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ON MAXIMAL FLOW PROBLEM IN A TRANSPORTATION NETWORK WITH A BUNDLE

TAKASHI KOBAYASHI

Faculty of Engineering, University of Tokyo

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Abstract

The author reduces the maximal flow problem in a transportation network with a bundle, which is a subset of its arcs and associated with a capacity, to a parametric programming problem. The latter is related to one of the usual transportation network problems, and a primal-dual algorithm for solving it is shown. This approach is very similar to Jewell's for the critical path analysis in a project with a divisible activity.

1. Introduction

Berge [1] presented the maximal flow problem in a transportation network with bundles as one of special problems of maximal flow. Bundle implies a subset of arcs of the network and is associated with a capacity.

The author reduces the maximal flow problem in a network with a bundle to a parametric programming problem which is related to one of the usual transportation network problems, and shows a primal-dual algorithm for solving the latter. It is very similar to Jewell's algorithm [4] used for the critical path analysis in a project with a divisible activity.

2. Mathematical Formulations of Problems

Let A denote the set of directed arcs, ordered pairs of nodes, of a transportation network with n nodes: 1 (the source), 2, $\cdots$, $n-1$, n (the sink). We shall consider the case when A consists of a bundle B whose capacity is C and its complement $\bar{B}$, each arc (i, j) of which is associated with capacity C_{ij}. We define $P(i)$ and $S(i)$ for node i as follows:

$$P(i)=\{j|(j, i)\varepsilon A\},$$

$$S(i)=\{j|(i, j)\varepsilon A\}.$$

Then, a mathematical formulation of the original problem is obtained:

$$\text{Maximize } \mu,$$

subject to

$$\sum_{j\in S(i)} x_{ij}-\sum_{j\in P(i)} x_{ji}=\begin{cases} \mu, & \text{for } i=1, \\ 0, & \text{for } i\neq 1,\ n, \\ -\mu, & \text{for } i=n, \end{cases}$$

$$C_{ij}\geqq x_{ij}\geqq 0 \qquad \text{for } (i, j)\varepsilon\bar{B},$$

$$x_{ij}\geqq 0 \qquad \text{for } (i, j)\varepsilon B,$$

$$\sum_{(i, j)\in B} x_{ij}\leqq C.$$

Here, we shall introduce a parameter λ and consider a parametric programming problem:

$P|\lambda$

$$\text{Maximize } \{\lambda\mu-\sum_{B} x_{ij}\},$$

subject to

$$\sum_{j\in S(i)} x_{ij}-\sum_{j\in P(i)} x_{ji}=\begin{cases} \mu, & \text{for } i=1, \\ 0, & \text{for } i\neq 1,\ n, \\ -\mu, & \text{for } i=n, \end{cases}$$

$$C_{ij}\geqq x_{ij}\geqq 0 \qquad \text{for } (i, j)\varepsilon\bar{B},$$

$$x_{ij}\geqq 0 \qquad \text{for } (i, j)\varepsilon B.$$

The relations between the solutions of two problems are given by the followings.

Theorem 1. Let (μ^*, X^*) be an optimal solution of $P|\lambda$ for some positive λ. If $\sum_B x^*_{ij}=C$, it is also optimal to the original problem.

Proof. (μ^*, X^*) is a feasible solution of the original since $\sum_B x^*_{ij}\leqq C$. Let (μ, X) be an optimal solution of the original. Then, (μ, X) is a feasible solution of $P|\lambda$. Hence,

$$\lambda\mu^*-\sum_B x^*_{ij}\geqq\lambda\mu-\sum_B x_{ij}.$$

So, $\lambda\mu^*\geqq\lambda\mu+\sum_B x^*_{ij}-\sum_B x_{ij}=\lambda\mu+(C-\sum_B x_{ij})\geqq\lambda\mu$.

As $\lambda>0$, $\mu^*\geqq\mu$. Therefore, (μ^*, X^*) is an optimal solution of the original problem.

Theorem 2. For λ greater than $|B|$, the number of elements of B, an optimal solution of $P|\lambda$, (μ^*, X^*) is also optimal to the original problem if $\sum_B x^*_{ij}<C$.

Proof. Obviously, (μ^*, X^*) is a feasible solution of the original. Assume that it is not optimal. Then, there exists a path from node 1 to node n, $(i_1=1, i_2, \cdots, i_{m-1}, i_m=n)$, such that

1) $(i_k, i_{k+1})\varepsilon B$,

2) $(i_k, i_{k+1})\varepsilon \bar{B}$ and $x^*_{i_k i_{k+1}}<C_{i_k i_{k+1}}$, or

3) $(i_{k+1}, i_k)\varepsilon A$ and $x^*_{i_{k+1} i_k}>0$.

