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ON THE EQUILIBRIUM PROBABILITIES OF GI/G/1

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1. Introduction

In the previous paper we have developped the equilibrium equations method on the most general queueing system [3]. As applications of the theory we could obtain the equilibrium probabilities and others in the simple cases as $M/G/1(\infty)$, $GI/M/m(\infty)$, $GI/M/m(N)$, $GI/E_l/1$, etc.

In the present note we shall give the solutions of $GI/M/m$ and $GI/G/1(\infty)$ system. Consider the many servers system. Let $A(\tau)$ be the distribution function of the inter-arrival time and $B(s)$ that of the service time, both satisfying the following conditions: (II) $A(\tau)$ and $B(s)$ are absolutely continuous except at most the sequences $\{\tau_\nu\}$ and $\{s_\nu\}$ respectively, where $\{\tau_\nu\}$and $\{s_\nu\}$ have not any finite cluster point. Furthermore, means

$$1/\lambda=\int\tau\, dA(\tau) \quad \text{and} \quad 1/\mu=\int s\, dB(s)$$

exist.

In [3] we assumed the conditions (I) and (III) in addition to (II), but in the simple cases as $GI/G/1$ and $GI/M/m$ (I) is necessary attained and (III) is simplified by $\lambda<\mu$ in the case $GI/G/1(\infty)$ or $\lambda<m\mu$ in the case $GI/M/m(\infty)$.

Let $\{t_n\}$ be the sequence of instants when customers arrive, and in the case $GI/G/1$ $\{t_n'\}$ that when services begin, and

in the case $GI/G/1$

$$P_0(\tau\,;t)=P_r\{\eta(t)=0, \tau\leqq\tau(t)<\tau+\delta\tau\}/\delta\tau\,,$$

$$P_n(\tau, s; t) = P_r\{\eta(t)=n, \tau \leqq \tau(t) < \tau+\delta\tau, s \leqq s(t) < s+\delta s\}/\delta\tau\delta s, \quad (n\geqq 1)$$

in the case $GI/M/m$

$$P_n(\tau; t) = P_r\{\eta(t)=n, \tau \leqq \tau(t) < \tau+\delta\tau\}/\delta\tau, \quad (n\geqq 0)$$

where

$$\tau(t) = \min_{t_n \leq t} (t-t_n)$$

and

$$s(t) = \min_{t_n' \leq t} (t-t_n'),$$

and $\eta(t)$ is a random variable denoting a number of customers in the system at the instant t. In these simple cases we have already obtained

Theorem. If the system satisfies the condition (II) and the inequality $\lambda<\mu$ (in the case $GI/G/1(\infty)$) or $\lambda<m\mu$ (in the case $GI/M/m(\infty)$), then

(i) in the aperiodic case, the limit

$GI/G/1$	$GI/M/m$
$\lim_{t\to\infty} P_0(\tau; t) = P_0(\tau)$ and $\lim_{t\to\infty} P_n(\tau, s; t) = P_n(\tau, s)$	$\lim P_n(\tau; t) = P_n(\tau)$

exist and satisfy the equation

(1) $$\sum_{n=0}^{\infty} p_n = 1,$$

where

$p_0 = \int P_0(\tau)d\tau$ and $p_n = \int P_n(\tau, s)d\tau ds.$	$p_n = \int P_n(\tau)d\tau$

In the periodic case with period ω, $P_0(\tau;t)$, $P_n(\tau,s;t)$, $P_0(\tau)$, $P_n(\tau,s)$ [or $P_n(\tau;t)$, $P_n(\tau)$] and p_n are to be replace merely by $\bar{P}_0(\tau;t)$, $\bar{P}_n(\tau,s;t)$, $\bar{P}_0(\tau)$, $\bar{P}_n(\tau,s)$ [or $\bar{P}_n(\tau;t)$, $\bar{P}_n(\tau)$] and $\bar{P}_n$, where

$$\bar{P}_0(\tau;t)=\frac{1}{\omega}\int_t^{t+\omega}P_0(\tau;t)dt,$$

$$\bar{P}_n(\tau,s;t)=\frac{1}{\omega}\int_t^{t+\omega}P_n(\tau,s;t)dt.$$

$$\bar{P}_n(\tau,t)=\frac{1}{\omega}\int_t^{t+\omega}P_n(\tau;t)dt.$$

(ii) $P_0(\tau)$ and $P_n(\tau,s)$ [or $P_n(\tau)$] satisfy the equilibrium equations shown in 3 [or in 2].

