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SOME EXTENSIONS OF THE M-MACHINE SCHEDULING PROBLEM

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§1 Introduction

In the early years of the research of job-shop scheduling problems, there had been two types among them. One of them is a so-called m-machine scheduling problem firstly studied and solved by S.M. Johnson for two and three machines cases [1]. The other one is a scheduling problem on two machines with time lags designed by L.G. Mitten [2].

In the above, m-machine scheduling problem means a problem to find the optimal sequence of n-items that should be processed on m-machines in minimal total elapsed time, where the processing requires that m-machines be used by the same numerical order for any item and n-items be sequenced identically on each machine.

As Johnson has indicated in his " Discussion " [3], scheduling problem with time lags has a similar construction with m-machine scheduling problem. So, in order to unify the problems of these two types, in this paper we shall consider a generalized problem which contains, as its special cases, both m-machine scheduling problem and scheduling problem with time lags for m-machines case.

Then we shall show Johnson's type criterions under some conditions for this generalized problem that contain former results in each problem of these two types and ours [4].

§2 Generalized Problem

A generalized problem considered in this paper is a problem to find the optimal sequence of n-items that are processed on m bottleneck machines having an intermediate imbottleneck machine among each two of them in minimum total elapsed time under next conditions for operations.

Firstly, the processing requires that m-bottleneck machines and $m-1$ intermediate imbottleneck machines must be used by the same numerical order for any item and n-items must be sequenced identically on each of $2m-1$ machines.

Secondly, only each of $m-1$ imbottleneck machines can be used by many items simultaneously.

In the following we shall give some results by using functional-equation approach formulated in our previous papers [4].

§3 Formulation by Functional-Equation Approach

Let m-bottleneck machines be $M_1, M_2, \cdots, M_m$ and an intermediate imbottleneck machine among M_k, M_{k+1} be $M_{k,k+1}$ and $m_{k,i}, H_{k,i}$ be the processing time of item i on machine $M_k, M_{k,k+1}$ respectively ($k=1, \cdots, m$ for $m_{k,i}$; $k=1, \cdots, m-1$ for $H_{k,i}$). Each item must be processed on $2m-1$ machines following the order $M_1, M_{1,2}, M_2, M_{2,3}, \cdots, M_{m-1,m}, M_m$.

When an optimal scheduling procedure is employed and after the processing of some definite sequence S of items, let machine M_{k+1} is committed t_k hours ahead for machine $M_k (k=1, \cdots, m-1)$ and we put $t=\sum_{k=1}^{m-1} t_k$.

After pre-subsequence S, let two items i, j be processed on machines.

If item i is processed firstly after S, then by defining $f_1(i, t)$= the time consumed in processing the item i, on $2m-1$ machines, it holds (Fig. 1)

$$f_1(i,t)=m_{1,\,i}+g(i,t), \tag{1}$$

where

$$g(i,t)=\sum_{k=2}^{m}\{m_{k,\,i}+H_{k-1,\,i}+\max\,[t_{k-1}^{(i)}-(m_{k-1,\,i}+H_{k-1,\,i}),\ 0]\} \tag{2}$$

and

$$t_1^{(i)}=t_1,\qquad t_k^{(i)}=t_k-\max(m_{k-1,\,i}+H_{k-1,\,i}-t_{k-1}^{(i)},\ 0),\quad (k=2,\cdots,m-1). \tag{3}$$

