{0.10, \quad 0.15, \quad 0.20, \quad 0.25, \quad 0.30\} .$$

We want to minimize the total expected number of undetected errors by distributing one device of type $U$ and another of type $V$ to each place, where we assume that each error is undetected, if and only if neither of the two devices assigned there can detect it, and that their detection is stochastically independent.

This is our 3-stage min. problem. To ease the computation, we put

$$\begin{bmatrix} 20\,W \\ 2\,U \\ 1/6\,V \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 11 & 12 & 14 \\ 2 & 3 & 5 & 5 & 6 \end{bmatrix},$$

where $2\,U$ means the probabilities of no detection for type $U$ multiplied by 2 and $1/6\,V$, those for type $V$ multiplied by 1/6.

We take the initial combination

$$\begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 14 & 12 & 11 & 8 & 7 \\ 6 & 5 & 5 & 3 & 2 \end{bmatrix}$$

and apply the fundamental procedures $A_3 : A_1A_2$, $A_1 : A_2A_3$ and $A_2 : A_3A_1$ successively, and obtain

$$(3.5) \qquad \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 12 & 11 & 7 & 14 & 8 \\ 6 & 5 & 5 & 2 & 3 \end{bmatrix}.$$

In this case, according to Theorem 2, we have

$$z^* = z^{(8)} = 144+165+140+140+144 = 733\,,$$

$$\pi = 144\cdot165\cdot140\cdot140\cdot144\,, \quad \pi^{1/5} = 146.32\,,$$

$$z_* = 5\cdot\pi^{1/5} = 731.60\,, \quad \delta = 1 \text{ and } N = 732\,.$$

Therefore,

$$732 \leqslant z^0 \leqslant 733 = z^*.$$

This relation of $z^*$ to $z^0$ does not approve the optimality of $z^*$. However, this may be satisfactory for practical use. And the value of $z^*$ is achieved by the combination (3.5).

*Note.* The above is an example for the optimal construction of parallel systems shown in Proposition 4. We have applied the "policy of equalization" to solve it. In this example, if we assume that each error is undetected whenever at least one of the two devices fails to detect it, the problem reduces to our type of max. problem and it can be solved immediately by Theorem 1 (the "policy of extremization"). This is the model of the optimal construction of series systems mentioned in Proposition 4.

### 3.2 Modified problem

We have described our $m$-stage rearrangement problem and its applications to multiple-system reliability in the preceding sections. In this section we shall show some problems which are closely related to the former but are considered to be modified ones; the modified max. problem in Example 5 and the modified min. problem in Example 6. In these examples we assume that hitting probabilities of hunters are stochastically independent and that we can neglect the probability that two or more hunting objects appear simultaneously at a place.

**Example 5.** (Modified max. problem.) There are $2n$ hunters whose hitting probabilities are, respectively,

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{2n} .$$

There are $m \geqslant n$ hunting-boxes where the expected numbers of hunting objects to appear are, respectively,

$$w_1 \geqslant w_2 \geqslant \cdots \geqslant w_m .$$

Now pair the $2n$ hunters with to make $n$ pairs and assign at most a pair of them to each box so as to maximize the total expected gatherings of the hunting, assuming that the object appearing is gathered if and only if both of the paired hunters hit it.

If a pair of hunters with hitting probabilities $a_j$ and $a_k$ is assigned to the box with expected value $w_i$, the expected gatherings at the box are $w_i \cdot a_j \cdot a_k$. And it is obvious that the hunters with probabilities $a_{2i-1}$ and $a_{2i}$ must be paired and assigned to the box with value $w_i$ $(i=1, \cdots, n)$. Therefore, the maximum of the total expected gatherings is

$$\sum_{i=1}^{n} w_i \cdot a_{2i-1} \cdot a_{2i} .$$

This means that no hunter is assigned to the $(m-n)$ boxes with smallest values $w_{n+1}, \cdots, w_m$.

This example proposes another model of the optimal construction and allocation of series systems. It is a modified type of our 3-stage max. problem and is solved also by the policy of extremization.

**Example 6.** (Modified min. problem.) There are five hunting-grounds where the expected numbers of the hunting objects are, respectively,

$$W = \{0.4, \quad 0.3, \quad 0.2, \quad 0.2, \quad 0.1\} .$$

And there are eight hunters whose hitting probabilities for an object are

$$U=\{0.8,\quad 0.6,\quad 0.5,\quad 0.4,\quad 0.4,\quad 0.3,\quad 0.3,\quad 0.2\}.$$

Allocate at most two hunters to each ground so as to maximize the total expected gatherings, assuming that an object when it appears is gathered whenever at least one of the hunters hits it.