(iii) the equilibrium equations have an unique non-trivial solution; that is, non negative and satisfying (1).

(iv) the equilibrium probabilities q_k at the arrival time $\{t_n\}$ are given by

$$q_0=\frac{1}{\lambda}\int P_0(\tau)\frac{dA(\tau)}{1-A(\tau)},$$

and

$$q_k=\frac{1}{\lambda}\int P_k(\tau,s)\frac{dA(\tau)}{1-A(\tau)}ds.$$

$$q_k=\frac{1}{\lambda}\int P_k(\tau)\frac{dA(\tau)}{1-A(\tau)}.$$

in the aperiodic case. In the periodic case P_k must be replaced by $\bar{P}_k$.

On the other hand, if the equilibrium equations satisfying the condition (II) have non trivial solution, then the system satisfies also $\lambda<\mu$ [or $\lambda<m\mu$].

Hence, it is sufficient to find a set of solution satisfying the equations.

2. *GI/M/m*

2.1. At first consider $GI/M/m(\infty)$ system which has been solved by L. Takacs [2]. The equilibrium equations in the aperiodic case are in the following:

for $\tau \neq 0, \tau_\nu$

(1) $$\left[\frac{d}{d\tau}+\lambda(\tau)+n\mu\right]P_n(\tau)=(n+1)\mu P_{n+1}(\tau) \quad (n=0,1,\cdots,m-1),$$

(2) $$\left[\frac{d}{d\tau}+\lambda(\tau)+m\mu\right]P_n(\tau)=m\mu P_{n+1}(\tau) \quad (n\geqq m),$$

with initial conditions

(3) $$P_n(0)=\int_0^\infty P_{n-1}(\tau)\frac{dA(\tau)}{1-A(\tau)} \quad (n\geqq 1),$$

and with

(4) $$P_n(\tau_\nu+0)=\frac{1-A(\tau_\nu+0)}{1-A(\tau_\nu-0)}P_n(\tau_\nu-0) \quad (\text{for all } n \text{ and } \nu),$$

where

$$\lambda(\tau)=A'(\tau)/[1-A(\tau)].$$

The equations (2) with (3) ($n\geqq m$) have a solution

(5) $$P_n(\tau)=k\alpha^{n-m}[1-A(\tau)]\exp[-m\mu(1-\alpha)\tau] \quad (n\geqq m),$$

where α is a root of a characteristic equation ($0<\alpha<1$)

(6) $$\alpha=A^*[m\mu(1-\alpha)],$$

A^* denoting the Laplace-Stiltje's tronsform of $A(\tau)$

$$A^*(z)=\int_0^\infty e^{-z\tau}dA(\tau).$$

The eqations (2.1) have a solution*

(7) $$P_n(\tau)=k[1-A(\tau)]\left[\sum_{\nu=0}^{m-n-1}(-1)^\nu\binom{n+\nu}{\nu}C_{n+\nu}e^{-(n+\nu)\mu\tau} +(-1)^{m-n}\prod_{\nu=1}^{m-n}\frac{m-\nu+1}{\nu-m\alpha}e^{-m\mu(1-\alpha)\tau}\right]$$
$$(n=0,1,\cdots,m-1),$$

where constants C_n are determined from (3) ($n<m$) by the equations

* It is assumed in (7) and (8) that $\nu-m\alpha$ ($\nu=1,2,\cdots,m-1$) are not zero. If one of $\nu-m\alpha$ is zero, (7) and (8) would be partially revised.

$$(8)\quad C_{m-n}A^{*}_{m-n}=\sum_{\nu=1}^{n-1}(-1)^{\nu-1}C_{m-n+\nu}\left[\binom{m-n+\nu}{\nu-1}+\binom{m-n+\nu}{\nu}A^{*}_{m-n+\nu}\right]$$
$$+(-1)^{n-1}\prod_{\nu=1}^{n-1}\frac{m-\nu+1}{\nu-m\alpha}\,\frac{n-(n-1)\alpha}{n-m\alpha}$$
$$(n=1,2,\cdots,m),$$

where

$$A_n^*=A^*(n\mu).$$

Adding (7) and (5) respectively, we have

$$\sum_{n=0}^{m-1}P_n(\tau)=k[1-A(\tau)]\left[C_0-\frac{1}{1-\alpha}e^{-m\mu(1-\alpha)\tau}\right],$$

and

$$\sum_{n=m}^{\infty}P_n(\tau)=\frac{k}{1-\alpha}[1-A(\tau)]e^{-m\mu(1-\alpha)\tau}.$$

