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A NOTE ON SEQUENCING PROBLEM WITH UNCERTAIN JOB TIMES

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In the present literature one finds many useful studies [1], [2], [3], [4] on the problem of finding an optimal ordering of a given set of jobs to be processed on a given group of machines. In all these studies, the processing times of all the jobs are assumed to be known and fixed. However, when the processing times are not deterministic but are random variables, one of the objectives to consider is the minimization of the expected time elapsed from the start of first job on the first machine till the processing of last job on last machine. This problem was discussed by Toji Makino [5] in case of two jobs—two machines and two jobs—three machines when the processing times follow exponential distributions and Erlang distributions. In this note we discuss the problem when there are more than two jobs to be processed on two machines and when the processing times are exponentially distributed. The case of 2 jobs with 3 machines is also considered.

1. Two Machine Case

Let the machine be A and B and without loss of generality, we take A to be the first machine and B to be the second machine.

A_i=processing time of job i on machine A, $i=1,2,\ldots,n$
B_i=processing time of job i on machine B, $i=1,2,\ldots,n$
X_i=idle time on machine B immediately before the ith item comes onto it.

Let us assume that A_i and B_i follow exponential distribution with parameters a_i and b_i respectively, *ie.*, Prob. $(A_i \leqslant t)=1-\exp(-t\,a_i)$. As is usual in the sequencing problems we assume the following:—

(a) No machine may process more than one job at any given time and each job once started must be processed to completion.
(b) All jobs are considered equal in importance, *i.e.* there are no due dates.
(c) Jobs are processed by the machines as soon as possible.
(d) In-process inventory is allowable.

Total time elapsed [1] $T(n)$ for the schedule

$$s=(1,2\cdots,n) \text{ is } T(n)=\sum_{i=1}^{n} B_i+\sum_{i=1}^{n} X_i \tag{1}$$

Where

$$\sum_{i=1}^{n} X_i=\max\left(A_1, \sum_{i=1}^{2} A_i-\sum_{i=1}^{1} B_i, \cdots, \sum_{i=1}^{n} A_i-\sum_{i=1}^{n-1} B_i\right)$$

and the objective will be to minimize

$$Y(s)=E\left[\sum_{i=1}^{n} X_i\right]$$

which is the expected cumulated idle time on machine B, the minimum being taken over all possible $n!$ sequences.

1.1 Case when $n=3$

$$
\begin{aligned}
Y(s)&=E(\sum_{i=1}^{3} X_i)=E[\max(A_1, A_1+A_2-B_1,\ A_1+A_2+A_3-B_1-B_2)]\\
&=E[\max(0, A_2-B_1,\ A_2+A_3-B_1-B_2)]+1/a_1\\
&=E[\max(B_1-A_2, 0, A_3-B_2)]+1/a_1+1/a_2-1/b_1\\
&=E[\max(Z_1, 0, Z_2)]+1/a_1+1/a_2-1/b_1
\end{aligned}
$$

Where $Z_1=B_1-A_2$, $Z_2=A_3-B_2$. Note that Z_1, Z_2 are independent random variables. Let $f_1(z_1), f_2(z_2)$ be the p.d.f.'s of Z_1, Z_2 respectively, then

$$
Y(s)=\int_0^{\infty}\lambda d\lambda\left[\frac{d}{d\lambda}\left\{\left(\int_{-\infty}^{\lambda} f_1(z_1)dz_1\right)\left(\int_{-\infty}^{\lambda} f_2(z_2)dz_2\right)\right\}\right]+\frac{1}{a_1}+\frac{1}{a_2}-\frac{1}{b_1} \tag{2}
$$

It can be easily shown that p.d.f. of $f(z)$ of R.V.Z=A−B where A and B are exponentially distributed with parameters a and b respectively is

$$
f(z)=\begin{cases}(ab/a+b)\exp(bz) & z<0\\ (ab/a+b)\exp(-az) & z>0\end{cases}
$$

From (2) we can, after some easy simplifications obtain

$$
Y(s)=\frac{b_1 b_2}{a_3(a_3+b_2)}\left[\frac{a_2+a_3+b_1}{(a_2+b_1)(a_3+b_1)}\right]-\frac{1}{a_2+b_2}+\frac{1}{a_1}+\frac{1}{a_2} \tag{3}
$$