By selecting a sufficiently small positive number ε and putting

$$\mu=\mu^*+\varepsilon,$$

$$x_{ij}=\begin{cases} x^*_{ij}+\varepsilon, & \text{if } (i,j) \text{ is a forward arc of the path (Case 1 or 2),} \\ x^*_{ij}-\varepsilon, & \text{if } (i,j) \text{ is a reverse arc of the path (Case 3),} \\ x^*_{ij}, & \text{otherwise,} \end{cases}$$

we can make (μ, X) a feasible solution of $P|\lambda$. Then, we have

$$\lambda\mu-\sum_B x_{ij}\geqq\lambda(\mu^*+\varepsilon)-\sum_B (x^*_{ij}+\varepsilon)=\lambda\mu^*-\sum_B x^*_{ij}+(\lambda-|B|)\varepsilon>\lambda\mu^*-\sum_B x^*_{ij},$$

which contradicts that (μ^*, X^*) is optimal to $P|\lambda$. Hence, (μ^*, X^*) is optimal to the original. The proof is completed.

From the theorems above, we may solve a parametric programming problem $P|\lambda$, instead of solving the original directly. $P|\lambda$ is one of the problems related to the following transportation network flow problem:

$D^*|\mu$ Minimize $\{\sum_{B} x_{ij}\}$,

subject to

$$\sum_{j\in S(i)} x_{ij} - \sum_{j\in P(i)} x_{ji} = \begin{cases} \mu & \text{for } i=1, \\ 0 & \text{for } i\neq 1, n, \\ -\mu & \text{for } i=n, \end{cases}$$

$$C_{ij} \geqq x_{ij} \geqq 0 \qquad \text{for } (i,j)\varepsilon B,$$

$$x_{ij} \geqq 0 \qquad \text{for } (i,j)\varepsilon B.$$

Note that μ is a parameter here.

According to Kurata's formulations [5], we shall introduce the dual problem of $P|\lambda$ (called $D|\lambda$), the restricted one based on (v), which is a part of an optimal solution of $D|\lambda$, (v, t) (called $RP|\lambda(v)$), and the restricted one based on (X), an optimal solution of $D^*|\mu$ (called $RP^*|\mu(X)$).

$D|\lambda$ Minimize $\{\sum_{\bar{B}} C_{ij} t_{ij}\}$,

subject to

$$v_n - v_1 = \lambda,$$

$$\left.\begin{array}{l} t_{ij} \geqq 0, \\ v_i - v_j + t_{ij} \geqq 0, \end{array}\right\} \text{for } (i,j)\varepsilon \bar{B},$$

$$v_i - v_j \geqq -1, \qquad \text{for } (i,j)\varepsilon B.$$

$RP|\lambda(v)$ Maximize μ,

subject to

$$\sum_{j\in S(i)} x_{ij} - \sum_{j\in P(i)} x_{ji} = \begin{cases} \mu & \text{for } i=1, \\ 0 & \text{for } i\neq 1, n, \\ -\mu & \text{for } i=n, \end{cases}$$

$$x_{ij} = 0 \qquad \text{if } (i,j)\varepsilon \bar{B} \text{ and } v_i - v_j > 0,$$

$$C_{ij} \geqq x_{ij} \geqq 0 \qquad \text{if } (i,j)\varepsilon \bar{B} \text{ and } v_i - v_j = 0,$$

$$x_{ij} = C_{ij} \qquad \text{if } (i,j)\varepsilon \bar{B} \text{ and } v_i - v_j < 0,$$

$$x_{ij} = 0 \qquad \text{if } (i,j)\varepsilon B \text{ and } v_i - v_j > -1,$$

$$x_{ij} \geqq 0 \qquad \text{if } (i,j)\varepsilon B \text{ and } v_i - v_j = -1,$$

where (v) is a part of an optimal solution of $D|\lambda$, (v, t).

$RP^*|\mu(X)$ Maximize λ,

subject to

$$\begin{aligned}
&v_n - v_1 - \lambda = 0,\\
&v_i - v_j \geqq 0 \text{ if } (i,j)\varepsilon\bar{B} \text{ and } x_{ij}=0,\\
&v_i - v_j = 0 \text{ if } (i,j)\varepsilon\bar{B} \text{ and } C_{ij} > x_{ij} > 0,\\
&v_i - v_j \leqq 0 \text{ if } (i,j)\varepsilon\bar{B} \text{ and } x_{ij} = C_{ij},\\
&v_i - v_j \geqq -1 \text{ if } (i,j)\varepsilon B \text{ and } x_{ij}=0,\\
&v_i - v_j = -1 \text{ if } (i,j)\varepsilon B \text{ and } x_{ij}>0,
\end{aligned}$$

where (X) is an optimal solution of $D^*|\mu$.