Hence from

$$\sum_{0}^{\infty}\int P_n(\tau)d\tau=1,$$

we have

$$(9)\qquad k=\lambda/C_0.$$

2.2. Now $P_n(\tau)$ are given, the following results are easy to obtain

$$(\text{i})\quad p_0=1-\frac{m\rho}{C_0}\left[\sum_{\nu=1}^{m-1}(-1)^{\nu}\frac{C_\nu}{\nu}(1-A_\nu^*)+\frac{(-1)^m}{m}\prod_{\nu=1}^{m}\frac{m-\nu+1}{\nu-m\alpha}\right],$$

$$p_n=\frac{m\rho}{C_0}\left[\sum_{\nu=0}^{m-n-1}(-1)^{\nu}\frac{C_{n+\nu}}{n+\nu}(1-A^{*}_{n+\nu})+\frac{(-1)^{m-n}}{m}\prod_{\nu=1}^{m-n}\frac{m-\nu+1}{\nu-m\alpha}\right]$$
$$(n=1,\cdots,m-1),$$

$$p_n=\frac{\rho}{C_0}\alpha^{n-m}\quad(n\geqq m).$$

$$(\text{ii})\quad q_n=\frac{n+1}{m\rho}p_{n+1}\quad(n\leqq m-2),$$

$$q_n=\frac{1}{\rho}p_{n+1}\quad(n\geqq m-1),$$

or

$$p_0=1-\rho-m\rho\sum_{\nu=1}^{m-1}q_{\nu-1}\left(\frac{1}{\nu}-\frac{1}{m}\right),$$

$$p_n=\frac{m\rho}{n}q_{n-1}\quad (n=1,2,\cdots,m-1),$$

$$p_n=\rho q_n.\quad (n\geqq m).$$

(iii) the probability not to wait is equal to

$$\sum_{\nu=0}^{m-1}q_\nu=1-\frac{\alpha}{C_0(1-\alpha)}.$$

(iv)
$$\sum_{\nu=0}^{m-1}\left(1-\frac{\nu}{m}\right)p_\nu=1-\rho.$$

(v) Let L be number of customers in the system and L_q that in the queue.

$$E\{L_q\}=\frac{\rho\alpha}{C_0(1-\alpha)^2},$$

$$E\{L\}=\frac{\rho\alpha}{C_0(1-\alpha)^2}+m\rho,$$

$$V(L_q)=\frac{\rho\alpha(1+\alpha)}{C_0(1-\alpha)^3}-\frac{\rho^2\alpha^2}{C_0^2(1-\alpha)^4}.$$

Let $L_q{}^I$ be a number of customers in the queue at arrival time.

$$E\{L_q{}^I\}=\frac{\alpha^2}{C_0(1-\alpha)^2},$$

$$V(L_q{}^I)=\frac{\alpha^2(1+\alpha)}{C_0(1-\alpha)^3}-\frac{\rho^2\alpha^2}{C_0^2(1-\alpha)^4},$$

(vi) Let W_q be the waiting time of customers. The distribution functions of W_q is given by

$$F(w_q)=1-\frac{\alpha}{C_0(1-\alpha)}e^{-m\mu(1-\alpha)w_q}.$$

Hence,

$$P_r\{W_q=0\}=1-\frac{\alpha}{C_0(1-\alpha)},$$

$$E\{W_q\} = \frac{\alpha}{m\mu C_0(1-\alpha)^2},$$

$$V(W_q) = \frac{1}{(m\mu)^2}\left[\frac{2\alpha}{C_0(1-\alpha)^3} - \frac{\alpha^2}{C_0^2(1-\alpha)^4}\right],$$

Let W be the time which customers spend in the system. Its dis-bution function is given by

$$F(w) = 1 - \left[1 + \frac{\alpha}{C_0(1-\alpha)(m-1-m\alpha)}\right]e^{-\mu w} + \frac{\alpha}{C_0(1-\alpha)(m-1-m\alpha)}e^{-m\mu(1-\alpha)w}$$