Adding two hypothetical hunters who have hitting probability 0, we can assume that ten hunters are paired and each ground is allocated to a pair of hunters. Therefore, an object when it appears runs away if and only if both of the paired hunters fail to hit it. In this case, if two hunters with failure probabilities $\bar{a}_j$ and $\bar{a}_k$ are assigned to the place with the number $w_i$, the expected number of objects that fly away at the place is $w_i \cdot \bar{a}_j \cdot \bar{a}_k$ $(i=1, \cdots, 5)$. We have only to get the assignment of hunters which minimize the sum of these products. This is our modified type of min. problem and can be approached by the "policy of equalization." That is, we want to make the products as equal as possible. To ease the computations we put

$$10\,W=\{4,\quad 3,\quad 2,\quad 2,\quad 1\}$$

$$10\,\bar{U}=\{2,\quad 4,\quad 5,\quad 6,\quad 6,\quad 7,\quad 7,\quad 8,\quad 10,\quad 10\},$$

where $10\,W$ are the expected numbers of objects multiplied by 10 and $10\,\bar{U}$, the failure probabilities of hunters (including the hypothetical ones) multiplied by 10. Rearranging them in the reverse orders to get the initial feasible combination:

$$\begin{bmatrix} W \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 2 & 2 & 1 \\ 2 & 4 & 5 & 6 & 6 \\ 7 & 7 & 8 & 10 & 10 \end{bmatrix},$$

and applying the procedures $A_1 : A_2 W$, $A_2 : A_1 W$ and $W : A_1 A_2$ succesively, we obtain

$$(3.6)\qquad \begin{bmatrix} 4 & 3 & 2 & 2 & 1 \\ 2 & 4 & 5 & 6 & 6 \\ 10 & 7 & 8 & 7 & 10 \end{bmatrix}.$$

No fundamental procedure can alter the combination (3.6). However, we can interchange the elements of $A_1$ and $A_2$ in the present problem. We interchange the elements 2 and 10 in the first column of (3.6) and apply the fundamental procedures but we cannot decrease the objective value. Next, interchanging the elements 4 and 7 in the second column of (3.6) and applying the fundamental procedures, we can decrease the objective value by 1 to obtain the combination

$$(3.7)\qquad \begin{bmatrix} 4 & 3 & 2 & 2 & 1 \\ 10 & 6 & 6 & 5 & 7 \\ 2 & 4 & 7 & 8 & 10 \end{bmatrix}.$$

Moreover, by interchanging the elements 6 and 7 in 3rd column of (3.7) and applying the fundamental procedure, it is shown that another combination

$$(3.8)\qquad \begin{bmatrix} 4 & 3 & 2 & 2 & 1 \\ 10 & 7 & 5 & 6 & 7 \\ 2 & 4 & 8 & 6 & 10 \end{bmatrix}$$

has the same objective value as (3.7).

The objective value can not be decreased by these procedures any more. We have

$$z^* = 80 + 84 + 80 + 72 + 70 = 386\,.$$

Applying Theorem 2 in order to obtain a lower bound of $z^0$,

$$\pi = 80 \cdot 84 \cdot 80 \cdot 72 \cdot 70$$

$$z_* = 5 \cdot \pi^{1/5} = 385.08\,, \quad \delta = 1 \text{ and } N = 386\,.$$

Hence, we get

$$z^* = 386 = z^0.$$

Thus, we have proved the combinations (3.7) and (3.8) are optimal. The latter combination is shown in the next table, where we use the original values and exclude the hypothetical hunters.

| grounds (in $w_i$) | 0.4 | 0.3 | 0.2 | 0.2 | 0.1 |
|---|---|---|---|---|---|
| hunters (in $a_i$) | 0.8 | 0.6<br>0.3 | 0.5<br>0.2 | 0.4<br>0.4 | 0.3 |

*Note.* Theorem 2 is obviously applicable to our type of modified min. problem except the fact that the objective value $z^*$ might be decreased by further interchange of elements which is permitted in our modified problem.

Under the same assumption as in Example 6, if we are given $m$ hunting-grounds especially with $w_1 = \cdots = w_m$ and $n$ hunters ($m \leqslant n < 2m$) with $a_1 \geqslant \cdots \geqslant a_n$, then, our optimal combination for hunters is given strictly by

$$\bar{a}_i \cdot \bar{a}_{2m-i+1} \quad (i=1, \cdots, m),$$

where $\bar{a}_i = 1 - a_i$ $(i=1, \cdots, n)$, and $\bar{a}_i = 1$ $(i=n+1, \cdots, 2m)$ correspond to hypothetical hunters with hitting probability 0. This is an example of the modified 2-stage min. problem, which is solved also by the policy of equalization.