Here, for brevity, by putting

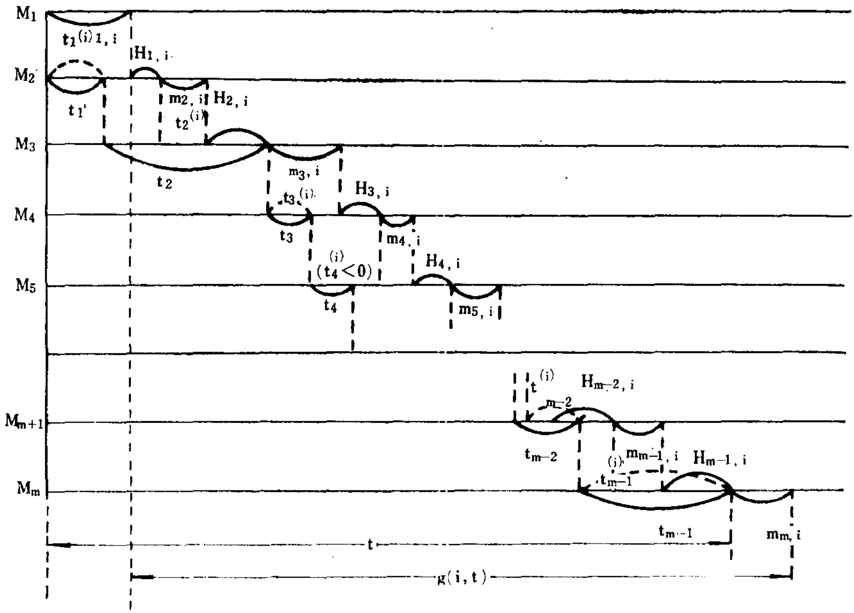

Fig. 1

(4)
$$\begin{cases} m_{k-1,i}+H_{k-1,i}=\underline{R}_{k-1,i} \\ m_{k,i}+H_{k-1,i}=\overline{R}_{k,i}, \end{cases} \qquad (k=2,\cdots,m)$$

they hold from (1), (2), (3) respectively

(1)′
$$f_1(i,t)=\underline{R}_{1,i}-H_{1,i}+g(i,t),$$

(2)′
$$g(i,t)=\sum_{k=2}^{m}\{\overline{R}_{k,i}+\max(t_{k-1}^{(i)}-\underline{R}_{k-1,i},\ 0)\},$$

(3)′
$$t_1^{(i)}=t_1, \qquad t_k^{(i)}=t_k-\max(R_{k-1,i}-t_{k-1}^{(i)},\ 0),$$
$$(k=2,\cdots,m-1)$$

where $t_k^{(i)}$ may be negative.

If we choose item j to follow after item i, then by defining $f_2(i,j,t)=$ the time consumed in processing two items i,j in this order after S on $2m-1$ machines, it holds

(5)
$$f_2(i,j,t)=\underline{R}_{1,i}-H_{1,i}+f_1(j,g(i,t)),$$

because $g(i,t)$ corresponds to t after the processing of item i.

On the other hand, if we interchange the order of these two items i,j, it holds similarly

(6)
$$f_2(j,i,t)=\underline{R}_{1,j}-H_{1,j}+f_1(i,g(j,t)).$$

Hence, when we process the same operations both after the order S,i,j and the order S,j,i, if the condition, that total elapsed time after the order which yields minimum f_2 is also smaller than that after the other order, be realized (we express this condition by (A)), we obtain the next theorem.

Theorem 1. When condition (A) follows, an optimal ordering is determined by the following rule:
Item i precedes item j if

(7)
$$f_2(i,j,t)<f_2(j,i,t).$$

If there is equality, either ordering is optimal.

Though, for the case when $m=2$ the condition (A) is always realized, for the cases when $m \geqq 3$ the condition (A) is not always realized.

§ 4 Value of $f_1(i, t)$ and $f_2(i, j, t)$

From (3)′, it holds

(8) $$t_k^{(i)} = \sum_{l=1}^{k} t_l - \max\Big[\sum_{l=1}^{k-1} t_l,\quad \sum_{l=1}^{k-2} t_l + \underline{R}_{k-1, i},\quad \sum_{l=1}^{k-3} t_l + \sum_{l=k-2}^{k-1} \underline{R}_{l, i},\quad \cdots, \sum_{l=1}^{2} t_l + \sum_{l=3}^{k-1} \underline{R}_{l, i},\quad t_1 + \sum_{l=2}^{k-1} \underline{R}_{l, i},\quad \sum_{l=1}^{k-1} \underline{R}_{l, i}\Big],$$
$$(k = 1, \cdots, m-1)$$

hence it holds from (1)′ and (2)′

(9) $$f_1(i, t) = m_{m, i} + \max\Big[\sum_{l=1}^{m-1} t_l,\quad \sum_{l=1}^{m-2} t_l + \underline{R}_{m-1, i},\quad \sum_{l=1}^{m-3} t_l + \sum_{l=m-2}^{m-1} \underline{R}_{l, i},\quad \cdots, \sum_{l=1}^{2} t_l + \sum_{l=3}^{m-1} \underline{R}_{l, i},\quad t_1 + \sum_{l=2}^{m-1} \underline{R}_{l, i},\quad \sum_{l=1}^{m-1} \underline{R}_{l, i}\Big]$$