Remark. (a) in the case M/M, $\alpha=\rho$

(b) in the case $GI/M/1(\infty)$,

$$C_0 = 1/(1-\alpha),$$

hence,

$$p_0 = 1-\rho$$

$$p_n = \rho\alpha^{n-1}(1-\alpha) \quad (n \geqq 1),$$

and

$$q_n = \alpha^u(1-\alpha) \quad (n \geqq 0).$$

In the case $GI/M/2(\infty)$,

$$C_0 = \frac{1-\alpha-A^*}{(1-\alpha)(1-2\alpha)A^*} \quad (A^* = A^*(\mu)),$$

hence,

$$p_0 = 1 - \frac{\rho(2-2\alpha-3A^*+2\alpha A^*)}{1-\alpha-A^*} \qquad q_0 = \frac{(1-\alpha)(1-2A^*)}{1-\alpha-A^*}$$

$$p_1 = \frac{2\rho(1-\alpha)(1-2A^*)}{1-\alpha-A^*} \qquad q_n = \frac{\alpha^{n-1}(1-\alpha)(1-2\alpha)A^*}{1-\alpha-A^*} \quad (n \geqq 1).$$

$$p_n = \frac{\rho\alpha^{n-2}(1-\alpha)(1-2\alpha)A^*}{1-\alpha-A^*} \quad (n \geqq 2).$$

(c) In the periodic case q_n are still meaningfull, which is obtained replacing P_n by $\bar{P}_n$.

2.3. Example. Suppose the inter-arrival distribution is uniform in the interval $\left(\frac{1-p}{\lambda}, \frac{1+p}{\lambda}\right)$ such as

$$A(\tau)=\begin{cases} 0, & \tau<(1-p)/\lambda \\ \dfrac{\lambda}{2p}\left(\tau-\dfrac{1-p}{\lambda}\right), & \dfrac{1-p}{\lambda}\leqq\tau\leqq\dfrac{1+p}{\lambda} \\ 1, & \tau>(1+p)/\lambda . \end{cases}$$

The results are shown in Table 1 for $U/M/m$ ($\rho=0.5$, $p=1.0$ and 0.1). Such tables for $\rho=0.05, 0.10, \cdots, 0.95$; $p=1.0, 0.9, \cdots, 0.1$; and $m=1, 2, \cdots, 5$ are available.

2.4. $GI/M/m(N)$. L. Takacs has treated in the case of $N=m$ [1]. The equilibrium equations are partially modified from those in 2.1; that is

$$\left[\frac{d}{d\tau}+\lambda(\tau)+n\mu\right]P_n(\tau)=(n+1)\mu P_{n+1}(\tau) \quad (n=0,1,\cdots,m-1), \tag{10}$$

$$\left[\frac{d}{d\tau}+\lambda(\tau)+m\mu\right]P_n(\tau)=m\mu P_{n+1}(\tau) \quad (n=m,\cdots,N-1), \tag{11}$$

Table 1. $U/M/m(\infty)$

(a) $\rho=0.5$, $p=1.0$.

m	1	2	3	4	5
$E(L)$	.782	1.164	1.604	2.068	2.546
$E(L_q)$	.282	.164	.104	.069	.047
$V(L_q)$	.521	.322	.210	.141	.097
WNOT	.639	.790	.866	.912	.940
p_0	.5000	.2909	.1574	.0827	.0428
p_1	.3196	.4181	.3697	.2760	.1876
p_2	.1153	.1860	.2884	.3219	.2981
p_3	.0416	.0671	.1179	.1976	.2524
p_4	.0150	.0242	.0425	.0779	.1365
p_5	.0054	.0087	.0153	.0281	.0528
p_6	.0020	.0032	.0055	.0101	.0190
p_7	.0007	.0011	.0020	.0037	.0069
p_8	.0003	.0004	.0007	.0013	.0025
p_9	.0001	.0001	.0003	.0005	.0009
p_{10}	.0000	.0001	.0001	.0002	.0003

(b) $\rho=0.5$. $p=0.1$.

