J. Operations Research Soc. of Japan
Vol. 10, Nos. 1 & 2 October 1967

THE CAPACIATED TRANSPORTATION PROBLEM IN LINEAR FRACTIONAL FUNCTIONALS PROGRAMMING

SURESH CHANDRA

Department of Mathematics, Indian Institute of Technology, Kanpur, India

(Received April 11, 1967)

Abstract

The present paper describes a method for solving the capaciated transportation problem in linear fractional functionals programming. The method takes advantage of the special structure of the problem and expresses it as a multi-index problem with linear fractional objective function. An algorithm is developed for the latter and hence the original problem can be solved.

Introduction

From practical point of view the capaciated transportation problems are very important. Such problems in linear programming can be solved efficiently by the Ford and Fulkerson's [1] primal dual algorithm for the capaciated Hitchcock problem. It is equally important to consider such problem with nonlinear objective function. This paper is a step in this direction, where the objective function taken is linear fractional. The mathematical formulation of the problem considered here, is as follows:—

" Given two $(n \times m)$ cost matrices $C=||c_{ij}||$ and $D=||d_{ij}||$, to determine an $(n \times m)$ solution matrix $x=||x_{ij}||$ such that the objective function

$$(1) \qquad Z = \frac{\left[\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} + \alpha\right]}{\left[\sum_{i=1}^{n} \sum_{j=1}^{m} d_{ij} x_{ij} + \beta\right]}$$

is maximized subject to the constraints

$$(2) \qquad \sum_{j=1}^{m} x_{ij} = b \quad (i=1,2,\cdots n),$$

$$(3) \qquad \sum_{i=1}^{m} x_{ij} = a_j \quad (j=1,2,\cdots m),$$

$$(4) \qquad 0 \leq x_{ij} \leq g_{ij}, \quad \text{for all } i \text{ and } j,$$

where α and β are given scalars and $G=||g_{ij}||$ is another given matrix, whose (i,j)th element represents the maximum number of units which may be shipped from ith origin to jth distination. The following assumptions are made here:—

(i) The denominater of the objective function Z is positive for all feasible solutions.

(ii) $\sum_{i=1}^{n} b = \sum_{j=1}^{m} a_j$, which is clearly a necessary condition for the problem to have a feasible solution. The condition is of course not sufficient because of the upper bound on x_{ij}. Hence it is further assumed that the set of S' of feasible solutions is not null.

Let us call the problem considered above as problem (A).

The paper has been divided into three sections. In Section I, the problem (A) has been reformulated as a multi-index problem with linear fractional objective function and general problem of this type is defined. In Section II and Section III, the main algorithm is developed for the general multi-index problem with linear fractional objective function. The algorithm is based on the Haley's [3] method for the multi index problem and Kantiswarup's [5] simplex-like technique for the linear fractional functionals programming.

Section I

The following reformulation of the problem (A) hes been made under the consideration that it has a special structure. We note that the constraint $0 \leq x_{ij} \leq g_{ij}$ can be written as

$$\sum_{k=1}^{2} y_{ijk} = E_{ij},$$

where $y_{ij1} = x_{ij}$ and y_{ij2} is the slack variable corresponding to this constraint. Now let us take:

$$c_{ij1} = c_{ij}, \qquad d_{ij1} = d_{ij},$$

$$c_{ij2} = 0, \qquad d_{ij2} = 0,$$

$$y_{ij1} = x_{ij}, \qquad y_{ij2} = \text{slack variable},$$

$$A_{j1} = a_j, \qquad A_{j2} = \sum_{i=1}^{n} g_{ij} - a_j,$$

$$B_{1i} = b_i, \qquad B_{2i} = \sum_{j=1}^{m} g_{ij} - b_i,$$

$$E_{ij} = g_{ij},$$

with these notations the problem (A) can be expressed as

$$(5) \qquad Z^* = \frac{\left[\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{2} c_{ijk} y_{ijk} + \alpha\right]}{\left[\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{2} d_{ijk} y_{ijk} + \beta\right]}$$

subject to

$$\sum_{i=1}^{n} y_{ijk}=A_{jk},$$

(6) $$\sum_{j=1}^{m} y_{ijk}=B_{ki},$$

$$\sum_{k=1}^{2} y_{ijk}=E_{ij},$$

$$y_{ijk}\geqslant 0,$$

$$\sum_{j=1}^{m} A_{jk}=\sum_{i=1}^{n} B_{ki},\quad \sum_{k=1}^{2} B_{ki}=\sum_{j=1}^{m} E_{ij},\quad \sum_{i=1}^{n} E_{ij}=\sum_{k=1}^{2} A_{jk},$$

$$\sum_{j=1}^{m}\sum_{k=1}^{2} A_{jk}=\sum_{i=1}^{n}\sum_{k=1}^{2} B_{ki}=\sum_{i=1}^{n}\sum_{j=1}^{m} E_{ij}.$$

We shall call this problem as problem (B).