Next, as the term

$$\overline{R}_{l+1, i} + \max(t_l^{(i)} - \underline{R}_{l, i},\ 0)$$

in the expression for $g(i, t)$ corresponds to t_l in that for t and as it holds

$$\sum_{l=1}^{k} \{\overline{R}_{l+1, i} + \max(t_l^{(i)} - \underline{R}_{l, i},\ 0)\} + \sum_{l=k+1}^{m-1} \underline{R}_{l, j}$$
$$= m_{k+1, i} - m_{1, i} + \max\Big[\sum_{p=1}^{k} t_p,\quad \sum_{p=1}^{k-1} t_p + \underline{R}_{k, i}, \cdots$$
$$\cdots, t_1 + \sum_{p=2}^{k} \underline{R}_{p, i},\quad \sum_{p=1}^{k} \underline{R}_{p, i}\Big] + \sum_{l=k+1}^{m-1} \underline{R}_{l, j} \quad (k = 0, \cdots, m-1),$$

so we have

(10) $$f_1(j, g(i, t)) = m_{m, j} + m_{m, i} - m_{1, i} + S(i, j, t),$$

where

$$S(i,j,t)=\max\Big[\sum_{p=1}^{m-1} t_p,\quad \sum_{p=1}^{m-2} t_p+\underline{R}_{m-1,\,i},\quad \cdots,$$

$$t_1+\sum_{p=2}^{m-1}\underline{R}_{p,\,i},\quad \sum_{p=1}^{m-1}\underline{R}_{p,\,i},$$

$$m_{m-1,\,i}+\underline{R}_{m-1,\,j}-m_{m,\,i}+\max\Big(\sum_{p=1}^{m-2} t_p,$$

$$\sum_{p=1}^{m-3} t_p+\underline{R}_{m-2,\,i},\quad \cdots,\quad t_1+\sum_{p=2}^{m-2}\underline{R}_{p,\,i},\quad \sum_{p=1}^{m-2}\underline{R}_{p,\,i}\Big),$$

$$\cdots,\ m_{2,\,i}+\sum_{l=2}^{m-1}\underline{R}_{l,\,j}-m_{,\,i}+\max(t_1,\underline{R}_{1,\,i}),$$

$$m_{1,\,i}+\sum_{l=1}^{m-1}\underline{R}_{l,\,j}-m_{m,\,i}\Big]. \tag{11}$$

So that, from (5) it holds

$$f_2(i,j,t)=m_{m,\,j}+m_{m,\,i}+S(i,j,t). \tag{12}$$

Similarly we obtain the valve of $f_2(j,i,t)$ by changing j for i and i for j,

$$f_2(j,i,t)=m_{m,\,i}+m_{m,\,j}+S(j,i,t). \tag{13}$$

Hence the criterion (7) is identical with

$$S(i,j,t)<S(j,i,t). \tag{14}$$

By omitting the term $\sum_{p=1}^{m-1} t_p$ in brackets both in $S(i,j,t)$ and in $S(j,i,t)$ we obtain the expressions $S'(i,j,t)$ and $S'(j,i,t)$ respectively.

Then, if

$$S'(i,j,t)<S'(j,i,t) \tag{15}$$

holds, it holds

$$S(i,j,t)\leqq S(j,i,t).$$

So that, when the condition (A) is realized we can use the inequality (15) as a criterion instead of (7).

§ 5 Case Where Condition (A) is realized

When

$$(16)\quad \begin{cases} \min_i \underline{R}_{k,i} \geqq \max_i \overline{R}_{k+1,i} \quad (k=1, \cdots, h-1), \quad \text{and} \\ \min_i \overline{R}_{k+1,i} \geqq \max_i \underline{R}_{k,i} \quad (k=h+1, \cdots, m-1) \end{cases}$$

hold, where h is a constant integer $(1 \leqq h \leqq m-1)$ $(m \geqq 3)$; let $\boldsymbol{m}_h$ be the set of all item i for which (16) hold, that is,

$$\boldsymbol{m}_h = \{i | \min_i \underline{R}_{k,i} \geqq \max_i \overline{R}_{k+1,i} (k=1, \cdots, h-1); \\ \min_i \overline{R}_{k+1,i} \geqq \max_i \underline{R}_{k,i} (k=h+1, \cdots, m-1)\}.$$

Here we consider the optimal sequencing of all items contained in the set $\boldsymbol{m}_k$, then for any item i in $\boldsymbol{m}_k$ they hold

$$(17)\quad \begin{cases} t_k = R_{k+1,l} \leqq \min_i R_{k,i} (k=1, \cdots, h-1) \\ t_k \geqq R_{k+1,l} \geqq \max_i R_{k,i} (k=h+1, \cdots, m-1), \end{cases}$$

hence the condition (A) is realized and also they hold in the brackets of (9)

$$\sum_{l=1}^{m-1} t_l \geqq \sum_{l=1}^{m-2} t_l + \underline{R}_{m-1,i}, \quad \sum_{l=1}^{m-3} t_l + \sum_{l=m-2}^{m-1} \underline{R}_{l,i}, \\ \cdots, \quad \sum_{l=1}^{h} t_l + \sum_{l=h+1}^{m-1} \underline{R}_{l,i}$$

and

$$\sum_{l=1}^{m-1} \underline{R}_{l,i} \geqq \sum_{l=1}^{h-1} t_l + \sum_{l=h}^{m-1} \underline{R}_{l,i}, \quad \sum_{l=1}^{h-2} t_l + \sum_{l=h-1}^{m-1} \underline{R}_{l,i}, \\ \cdots, \quad t_1 + \sum_{l=2}^{m-1} \underline{R}_{l,i},$$