Here s is the sequence $(1,2,3)$ of jobs. Consider the sequence s^1 obtained from s by interchanging the first two jobs in s, *i.e.* $s^1=(2,1,3)$. s is preferable to s^1 *i.e.* job 1 precedes job 2 if $Y(s)-Y(s^1)\leqslant 0$. $Y(s^1)$ is obtained by interchanging subscript 1, and 2 of a, and b in (3). It can be shown that s is preferable to s^1 if and only if

$$
a_1+b_2\geqslant a_2+b_1 \tag{4}
$$

Similarly it can be verified that job j precedes job $j+1$ if

$$
a_j+b_{j+1}\geqslant a_{j+1}+b_j \tag{5}
$$

Further, the above relation is transitive *i.e.* if 1 precedes 2, and 2 precedes 3, then 1 precedes 3 for 1 precedes $2 \rightarrow a_1+b_2 \geqslant a_2+b_1$ and 2 precedes $3 \rightarrow a_2+b_3 \geqslant a_3+b_2$. By adding we get $a_1+b_3 \geqslant a_3+b_1 \rightarrow 1$ precedes 3. It can be recalled that in the case of fixed A's and B's the criterion for the precedence of j over $(j+1)$ is

$$\min (A_j, B_{j+1}) < \min (A_{j+1}, B_j).$$

It is interesting to note that in the random case when the process times are exponentially distributed, $E[\min (A_j, B_{j+1})] < E[\min (A_{j+1}, B_j)] \rightarrow a_j+b_{j+1} > a_{j+1}+b_j$ which is the criteria we have obtained for the precedence of j over $(j+1)$ in the optimal sequence. The criterion for the interchange also holds for $n=4$. We believe that the criterion is true for any n. We have however not been able to prove this. Toji Makino [5] considered the 2 jobs—2 machine case and in deriving the criterion used the moment generating functions to obtain the expected total elapsed time. Our approach is simpler than his for $n=2$, and we considered the cases $n=3$ & $n=4$ as well.

2. Three Machine Case

Let us take C to be the third machine and C_i to be the processing time of job i on machine C, $i=1,2,\cdots,n$. Let C_i follow exponential distribution with parameters c_i. T_i=idle time on the machine C immediately before the ith item comes onto it. Let $Y(s)$ denote again the total expected idle time on machine C for the sequence s.

2.1 Case when $n=2$

It is easy to see that

$$\begin{aligned} Y(s)=E(\sum_{i=1}^{2} T_i)&=E[\max (A_1+B_1, A_1+A_2+B_2-C_1, B_1+B_2+A_1-C_1)] \\ &=E[\max (0, A_2+B_2-B_1-C_1, B_2-C_1)]+1/a_1+1/b_1 \\ &=E[\max (C_1-B_2, A_2-B_1, 0)]+1/a_1+1/b_1+1/b_2-1/c_1 \end{aligned}$$

Let $Z_3=C_1-B_2, Z_4=A_2-B_1$ and $f_3(z_3), f_4(z_4)$ be p.d.f's of Z_3, Z_4 respectively

Then

$$Y(s)=\int_0^{\infty} \lambda d\lambda \left[\frac{d}{d\lambda} \left\{ \left(\int_{-\infty}^{\lambda} f_3(z_3)dz_3 \right) \left(\int_{-\infty}^{\lambda} f_4(z_4)dz_4 \right) \right\} \right]$$

$$+\frac{1}{a_1}+\frac{1}{b_1}+\frac{1}{b_2}-\frac{1}{c_1} \tag{2}$$

$$=\frac{-b_2 b_1}{(b_2+c_1)(a_2+b_1)(a_2+c_1)}+\frac{b_1}{a_2(a_2+b_1)}+\frac{b_2}{c_1(b_2+c_1)}$$

$$+\frac{1}{a_1}+\frac{1}{b_1}+\frac{1}{b_2}-\frac{1}{c_1} \tag{7}$$

Here $s=(1,2)$. For comparison with the schedule $s^1=(2,1)$, $Y(s^1)$ can be easily obtained from (7) by interchanging the subscripts 1,2 of a, b and c. Expression (7) for the expected total idle time is much simpler than one given by T. Makino [5].

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