m	1	2	3	4	5
$E(L)$	.629	1.065	1.536	2.020	2.512
$E(L_q)$	.129	.066	.036	.021	.012
$V(L_q)$	.178	.095	.053	.031	.018
WNOT	.795	.895	.942	.966	.980
p_0	.5000	.2550	.1223	.0572	.0264
p_1	.3977	.4901	.3963	.2682	.1647
p_2	.0814	.2028	.3406	.3731	.3269
p_3	.0167	.0415	.1120	.2204	.2943
p_4	.0034	.0085	.0229	.0645	.1397
p_5	.0007	.0017	.0047	.0132	.0381
p_6	.0001	.0004	.0010	.0027	.0078
p_7	.0000	.0001	.0002	.0006	.0016
p_8	.0000	.0000	.0000	.0001	.0003
p_9	.0000	.0000	.0000	.0000	.0001
p_{10}	.0000	.0000	.0000	.0000	.0000

Third rows "WNOT" denote the probabilities not to wait.

(12) $$\left[\frac{d}{d\tau}+\lambda(\tau)+m\mu\right]P_N(\tau)=0,$$

with initial conditions

(13) $$P_n(0)=\int P_{n-1}(\tau)\frac{dA(\tau)}{1-A(\tau)}\quad(n=1,2,\cdots,N-1),$$

(14) $$P_N(0)=\int[P_N(\tau)+P_{N-1}(\tau)]\frac{dA(\tau)}{1-A(\tau)},$$

and with (4). We could obtain the following solution as the same manner,

$P_n(\tau)$ $(n\leq N-1)$ are the same as (5), (7) and (8),

$$P_N(\tau)=\frac{k\alpha^{N-m}}{1-A_m^*}[1-A(\tau)]e^{-m\mu\tau}.$$

hence

(15) $$k=1\Big/\left[\frac{C_0}{\lambda}-\frac{\alpha^{N-m+1}}{1-\alpha}\right].$$

3. $GI/G/1(\infty)$

3.1. The equilibrium equations are as follows:

(1) $$\left[\frac{d}{d\tau}+\lambda(\tau)\right]P_0(\tau)=\int P_1(\tau, s)\frac{dB(s)}{1-B(s)},$$

(2) $$\left[\frac{\partial}{\partial\tau}+\frac{\partial}{\partial s}+\lambda(\tau)+\mu(s)\right]P_n(\tau, s)=0 \quad (n\geqq 1),$$

where

$$\lambda(\tau)=A'(\tau)/[1-A(\tau)]$$

and

$$\mu(s)=B'(s)/[1-B(s)].$$

These equations hold except $\tau=0$, τ_ν, $s=0, s_\nu$ and at most denombrable number of lines, and are accompanied by the boundary conditions

(3) $$P_1(0, s)=0 \quad (\text{for } s>0),$$

(4) $$P_n(0, s)=\int P_{n-1}(\tau, s)\frac{dA(\tau)}{1-A(\tau)} \quad (n\geqq 2),$$

(5) $$P_n(\tau, 0)=\int P_{n+1}(\tau, s)\frac{dB(s)}{1-B(s)} \quad (n\geqq 1),$$

and

(6) $P_1(\tau, s)$ has a singular component $\delta(s-\tau)[1-A(\tau)][1-B(s)]\int P_0(\tau)\times\frac{dA(\tau)}{1-A(\tau)}$, where δ is Dirac's delta function.

$P_n(\tau, s)$ have jumps at τ_ν and s_ν such that

(7) $$\begin{cases} P_n(\tau_\nu+0, s)=\dfrac{1-A(\tau_\nu+0)}{1-A(\tau_\nu-0)}P_n(\tau_\nu-0, s), \\ P_n(\tau, s_\nu+0)=\dfrac{1-B(s_\nu+0)}{1-B(s_\nu-0)}P_n(\tau, s_\nu-0), \end{cases}$$

To solve this difference and partial differential equation with boundary conditions is not easy in general, for the equations have no initial condition as difference equations, so that the solution of the equations (1) ~(7) is not unique analytically (in the ordinary sense). It is remarkable that the equations have, however, unique solution in the probability sense, that is non negative and satisfying (1.1). To avoid this difficulty we shall use the successive approximation method.