So that (9) becomes

$$f_1(i,t)=m_{m,\,i}+\max\,[\sum_{l=1}^{m-1}t_l,\quad \sum_{l=1}^{m-1}\underline{R}_{l,\,i}].$$

Next, similarly by (17), (8) becomes

$$t_k^{(i)}=\sum_{p=1}^{k}t_p-\max\,[\sum_{p=1}^{k-1}t_p,\quad \sum_{p=1}^{k-1}\underline{R}_{p,\,i}]\quad (k=1,\cdots,m-1).$$

Therefore it holds from (11) by the transformations of the terms

$$S(i,j,t)=\max\,[\sum_{p=1}^{m-1}t_p,\quad \sum_{p=1}^{m-1}\underline{R}_p,\quad m_{1,\,i}-m_{m,\,i}+\sum_{p=1}^{m-1}\underline{R}_{p,\,j}]. \tag{19}$$

Hence by definition

$$\begin{aligned}S'(i,j,t)&=\max\,[\sum_{p=1}^{m-1}\underline{R}_{p,\,i},\quad m_{1,\,i}-m_{m,\,i}+\sum_{p=1}^{m-1}R_{p,\,j}]\\&=\sum_{p=1}^{m-1}\underline{R}_{p,\,i}+\sum_{p=1}^{m-1}\underline{R}_{p,\,j}-\min\,[\sum_{p=1}^{m-1}\underline{R}_{p,\,j},\\&\qquad\sum_{p=1}^{m-1}\underline{R}_{p,\,i}-m_{1,\,i}+m_{m,\,i}]\\&=\sum_{p=1}^{m-1}\underline{R}_{p,\,i}+\sum_{p=1}^{m-1}\underline{R}_{p,\,j}-\min\,[\sum_{p=1}^{m-1}\underline{R}_{p,\,j},\quad \sum_{p=2}^{m}\overline{R}_{p,\,i}].\end{aligned}\tag{20}$$

So that the criterion (7) in Theorem 1 becomes

$$\min\,[\sum_{p=1}^{m-1}\underline{R}_{p,\,i},\quad \sum_{p=2}^{m}\overline{R}_{p,\,j}]<\min\,[\sum_{p=1}^{m-1}\underline{R}_{p,\,j},\quad \sum_{p=2}^{m}\overline{R}_{p,\,i}]$$

Hence we obtain the next theorem.

Theorem 2. When h is a constant integer $(1\leqq h\leqq m-1)$ $(m\geqq 3)$, for a set

$$\boldsymbol{m}_h=\{i|\min_i \underline{R}_{k,\,i}\geqq \max_i \overline{R}_{k+1,\,i}(k=1,\cdots,h-1)\,;$$

$$\min_i \overline{R}_{k+1,\,i}\geqq \max_i \underline{R}_{k,\,i}(k=h+1,\cdots,m-1)\}$$

an optimal ordering is determined by the following rule: Item i precedes item j if

(21) $$\min\left[\sum_{p=1}^{m-1}\underline{R}_{p,\,i},\quad \sum_{p=2}^{m}\overline{R}_{p,\,j}\right] < \min\left[\sum_{p=1}^{m-1}\underline{R}_{p,\,j},\quad \sum_{p=2}^{m}\overline{R}_{p,\,i}\right]$$

holds. If there is equality, either ordering is optimal.

By substituting $\underline{R}_{k,\,i}=m_{k,\,i}+H_{k,\,i}(k=1,\cdots,m-1)$ and $\overline{R}_{k,\,i}=m_{k,\,i}+H_{k-1,\,i}$ $(k=2,\cdots,m)$ in the formulus in theorem 2, we have the next theorem.