The problem (B) is termed as the multi-index problem with linear fractional objective function. The general problem of this type (Call as problem (C)) is

(7) Minimize $$Z=\frac{\left[\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} c_{ijk}\,x_{ijk}+\alpha\right]}{\left[\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} d_{ijk}\,x_{ijk}+\beta\right]}$$

subject to

$$\sum_{i=1}^{m} x_{ijk}=A_{jk},$$

(8) $$\sum^{m} x_{ijk}=B_{ki},$$

$$\sum_{k=1}^{p} x_{ijk}=E_{ij},$$

$$x_{ijk}\geqslant 0,$$

$$(9)\qquad \sum_{j=1}^{m} A_{jk}=\sum_{i=1}^{n} B_{ki},\quad \sum_{k=1}^{p} B_{ki}=\sum_{j=1}^{m} E_{ij},\quad \sum_{i=1}^{n} E_{ij}=\sum_{k=1}^{p} A_{jk},$$

$$\sum_{j=1}^{m}\sum_{k=1}^{p} A_{jk}=\sum_{k=1}^{p}\sum_{i=1}^{n} B_{ki}=\sum_{i=1}^{n}\sum_{j=1}^{m} E_{ij},$$

with the assumption that the denominator of the objective function is positive for all feasible solutions of problem (C).

We notice that in the formulation (B) no other restriction is added and the same objective function is minimized, so the formulation is valid. The variables with $p=1$ give the solution to the main problem and those with $p=2$, represent the slack in the constraints.

Section II

In this section and next section we shall develope an algorithm for the problem (C). Now onwords we are considering problem (C) only.

We determine two sets of shadow costs namely $\{u_{jk}, v_{ki}, w_{ij}\}$ and $\{u^*_{jk}, v^*_{ki}, w^*_{ij}\}$ from the equations

$$c_{ijk}=u_{jk}+v_{ki}+w_{ij}$$

and

$$d_{ijk}=u^*_{jk}+v^*_{ki}+w^*_{ij}$$

where $\{i, j, k\}$ takes suffixes over the basic variables. Suppose

$$(9)\qquad c'_{ijk}=c_{ijk}-(u_{jk}+v_{ki}+w_{ij})$$

and

$$(10)\qquad d'_{jjk}=d_{ijk}-(u^*_{jk}+v^*_{ki}+w^*_{ij})$$

It can be seen that there are $(mn+pm+np)$ shadow costs in each set and $(mn+pm+np-m-n-p+1)$ equations (corresponding to basic variables) for each set. So $(m+n+p-1)$ shadow costs are to be put arbitrary zero. The remaining shadow costs for each set can now be evaluated since for each set the number of equations and number of unknowns are same.

The method for the computation of the two sets of shadow costs $\{u_{jk}, v_{ki}, w_{ij}\}$ and $\{u^*_{jk}, v^*_{ki}, w^*_{ij}\}$ is same as given in [3] for the multi-index problem of linear programming. Thus having determined $\{u_{jk}, v_{ki}, w_{ij}\}$ and $\{u^*_{jk}, v^*_{ki}, w^*_{ij}\}$ it is easy to compute c'_{ijk} and d'_{ijk} given by equations (9) and (10) respectively.

Section III

In this section we shall find an improved basic feasible solution which requires the determination of entering and departing vectors. We shall first express the objective function of problem (C) in terms of the non-basic variables.

We have

$$Z=\frac{\left[\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} c_{ijk}\, x_{ijk}+\alpha\right]}{\left[\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} d_{ijk}\, x_{ijk}+\beta\right]}=\frac{N}{D} \quad \text{(Say)}. \tag{11}$$

Then

$$\begin{aligned} N= & \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} c_{ijk}\, x_{ijk}+\sum_{j=1}^{m}\sum_{k=1}^{p}(A_{jk}-\sum_{i=1}^{n} x_{ijk})\, u_{jk} \\ & +\sum_{k=1}^{p}\sum_{i=1}^{n}(B_{ki}-\sum_{j=1}^{m} x_{ijk})\, v_{ki} \\ & +\sum_{i=1}^{n}\sum_{j=1}^{m}(E_{ij}-\sum_{k=1}^{p} x_{ijk})\, w_{ij}+\alpha \end{aligned}$$

as

$$\sum_{i=1}^{n} x_{ijk}=A_{jk}, \quad \sum_{j=1}^{m} x_{ijk}=B_{ki}, \quad \sum_{k=1}^{p} x_{ijk}=E_{ij}.$$