### 3.3 Reliability of series-parallel systems

There are $m$ kinds of elements denoted by $A_1, A_2, \cdots, A_m$ and we are given more than $n$ elements for each kind. We classify each kind of them into $n$ subclasses:

$$A_i = \{A_{i1}, A_{i2}, \cdots, A_{in}\}, \quad (i=1, \cdots, m).$$

We assume that every subclass contains at least one element, and if two or more elements are contained in a subclass we combine them in parallel to constitute a composite unit. Thus we have $n$ composite units for each kind, including those units that are composed of only one element. Now we construct each of $n$ systems by series connection of $m$ kinds of composite units just as the series systems in 3.1. Let us call them "series-parallel systems" and denote them $(SP)_1, \cdots, (SP)_n$.

There arise some reliability problems concerning the optimal construction and assignment of these systems as in the previous sections. To state the problems we define some notations as follows:

(3.9) $x_{ij}$=failure probability of the composite unit $A_{ij}$, $(i=1, \cdots, m;\ j=1, \cdots, n)$,

(3.10) $\pi_i = \prod_{j=1}^{n} x_{ij}, \quad (i=1, \cdots, m)$,

$Z$=the number of the systems that can operate without failure.

The probability $x_{ij}$ is the product of unreliabilities of those elements which compose $A_{ij}$ under the assumption of independent failures of them. And $\pi_i$'s are the positive constants in our problem. $Z$ is apparently a random variable.

The probability that none of the $n$ systems fails is

$$P(Z=n) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1-x_{ij}), \tag{3.11}$$

which depends on how to classify the set $A_i$ to obtain $A_{i1}, \cdots, A_{in}$ $(i=1, \cdots, m)$. It is maximized if we can take

$$x_{ij} = \pi_i^{1/n}, \quad (j=1, \cdots, n) \tag{3.12}$$

for each $i$. Therefore,

$$P(Z=n) \leq \prod_{i=1}^{m} (1-\pi_i^{1/n})^n \tag{3.13}$$

The "policy of equalization" is meant also by (3.12) and the maximum of $P(Z=n)$ can be approached by the iterative procedures as in Example 6, that is, by choosing $x_{i1}, \cdots, x_{in}$ as equally as possible, although the exact equalities (3.12) cannot always be achieved since $x_{ij}$'s are discrete in our problem.

If we can find such constants $\lambda_{i1}, \cdots, \lambda_{in}$; $\mu_{i1}, \cdots, \mu_{in}$; $k_i, l_i$ $(0 \leqslant k_i < l_i - 1 \leqslant n)$ that satisfy

$$(3.14) \quad \begin{cases} 0=\lambda_{i0}<\lambda_{i1}\leqslant\cdots\leqslant\lambda_{ik_i}\leqslant\pi_{k_il_i}\leqslant\lambda_{ij}, & (j=k_i+1, \cdots, n) \\ \mu_{ij}\leqslant\pi_{k_il_i}\leqslant\mu_{il_i}\leqslant\cdots\leqslant\mu_{in}<\mu_{i,n+1}=1, & (j=1, \cdots, l_i-1) \end{cases}$$

and

$$(3.15) \quad \mu_{ij}\leqslant x_{ij}\leqslant\lambda_{ij}, \quad (j=1, \cdots, n)$$

for each $i$, then, the upper bound of $P(Z=n)$ is improved by

$$(3.16) \quad P(Z=n)\leqslant \prod_{i=1}^{m} \prod_{j=1}^{k_i} (1-\lambda_{ij})\cdot \prod_{j=l_i}^{n} (1-\mu_{ij})\cdot(1-\pi_{k_il_i})^{l_i-k_i-1}$$

$$\leqslant \prod_{i=1}^{m} (1-\pi_i^{1/n})^n,$$

where the constant $\pi_{k_il_i}$ is clearly defined by the constants $\lambda_{i0}, \cdots, \lambda_{in}$; $\mu_{i1}, \cdots, \mu_{i,n+1}$ and $\pi_i$ as in (2.21) for each $i$. We summarize this result in the next proposition.

**Proposition 5.** "The probability $P(Z=n)$ defined by (3.11) can be increased by the policy of equalization applied on $x_{i1}, \cdots, x_{in}$ for each $i$. Under the constraints (3.10) and (3.15), where the constants satisfy (3.14), the upper bounds are given by (3.16)."