Theorem 2′. When h is a constant integer $(1\leqq h\leqq m-1)$ $(m\geqq 3)$, for a set

$$\boldsymbol{m}_h=\{i|\min_i(m_{k,\,i}+H_{k,\,i})\geqq\max_i(m_{k+1,\,i}+H_{k,\,i})\ (k=1,\cdots,h-1);$$
$$\min_i(m_{k+1,\,i}+H_{k,\,i})\geqq\max_i(m_{k,\,i}+H_{k,\,i})\ (k=h+1,\cdots,m-1)\}$$

an optimal ordering is determined by the following rule:

Item i precedes item j if

(21)′ $$\min\left[\sum_{p=1}^{m-1}(m_{p,\,i}+H_{p,\,i}),\quad \sum_{p=1}^{m-1}(m_{p+1,\,j}+H_{p,\,j})\right] < \min\left[\sum_{p=1}^{m-1}(m_{p,\,j}+H_{p,\,j}),\quad \sum_{p=1}^{m-1}(m_{p+1,\,i}+H_{p,\,i})\right]$$

holds. If there is equality, either ordering is optimal.

From Theorem 2 for $h=m-1$ and $h=1$ respectively, we obtain the next corollary.

Corollary. Either for a set $\{i|\min_i\underline{R}_{k,\,i}\geqq\max_i\overline{R}_{k+1,\,i}(k=1,\cdots,m-2)\}$ or for a set $\{i|\min_i\overline{R}_{k+1,\,i}\geqq\max_i\underline{R}_{k,\,i}(k=2,\cdots,m-1)\}$, an optimal ordering is determined by the same rule as in the theorem 2 or the theorem 2′.

§6 Another Criterion for Set $\boldsymbol{m}_h$ in Theorem 2

If we express the set $\boldsymbol{m}_h$ in the theorem 2 as a direct sum,

(22) $$\boldsymbol{m}_h=\boldsymbol{m}_h^{(1)}+\boldsymbol{m}_h^{(2)}$$

where

$$(23)\qquad \begin{cases} \boldsymbol{m}_h^{(1)}=\{i|i\in\boldsymbol{m}_h\,;\ m_{1,i}\leqq m_{m,i}\} \\ \boldsymbol{m}_h^{(2)}=\{i|i\in\boldsymbol{m}_h\,;\ m_{1,i}> m_{m,i}\} \end{cases}$$

then, being

$$\sum_{p=2}^{m-1}\underline{R}_{p,i}=m_{1,i}+\sum_{p=2}^{m-1}m_{p,i}+\sum_{p=1}^{m-1}H_{p,i},$$

and

$$\sum_{p=2}^{m}\overline{R}_{p,i}=m_{m,i}+\sum_{p=2}^{m-1}m_{p,i}+\sum_{p=1}^{m-1}H_{p,i},$$

they hold

$$\sum_{p=1}^{m-1}\underline{R}_{p,i}\leqq\sum_{p=1}^{m}\overline{R}_{p,i}\qquad\text{for}\quad i\in\boldsymbol{m}_h^{(1)},$$

$$\sum_{p=1}^{m-1}\underline{R}_{p,i}>\sum_{p=2}^{m}\overline{R}_{p,i}\qquad\text{for}\quad i\in\boldsymbol{m}_h^{(2)}.$$

Hence, by the modification of the criterion (21) we have the next theorem.

Theorem 3. An optimal ordering of the items of the set $\boldsymbol{m}_h=\boldsymbol{m}_h^{(1)}+\boldsymbol{m}_h^{(2)}$ in the theorem 2 is determined by the following rule:

(i) An optimal ordering is then the set $\boldsymbol{m}_h^{(1)}$ followed by the set $\boldsymbol{m}_h^{(2)}$.

(ii) In the set $\boldsymbol{m}_h^{(1)}$, item i precedes item j if $\sum_{p=1}^{m-1}\underline{R}_{p,i}<\sum_{p=1}^{m-1}\underline{R}_{p,j}$ holds.

(iii) In the set $\boldsymbol{m}_h^{(2)}$, item i precedes item j if $\sum_{p=2}^{m}\overline{R}_{p,j}<\sum_{p=2}^{m}\overline{R}_{p,i}$ holds.

If there is equality in (ii), (iii), either ordering is optimal.

Remark: In each theorem above mentioned, condition that corresponds to (16) in § 5 can be omitted when $m=2$ and then we obtain the theorem similar to Johnson's result [1] for the case $m=2$.

§ 7 Numerical Example

In this section we give an example of the theorem 2, 2′ or of the theorem 3 in our problem.

Example: we put $m=4$, $n=6$ in the case of Theorem 2. The table of processing times is given in Table 1.1.

Table 1.1. Processing Time (Hour)

i	$m_{1,i}$	$H_{1,i}$	$m_{2,i}$	$H_{2,i}$	$m_{3,i}$	$H_{3,i}$	$m_{4,i}$
1	4	5	3	7	1	7	3
2	4	4	3	8	2	5	5
3	10	6	1	6	3	4	5
4	8	7	1	9	2	6	3
5	5	4	2	10	2	5	4
6	4	6	2	9	3	6	5

Find an optimal ordering of the six items and its minimal total elapsed time.