Suppose

$$\text{(8)} \quad \begin{cases} P_0(\tau)=[1-A(\tau)]Q_0(\tau) \\ P_n(\tau, s) = \begin{cases} [1-A(\tau)][1-B(s)]Q_n(\tau-s) & (\tau>s) \\ [1-A(\tau)][1-B(s)]R_n(s-\tau) & (\tau\leqq s) \end{cases} \end{cases}$$

where $Q_n(\tau)$ and $R_n(s)$ are functions of one variable which are to be determined. The functions P_n given in (8) satisfy all the equations (2). (1), (3)~(6) are transformed by the replacement (8) to the difference and integral equations

$$\text{(9)} \quad \frac{d}{d\tau} Q_0(\tau)=\int_\tau^\infty R_1(s-\tau)dB(s)+ \int_0^\tau Q_1(\tau-s)dB(s)\,,$$

$$\text{(10)} \quad R_1(s)=\delta(s) \int Q_0(\tau)dA(\tau)\,,$$

$$\text{(11)} \quad R_n(s)=\int_0^s R_{n-1}(s-\tau)dA(\tau) + \int_s^\infty Q_{n-1}(\tau-s)dA(\tau) \quad (n\geqq 2)\,,$$

and

$$\text{(12)} \quad Q_n(\tau)=\int_\tau^\infty R_{n+1}(s-\tau)dB(s)+\int_0^\tau Q_{n+1}(\tau-s)dB(s) \quad (n\geqq 1)\,.$$

It is always possible to determine $Q_0(\tau)$ which satisfies (9) and (10) after $Q_1(\tau)$ and $R_1(s)$ are obtained. Writing aside (9) and a numerical factor in the right hand aside of (10), write

$$\text{(10')} \quad R_1(s)=\delta(s)\,,$$

we shall find the solution of (10'), (11) and (12) by the successive approximation, and a numerical factor by which all the functions R_n and

Q_n should be multiplied.

3.2. Let

$$(13)\quad \begin{cases} R_2^{(0)}(s) = \int_0^s R_1(s-\tau)dA(\tau), \\ Q_1^{(0)}(\tau) = \int_\tau^\infty R_2^{(0)}(s-\tau)dB(s). \end{cases}$$

After $R_n^{(\nu)}(s)$ $(n=2, 3, \cdots, \nu+2)$ and $Q_n^{(\nu)}(\tau)$ $(n=1, 2, \cdots, \nu+1)$ have been determined, let us define in $(\nu+1)$-th cycle $R_n^{(\nu+1)}(s)$ and $Q_n^{(\nu+1)}$ in the following:

$$(14)\quad \begin{cases} R_2^{(\nu+1)}(s) = \int_0^s R_1(s-\tau)dA(\tau) + \int_s^\infty Q_1^{(\nu)}(\tau-s)dA(\tau), \\ R_n^{(\nu+1)}(s) = \int_0^s R_{n-1}^{(\nu+1)}(s-\tau)dA(\tau) + \int_s^\infty Q_{n-1}^{(\nu)}(\tau-s)dA(\tau), \\ \qquad (n=3, 4, \cdots, \nu+2) \\ R_{\nu+3}^{(\nu+1)}(s) = \int_0^s R_{\nu+2}^{(\nu+1)}(s-\tau)dA(\tau), \end{cases}$$

$$(15)\quad \begin{cases} Q_{\nu+2}^{(\nu+1)}(s) = \int_\tau^\infty R_{\nu+3}^{(\nu+1)}(s-\tau)dB(s), \\ Q_n^{(\nu+1)}(\tau) = \int_\tau^\infty R_{n+1}^{(\nu+1)}(s-\tau)dB(s) + \int_0^\tau Q_{n+1}^{(\nu+1)}(\tau-s)dB(s) \\ \qquad (n=\nu+1, \nu, \cdots, 2, 1) \end{cases}$$

To prove the convergence of $Q_n^{(\nu)}(\tau)$ and $R_n^{(\nu)}(s)$ as $\nu\to\infty$, we shall show at first

(i) (16) $\qquad R_n^{(\nu)}(s) \geqq 0$ and $Q_n^{(\nu)}(\tau) \geqq 0$,

(ii) (17) $\qquad R_n^{(\nu+1)}(s) \geqq R_n^{(\nu)}(s)$ and $Q_n^{(\nu+1)}(\tau) \geqq Q_n^{(\nu)}(\tau)$

(iii) there exist upper bounds such that

(18) $\qquad R_n^{(\nu)}(s) \leqq R_n^*(s)$ and $Q_n^{(\nu)}(\tau) \leqq Q_n^*(\tau)$,

for all $n \geqq 1$, $\tau \fallingdotseq \tau_\nu$, $s \fallingdotseq s_\nu$, and $\nu \geqq 1$.