This can be proved just as in Lemma 2. The upper bounds may be useful for evaluating the efficiency of the achieved value of $P(Z=n)$.

*Note.* The probability $P(Z=0)$ that all of the $n$ series- parallel systems fail is obviously evaluated by

$$P(Z=0)=\prod_{j=1}^{n}\left(1-\prod_{i=1}^{m}(1-x_{ij})\right)$$

$$\geqslant 1-\prod_{i=1}^{m}(1-\pi_i)$$

where in the right hand side $\Pi_{i=1}^{m}(1-\pi_i)$ is the reliability of the unique series-parallel system which is got in the special case of $n=1$. This means that it is inefficient under our assumptions to construct multiple systems in order to maximize the probability that at least a system can survive. In this case, the merit of multiple redundant systems can not be observed unless another measure of utility, such as maintainability, is introduced.

To proceed to the second problem on the systems $(SP)_1, \cdots, (SP)_n$, assume moreover that the unreliabilities of given elements are so small that, if the maximum of them is denoted by $\theta$, then,

(3.17) $\quad m\theta^2$ is negligible compared to $\theta$.

The $n$ systems which are assumed presently to be error-detecting devices are going to be assigned, one by one, to $n$ spots where the expected numbers, per unit time, of errors are known to be $w_1, \cdots, w_n$, respectively, and we assume also that an error is undetected if and only if the device assigned to the place is in failure.

Now our present problem is as follows. "Construct $n$ devices $(SP)_1, \cdots, (SP)_n$ and assign them to each spot so as to minimize the total expected number of undetected errors, under the conditions imposed above."

Let the $m$-stage device constructed by the composite units $A_{1j}, \cdots, A_{mj}$ be assigned to the spot with the expected number of errors $w_j$ ($j=1, \cdots, n$). The objective value $z$ of our problem is

$$z=\sum_{j=1}^{n} w_j\left(1-\prod_{i=1}^{m}(1-x_{ij})\right). \tag{3.18}$$

Owing to the assumption (3.17) this is reduced to

$$z=\sum_{i=1}^{m}\sum_{j=1}^{n} w_j x_{ij},$$

where $x_{i1}, \cdots, x_{in}$ is restricted to the condition (3.10) for each $i$. Hence, our problem is separated into $m$ subproblems to minimize

$$z_i = \sum_{j=1}^{n} w_j x_{ij}, \tag{3.19}$$

respectively, for $i=1, \cdots, m$. Each of these is no other problem than the modified minimizing one clarified in Example 6 of the previous section. Therefore, it is desirable to choose $w_1 x_{i1}, \cdots, w_n x_{in}$ as equally as possible, according to the policy of equalization, for each $i$. We note that Theorem 2 is also applicable in our subproblems to evaluate lower bounds of the objective values. In this case the constant $\pi$ of the theorem should be replaced by

$$\pi_i^* = \prod_{j=1}^{n} w_j x_{ij} = \pi_i \cdot \prod_{j=1}^{n} w_j$$

for each $i$. Summarizing these results, we have the following proposition:

**Proposition 6.** "Under the assumption (3.17) the problem of finding the optimal construction and the optimal assignment, of the $n$ devices $(SP)_1, \cdots, (SP)_n$ to the $n$ spots with expected values $w_1, \cdots, w_n$, that minimize the total expected loss (3.18) reduces to the modified problems of minimizing the objective (3.19) for each $i$ under the constraints (3.9~10)."

*Note.* If any error occurs at one of the $n$ spots with probabilities $w_1, \cdots, w_n$, respectively, then, the objective value (3.18) can be interpreted as the probability for an error to be undetected. The above proposition shows how to minimize the probability under our circumstances. And if, alternatively, $w_1 = \cdots = w_n = 1$, then, the objective value (3.18) is considered to be the expected number of inoperative systems.

## Conclusive Remark

We have described our type of multi-stage rearrangement problem and its applications to multiple-system reliability. Our model stands for the situations where we are given a finite number of personnel and/or machinery components with similar functions but with different abilities and where we want to assign them to each position in order to obtain maximum effect. These situations may occur in the various systems under preventive maintenance or individual replacement. It is for the convenience of intuitive interpretation that we have used special terminologies, such as "hunter," "hunting object," "hunting-ground," "error" and "error-detecting device," in the propositions and examples. We believe that our models and methods can be applicable to the various cases, beyond the special terminologies, where reliability resources should be distributed on account of their reliabilities and usage intensities in order to obtain maximum utility.

**Acknowledgement.** The author is grateful to Dr. H. Oodaira for his information on Chapter $X$ of [4].

## References

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