Solution: Being

$$8=\min_i (m_{1,i}+H_{1,i}) \geqq \max_i (m_{2,i}+H_{1,i})=8$$

and

$$9=\min_i (m_{4,i}+H_{3,i}) \geqq \max_i (m_{3,i}+H_{3,i})=9,$$

$$\boldsymbol{m}_2=(i=1,2,3,4,5,6)\ (h=2).$$

It will be shown below two types of solutions.

(i) First we use the criterion (21) or (21)′ of theorem 2 or 2′. Table of $A=\sum_{p=1}^{3}(m_{p,i}+H_{p,i})$ and $B=\sum_{p=2}^{4}(m_{p,i}+H_{p-1,i})$ is given as Table 1.2.

By the well known rule for two machine's case we obtain an optimal ordering (2, 6, 4, 5, 1, 3) and minimal total elapsed time is 60 hours.

(ii) Next we use the criterion of Theorem 3. In this case

$$\boldsymbol{m}_2^{(1)}=\{i=2,6\},\quad \boldsymbol{m}_2^{(2)}=\{i=1,3,4,5\}.$$

Hence same results follow as in (i).

Table 1.2.

i	A	B
1	27	26
2	26	27
3	30	25
4	33	28
5	28	27
6	30	31

§ 8 *m*-Machine Scheduling Problem

In the above sections, if there is no intermediate imbottleneck machines, then $H_{k,i}\equiv 0\,(k=1,\cdots,m-1)$ and our problem becomes m-machine scheduling problem and by putting

$$R_{k,i}=m_{k,i},\quad R_{k+1,i}=m_{k+1,i}\quad (k=1,\cdots,m-1)$$

in Theorem 2, 2′, Corollary and Theorem 3, we obtain the theorems for m-machine scheduling problem, some of them are the results in our previous paper [4] and the others are new results.

§ 9 Scheduling Problem with Time Lags

9.1 Formulation as a Generalized Problem

The Scheduling problem with time lags is a problem of deciding the order of n jobs (lots) in which n jobs should be processed by m machines in order to minimize the time required to complete all the operations where the processing requires that m machines $M_1, M_2\cdots M_m$ be used by this order for any job and n jobs be requenced identically on each machine and moreover job i be started on machine M_{k+1}, $D_{k,i}$ time units (start lag) after it has been started on machine M_k and job i may not be completed on machine M_{k+1} sooner than $E_{k,i}$ time units (stop lag) after its completion on machine M_k $(k=1,\cdots,m-1)$.

For this problem, if we put

(24) $$H_{k,i}=\max[D_{k,i}-m_{k,i}, E_{k,i}-m_{k+1,i}] \quad (k=1,\cdots,m-1)$$

where $m_{k,i}$ is a processing time of job i on M_k $(k=1,\cdots,m)$.

Then, when $H_{k,i}\geqq 0, H_{k,i}$ means a processing time on intermediate imbottleneck machine $M_{k,k+1}$ between M_k and M_{k+1} and when $H_{k,i}<0$, $H_{k,i}$ means the time in which job i is simultaneously processed on M_k and on M_{k+1}. For both cases m machines are the bottleneck machines. (Fig. 2)

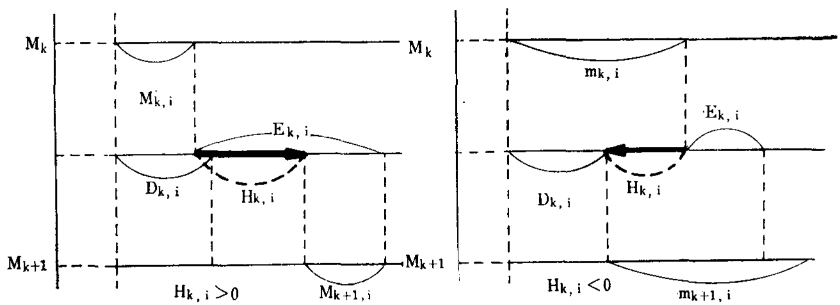

Fig. 2. Operations of Job i at M_k and M_{k+1}.

Hence scheduling problem with time lags can be regarded as the same problem with our problem in § 2.

In the following we shall use same notations as above.

But even through $H_{k,i}$ and $\underline{R}_{k,i}, \bar{R}_{k+1,i}$ may be negative in this case, the formulation in §§ 3, 4 and 5 are justified as well.