Thus

$$N=\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p}(c_{ijk}-u_{jk}-v_{ki}-w_{ij})x_{ijk}$$
$$+\left[\sum_{j=1}^{m}\sum_{k=1}^{p}A_{jk}u_{jk}+\sum_{i=1}^{n}\sum_{k=1}^{p}B_{ki}v_{ki}+\sum_{i=1}^{n}\sum_{j=1}^{m}E_{ij}w_{ij}+\alpha\right]$$

or

$$N=\left[\sum_{i,j,k\in S}c'_{ijk}\,x_{ijk}+V_1\right],$$

where $\sum_{i,j,k\in S}$ denotes the summation extending over the set of non-basic variables, and

$$V_1=\left[\sum_{j=1}^{m}\sum_{k=1}^{p}A_{jk}u_{jk}+\sum_{i=1}^{n}\sum_{k=1}^{p}B_{ki}v_{ki}+\sum_{i=1}^{n}\sum_{j=1}^{m}E_{ij}w_{ij}+\alpha\right].$$

Similarly

$$D=\left[\sum_{i,j,k\in S}d_{ijk}\,x_{ijk}+V_2\right],$$

where

$$V_2=\left[\sum_{j=1}^{m}\sum_{k=1}^{p}A_{jk}u^*_{jk}+\sum_{i=1}^{n}\sum_{k=1}^{p}B_{ki}v^*_{ki}+\sum_{i=1}^{n}\sum_{j=1}^{m}E_{ij}w^*_{ij}+\beta\right].$$

Therefore from (11) we have

$$Z=\frac{\left[\sum_{i,j,k\in S}c_{ijk}\,x_{ijk}+V_1\right]}{\left[\sum_{i,j,k\in S}d_{ijk}\,x_{ijk}+V_2\right]}. \tag{12}$$

Now from (12), differentiating Z with respect to the non-basic variable x_{ijk} (i,j,k ranging over the set S), we have

(13) $$\left(\frac{\partial Z}{\partial x_{ijk}}\right)=\frac{c'_{ijk}\left[V_2+\sum_{i,j,k\in S}d'_{ijk}\,x_{ijk}\right]-d'_{ijk}\left[V_1\sum_{i,j,k\in S}c'_{ijk}\,x_{ijk}\right]}{\left[\sum_{i,j,k\in S}d'_{ijk}\,x_{ijk}+V_2\right]^2}.$$

Let $\left(\frac{\partial Z}{\partial x_{ijk}}\right)_0$ denotes the value of $\left(\frac{\partial Z}{\partial x_{ijk}}\right)$ at the basic feasible solution, then from (13) we have

$$\left(\frac{\partial Z}{\partial x_{ijk}}\right)_0=\frac{[V_2\, c'_{ijk}-V_1\, d'_{ijk}]}{[V_2]^2}.$$

Since at a basic feasible solution all non-basic variables are at zero level. Now due to similar arguments as in [6], the absence of degeneracy the optimality criterion comes out to be

$$(14) \qquad \Delta_{ijk}=[V_2\, c'_{ijk}-V_{1ijk}\, d'_{ijk}] \geqslant 0$$

for all non-basic cells (i, j, k), for basic cells Δ_{ijk} is obviously zero. If all Δ_{ijk} are not$\geqslant 0$, let $\Delta_{rst}=\min \{\Delta_{ijk} | \Delta_{ijk} < 0\}$, then the inclusion of x_{rst} instead of one of the current basic variables will improve the value of Z.

Thus the whole iterative procedure of the multi-index problem given in [3, 4] can be taken over the problem (C). The only change will be in the optimately criterison (14). The methods for finding the initial basic feasible solution, shadow costs and treating degeneracy remain same as for the Haley's multi-index problem.

Acknowledgements

I am extremely grateful to Prof. J.N. Kapur and Dr. S.K. Gupta, Department of Mathematics, Indian Institute of Technology, Kanpur, for their continued help and inspiring guidance.

References

[1] Ford; L.R., Jr and D.R. Fulkerson, " A Primal Dual Algorithm for the Capaciated Hitckcock Problem " *Naval Research Logistics Quarterly*, 4 (1957), 47–54.

[2] Hadley, G., " *Linear Programming*," Addision-Wesely Publication Company Inc., (1963).

[3] Haley, K.B., " The Solid Transportation Problem " *Operations Research*, 10 (1962), 448–463.

[4] Haley, K.B., "The Multi-Index Problem" *Operations Research*, **11** (1963), 368-379.

[5] Kantiswarup, "Linear Fractional Functionals Programming" *Operations Research*, **13**, (1965) 1029-1036.

[6] Kantiswarup, "Some Aspects Linear Fractional Functionals Programming" *Australian Journal of Statistics*, 7 (1965), 90.