The following theorems are proved by Kurata's propositions.

Theorem 3. (μ, X) is a feasible solution of $RP|\lambda(v)$ if and only if it is an optimal solution of $P|\lambda$.

Theorem 4. Let (λ, v) be an optimal solution of $RP^*|\mu(X)$, and $t_{ij}=\max(v_j - v_i, 0)$ for $(i,j)\varepsilon\bar{B}$.

Then, 1) (v, t) is optimal to $D|\lambda$, and

2) (μ, X) is a feasible solution of $RP|\lambda(v)$ and also an optimal one of $P|\lambda$.

The next theorem is available when we wish to get an optimal solution for a special value of C.

Theorem 5. Let $(\mu_1 . X_1)$ and (μ_2, X_2) be both optimal solutions of $P|\lambda$. Then, for $0<\theta<1$, $[\theta\mu_1+(1-\theta)\mu_2,\ \theta X_1+(1-\theta)X_2]$ is also optimal to $P|\lambda$.

3. Algorithm

Since it is easily seen that $(X=O)$ is an optimal solution of $D^*|\mu=0$, we start with $\mu=0$, $X=O$, and continue to solve $RP^*|\mu(X)$, or $RP|\lambda(v)$ alternatively until it occurs that $\lambda>|B|$ or that $\sum_B x_{ij} \geqq C$. The flow chart of the algorithm is shown in Fig. 1.

To solve $RP^*|\mu(X)$, Iri's Θ matrix method [3] is applicable, and the procedure for solving $RP|\lambda(v)$ given its feasible solution can be constructed similarly as the usual maximal flow labeling procedure [2]. Here, it is very useful that an optimal solution of $RP|\lambda(v)$ is a feasible solu-

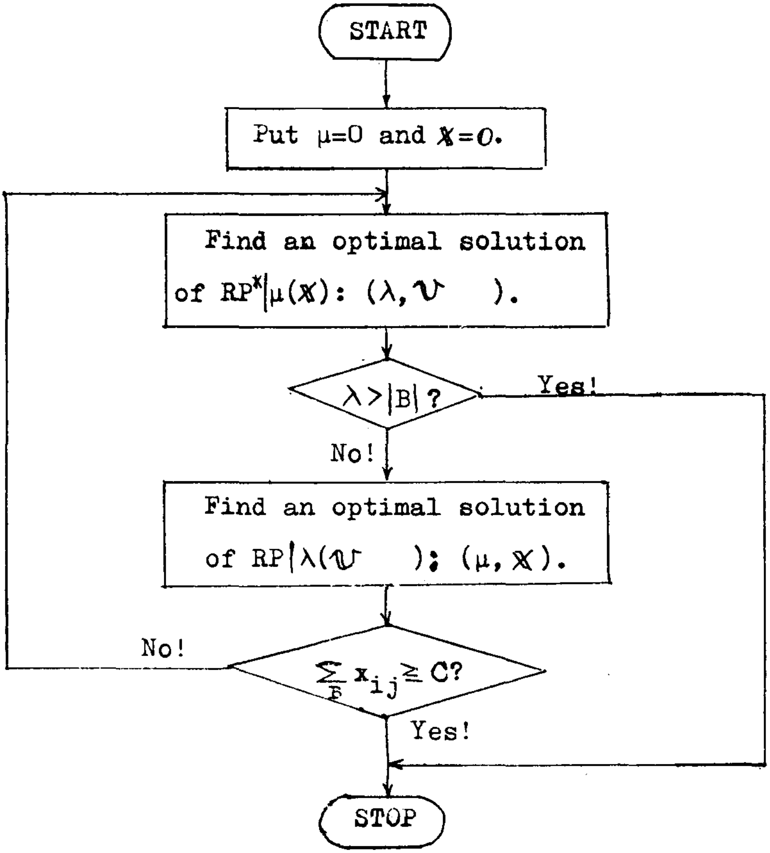

Fig. 1.

tion in the next step (Theorem 4). Details of these procedures are deleted.

4. Remarks

In a network with two or more bundles, it is much difficult to solve the problem by this approach.

It is advisable that the labeling procedure and Θ matrix method are programmed in more general forms so as to be applied to the transportation network problem, the critical path method, and some other problems to arise.

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