9.2 Theorems Derived from That of Generalized Problem

By means of the facts indicated in **9.1**, substituting the right hand side of equation (24) for $H_{k,i}$ in Theorems 2, 2′, Corollary and Theorem 3, they hold

$$\underline{R}_{k,i}=\max(D_{k,i}, E_{k,i}-m_{k+1,i}+m_{k,i}),$$

$$\bar{R}_{k+1,i}=\max(D_{k,i}-m_{k,i}+m_{k+1,i}, E_{k,i}),$$

and we obtain the theorems for scheduling problem with time lags. Especially in the theorem corresponding to Theorem 3 Mitten's result for two machines can be obtained by putting $m=2$.

9.3 Theorem Characteristic to Present Problem

Here we give a theorem of some interest which has a simple criterion under slightly complicated conditions.

Firstly, condition (16) in section 5 can be written as next form;

$$\min_i [\max(D_{k,i}, E_{k,i}-m_{k+1,i}+m_{k,i})] \geqq \max_i [\max(D_{k,i}-m_{k,i}+m_{k+1,i}, E_{k,i})] \quad (k=1,\cdots,h-1), \tag{25}$$

$$\min_i [\max(D_{k,i}-m_{k,i}+m_{k+1,i}, E_{k,i})] \geqq \max_i [\max(D_{k,i}, E_{k,i}-m_{k+1,i}+m_{k,i})] \quad (k=h+1,\cdots,m-1); \tag{26}$$

where h is a constant integer $(1 \leqq h \leqq m-1)$.

If we consider two sets of jobs $L^{(1)}, L^{(2)}$, that is,

$$L^{(1)}=\{i|D_{k,i}-m_{k,i} \geqq E_{k,i}-m_{k+1,i} (k=1,\cdots,m-1); m_{1,i} \leqq m_{m,i}\},$$

$$L^{(2)}=\{i|D_{k,i}-m_{k,i} < E_{k,i}-m_{k+1,i} (k=1,\cdots,m-1); m_{1,i} > m_{m,i}\},$$

then, for $i \in L^{(1)}$, (25) and (26) reduces respectively to

$$\min_i D_{k,i} \geqq \max_i (D_{k,i}-m_{k,i}+m_{k+1,i}) \quad (k=1,\cdots,h-1) \tag{27}$$

and

$$\min_i (D_{k,i}-m_{k,i}+m_{k+1,i}) \geqq \max_i D_{k,i} \quad (k=h+1,\cdots,m-1); \tag{28}$$

also for $i \in L^{(2)}$, (25) and (26) reduces respectively to

$$\min_i (E_{k,i}-m_{k+1,i}+m_{k,i}) \geqq \max_i E_{k,i} \quad (k=1,\cdots,h-1) \tag{29}$$

and

(30) $$\min_i E_{k,i} \geqq \max_i (E_{k,i} - m_{k+1,i} + m_{k,i}) \quad (k=h+1, \cdots, m-1)$$

where h is a constant integer $(1 \leqq h \leqq m-1)$.

Moreover criterion (21) in theorem 2 can be written as follows: that is, for $i, j \in L^{(1)}$ (21) becomes

$$\min \left[\sum_{p=1}^{m-1} D_{p,i}, \sum_{p=1}^{m-1} (D_{p,j} + m_{p+1,j} - m_{p,j}) \right]$$

(31) $$< \min \left[\sum_{p=1}^{m-1} D_{p,j}, \sum_{p=1}^{m-1} (D_{p,i} + m_{p+1,i} - m_{p,i}) \right].$$

But, as they hold for terms in the brackets of (31)

$$\sum_{p=1}^{m-1} (D_{p,j} + m_{p+1,j} - m_{p,j}) = \sum_{p=1}^{m-1} D_{p,j} + m_{m,j} - m_{1,j},$$

$$\sum_{p=1}^{m-1} (D_{p,i} + m_{p+1,i} - m_{p,i}) = \sum_{p=1}^{m-1} D_{p,i} + m_{m,i} - m_{1,i},$$

so, for $i, j \in L^{(1)}$ (21) is identical with

(32) $$\sum_{p=1}^{m-1} D_{p,i} < \sum_{p=1}^{m-1} D_{p,j}.$$

Also for $i, j \in L^{(2)}$, by the some reasons as above, (21) is identical with

(33) $$\sum_{p=1}^{m-1} E_{p,j} < \sum_{p=1}^{m-1} E_{p,i}.$$

Finally for $i \in L^{(1)}$ and $j \in L^{(2)}$, being inequalities

$$\sum_{p=1}^{m-1} D_{p,i} \leqq \sum_{p=1}^{m-1} D_{p,i} + m_{m,i} - m_{1,i}, \qquad \sum_{p=1}^{m-1} E_{p,j} - m_{m,j} + m_{1,j} > \sum_{p=1}^{m-1} E_{p,j}$$

hold, obviously (21) holds. So that job i precedes job j.