(i) and (ii) will be easily proved by the mathematical induction method. To prove (iii), remember that the equilibrium equations have non trivial solution if $\lambda<\mu$, so that (10′), (11) and (12) have unique non-trivial solution $Q_n^*(\tau)$ ($n\geqq 1$) and $R_n^*(s)$ ($n\geqq 2$). It would be proved by the induction that these functions satisfy the inequalities (18).

Putting

$$p_n^{(\nu)}=\int_0^\infty\int_s^\infty[1-A(\tau)][1-B(s)]Q_n^{(\nu)}(\tau-s)d\tau ds$$
$$+\int_0^\infty\int_0^s[1-A(\tau)][1-B(s)]R_n^{(\nu)}(s-\tau)d\tau ds,$$

and

$$\text{sum}\,(\nu)=\sum_1^{\nu+2}p_n^{(\nu)},$$

the successive approximation should be interrupted when sum $(\nu)-$ sum

Table 2.

Equlb. Prob.	*M/M*				*U/M*			
	$\rho=0.25$		$\rho=0.5$		$\rho=0.2$		$\rho=0.5$	
	Est.	True	Est.	True	Est.	True	Est.	True
1	.187498	.1875	.2506	.25	.177468	.177463	.3210	.3196
2	.046873	.046875	.1251	.1250	.019996	.019997	.1153	.1153
3	.011718	.011719	.0625	.0625	.002252	.002253	.0412	.0416
4	.002929	.002930	.0311	.0313	.000253	.000254	.0147	.0150
5	.000732	.000732	.0155	.0156	.000028	.000029	.0052	.0054
6	.000182	.000183	.0077	.0078	.000003	.000003	.0018	.0020
7	.000046	.000046	.0038	.0039			.0002	.0007
8	.000011	.000011	.0019	.0020			.0001	.0001
9	.000003	.000003	.0009	.0010				
10	.000001	.000001	.0005	.0005				
11			.0002	.0002				
12			.0001	.0001				
13			.0001	.0001				
Cycles	12		18		5		9	
Error	2×10^{-6}		10^{-3}		2×10^{-6}		5×10^{-4}	

($\nu-1$) becomes smaller than the desirable number. The calculation will be completed by the adjustment to satisfy (1.1) or $\sum_{0}^{\infty} p_n=\rho$.

3.3. Exemples. (1) $M/M/1$ and $U/M/1$. At first $M/M/1$ and $U/M/1$ were calculated as a verification where U denotes the uniform distribution $A(\tau)=(\lambda/2)\tau(0\leqq\tau\leqq 2/\lambda)$ and $=1(\tau>2/\lambda)$, since their true probabilities are known. The results are shown in Table 2. Table shows that for the adquate value of ρ (not so high) the convergence is pretty well.

(2) $U/U/1$. The results are illustrated in Table 3 for $\rho=1/10$, 1/5, 1/4, 1/3 and 1/2.

Table 3.

Equilibrium Probabilities of $U/U/1$ systems

ρ	0.1		0.2		0.25		1/3		0.5	
	p	q	p	q	p	q	p	q	p	q
0	.9	.989639	.8	.95684	.75	.93099	.66667	.87189	.5	.67704
1	.096486	.010000	.18508	.04000	.22591	.06250	.28780	.11111	.37942	.24846
2	.003420	.000351	.01405	.00298	.02227	.00602	.04059	.01518	.09679	.06005
3	.000092	.000009	.00082	.00017	.00170	.00045	.00447	.00165	.01948	.01185
4	.000002	.000000	.00004	.00000	.00011	.00003	.00043	.00016	.00355	.00215
5					.00001	.00000	.00004	.00001	.00062	.00038
6									.00011	.00007
7									.00002	.00001
$E\{L\}$	.1036	.0107	.2158	.0465	.2760	.0760	.3843	.1471	.6495	.4150
Cycles	5		5		6		8		15	
Error	10^{-7}		2×10^{-5}		2×10^{-5}		2×10^{-5}		3×10^{-5}	

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