Hence we obtain the next theorem.

Theorem 4. Let, for a constant integer h $(1 \leqq h \leqq m-1)$,

$$L^{(1)}=\{i\,|\,D_{k,\,i}-m_{k,\,i}\geqq E_{k,\,i}-m_{k+1,\,i}(k=1,\cdots,m-1)\,;\,m_{1,\,i}\leqq m_{m,\,i}\,;\,\min_i D_{k,\,i}\geqq \max_i (D_{k,\,i}-m_{k,\,i}+m_{k+1,\,i})\ (k=1,\cdots,h-1)\text{ and }\min_i (D_{k,\,i}-m_{k,\,i}+m_{k+1,\,i})\geqq \max_i D_{k,\,i}(k=h+1,\cdots,m-1)\},$$

$$L^{(2)}=\{i\,|\,D_{k,\,i}-m_{k,\,i}<E_{k,\,i}-m_{k+1,\,i}(k=1,\cdots,m-1)\,;\ m_{1,\,i}>m_{m,\,i}\,;\,\min_i (E_{k,\,i}-m_{k+1,\,i}+m_{k,\,i})\geqq \max_i E_{k,\,i}(k=1,\cdots,h-1)\text{ and }\min_i E_{k,\,i}\geqq \max_i (E_{k,\,i}-m_{k+1,\,i}+m_{k,\,i})\ (k=h+1,\cdots,m-1)\}.$$

Then for the set of jobs $L=L^{(1)}+L^{(2)}$, an optimal ordering is determined by the following rules:

(i) An optimal ordering is then $L^{(1)}$ followed by $L^{(2)}$.

(ii) In $L^{(1)}$, job i precedes job j if $\sum_{p=1}^{m-1} D_{p,i}<\sum_{p=1}^{m-1} D_{p,\,j}$ holds.

(iii) In $L^{(2)}$, job i precedes job j if $\sum_{p=1}^{m-1} E_{p,\,j}<\sum_{p=1}^{m-1} E_{p,\,i}$ holds.

If there is equality in (ii), (iii), either ordering is optimal.

Especially for $m=2$ in Theorem 4, next corollary follows.

Corollary 1. In the case of two machines with time lags, let

$$L^{(1)}=\{i\,|\,D_{1,\,i}-m_{1,\,i}\geqq E_{1,\,i}-m_{2,\,i}\,;\ m_{1,\,i}\leqq m_{2,\,i}\},$$

$$L^{(2)}=\{i\,|\,D_{1,\,i}-m_{1,\,i}<E_{1,\,i}-m_{2,\,i}\,;\ m_{1,\,i}>m_{2,\,i}\}.$$

Then, for the set of jobs $L=L^{(1)}+L^{(2)}$, an optimal ordering is determined by the following rule:

(i) An optimal ordering is then $L^{(1)}$ followed by $L^{(2)}$.

(ii) In $L^{(1)}$, job i precedes job j if $D_{1,\,i}<D_{1,\,j}$ holds.

(iii) In $L^{(2)}$, job i precedes job j if $E_{1,\,j}<E_{1,\,i}$ holds.

If there is equality in (ii), (iii) either ordering is optimal.

Moreover, when stop lag $E_{1,\,i}$ is equal to start lag $D_{1,\,i}$ for any job

i, the above corollary 1 reduces to Mitten's result in this situation; that is,

Corollary 2. In the case of two machines with time lags where stop lag $E_{1,k}$ is equal to start lag $D_{1,i}$ for any job i.

Let

$$L^{(1)}=\{i|m_{1,i}\leqq m_{2,i}\},\quad L^{(2)}=\{i|m_{1,i}>m_{2,i}\},$$

then an optical ordering of all given jobs is determined by the following rule:

(i) An optimal ordering is then $L^{(1)}$ followed by $L^{(2)}$.

(ii) In $L^{(1)}$, job i precedes job j if $D_{1,i}<D_{1,j}$ holds.

(iii) In $L^{(2)}$, job i precedes job j if $D_{1,i}>D_{1,j}$ holds.

If there is equality in (ii), (iii), either ordering is optimal.

§10 Remained Work

Some other consideration of our generalized problem that look for optimal ordering under no conditions is possible.

By doing this, general algorithm containing special algorithm for optimal sequencing in each problem of two types can be obtained.

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